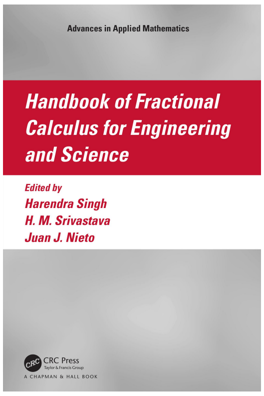

Handbook of Fractional Calculus for Engineering and Science

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Analytical and Numerical Methods to Solve the Fractional Model of the Vibration Equation

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Analytical and Numerical Methods to Solve the Fractional Model of the Vibration Equation

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CONTENTS

1.1 Introduction

Fractional partial differential equations (FPDEs) are found in many different branches of science and engineering, such as hydrodynamics, electroanalytical chemistry, quantum science, viscoelastic mechanics, signal image processing, chain-breaking of polymer materials, molecular spectrum, and

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anomalous diffusion process of ions in nerve cells [1–4]. Also, PDEs with a fractional order have been used to simulate the flow and filtration of a fluid in a porous fractal medium. The use of fractional derivatives (FDs) for modeling real physical processes or environments leads to the appearance of equations containing derivatives and integrals of fractional order in addition to the classical ones. Researchers have focused their efforts on fractional-order physical models [5, 6] because of the material's dynamic and viscoelastic behavior [7]. As a result, the model of fractional order is widely employed to model the frequency apportionment of structural damping mechanisms [8, 9]. An intrinsic multiscale existence of these operators is an interesting feature. As a consequence, memory effects (i.e., a system's response is a function of its previous history) are enabled by time-fractional operators, while non-local and scale effects are enabled by space-fractional operators. Fractional analysis is used in many areas of science, including nonlinear biological processes, solid-state mechanics, field theory, control theory, friction, and fluid dynamics [10]. For the study of fractionally damped viscoelastic material, Josefson and Enelund [11] employed the finite element scheme. In problems of linear viscoelasticity, fractional calculus has been widely used [12–14]. In the design of buildings and other facilities, concrete structures are used, because they are sturdy, reliable, and robust. At the same time, the surface of concrete structures is susceptible to major damage. Therefore, a composite with enhanced operating characteristics is currently being developed, based on a concrete-blend, polymer concrete, which is characterized by greater tolerance to moisture, chemical compounds, low temperatures, and toughness relative to concrete. It is possible to simulate polymer concrete as a collection of solid filler granules contained in a viscoelastic medium [15–18]. The fractional oscillator equation describes transverse movement under the control of the force of gravity or the exterior force of a filler granule. Thus, the substitution of concrete for polymer concrete equates to the substitution of second-order differential equations with FDEs. Special attention is given to the use of fractional calculus to establish better mathematical models of many real-world issues. Many scholars have described the theoretical evolution and implementation of fractional calculus. FDEs are more pragmatic to display natural phenomena and have been utilized in many branches of applied mathematics [19–23]. In some significant and groundbreaking books [12, 24–27], the FDs method has been applied to a wide range of physical problems in science and engineering and in physical models. The dynamic formulation of the problem with viscoelastic damping and its application in science and engineering are examined in the next section.

The chapter is organized as follows. In Section 1.2, we discuss the model's construction and its application in science and engineering. FVEs are presented in Section 1.3. In Section 1.4, we derive some simple definitions and essential lemmas. In Section 1.5, the Fourier method for solving FVEs is discussed. We also describe the ADI scheme for time–space FVEs in Section 1.5. Section 1.6 considers numerical solutions using the finite difference scheme.

The suggested method in Section 1.7 shows the proposed scheme's stability and convergence. To validate our theoretical findings, numerical experiments are performed in Section 1.8. Finally, a brief conclusion is presented in the last section.

1.2 The Problem Formulation for FVEs with Viscoelastic Damping and its Application in Science and Engineering

In the description of vibration models, fractional differentiation operators are commonly utilized. It is well known that FD equations accurately describe the motion of vibrations with elastic and viscoelastic components [28, 29]. Damped vibrations with fractional damping are also defined by these equations; where damping is characterized as the restraint of oscillatory or vibration motion, it decreases, restricts, and prevents an oscillatory mechanism from oscillating. When a damping force becomes viscoelastic, it incorporates viscous and elastic properties to suppress or damp the system's vibration. Likewise, the damping force is known as viscous-viscoelastic when the model reaches viscous rubbing at low speed and reaches pure viscoelastic friction at high speed, in particular the vibrations of an aircraft wing in a supersonic gas flow, vibrations of nanoscale sensors, etc. [30]. Figure 1.1 depicts free oscillation with viscoelastic damping, the model under discussion in this chapter.

The viscoelastic vibration mathematical model is described as:

$$mw'' + bD^{\gamma}w + aw = F, \tag{1.1}$$

where m is the mass of the system, γ, b are the fractional order viscoelastic vibration and coefficient of the damping, respectively. a is a natural frequency and D^{γ}, F are the FD operator and external force respectively. The findings of [31] demonstrate that the outcome of solving problems can be used to simulate alteration in the deformation-strength properties of polymer concrete under the effect of gravity force (2.1). The researchers looked at samples

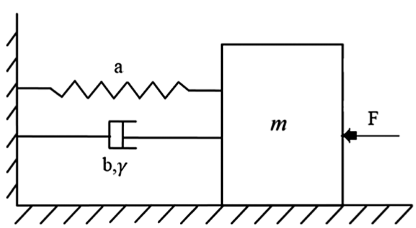

FIGURE 1.1
Viscoelastic damping of the vibration model.

of polyester resin-based polymer concrete (chloride-1, Diane, 1-dichloro-2, diacyl, and 2-diethylene).

FDs are frequently employed to characterize viscoelastic characteristics of sophisticated materials, as well as dissipative forces in structural dynamics [32]. In [33], the analytical solution for linear fractional vibration dependent on various fractional calculus constitutive problems is given using the Laplace transform technique. In the fractional calculus model of viscoelastic conduct, fractional-order derivatives are used to link stress and strain fields in viscoelastic materials. The evolution of such models has previously been discussed, and it has been proved that FD-constitutive equations have been related to molecular theories explaining the macroscopic behavior of viscoelastic media. The improvement of new damping mechanisms in technology and engineering focused on a continuum of damping factors spread uninterruptedly during the relaxation or creep periods instead of a single method of damping elements has reignited interest in viscoelastic models and their application to complex problems [34]. Previous efforts to explain the mechanical characteristics of viscoelastic solids have failed because the mathematical models that describe the action of these materials have not been precisely related to the underlying physical concepts. To explain the mechanical properties of these components, engineers were forced to use phenomenological (empirical) methods. Also on an electronic machine, the order of the extended formulas for systems of engineering importance could be very high, and the scale of the matrices prohibitive to control. FD strain-stress fundamental relations for viscoelastic solids do not just explain the mechanical characteristics of certain components but also contribute to closed-form formulations of numerical simulation motion equations for viscoelastically damped systems. In the study of vibration damped vibrations, numerical techniques are often used. In 1983, Bagley and Torvik were among the first to use numerical approaches to solve dynamic models involving viscoelasticity damped models.

1.3 Mathematical Model of FVEs

First, we note that FDs in space can be utilized to simulate irregular diffusion or scattering, and FDs in time can be utilized to stimulate certain processes with memory. Let us study the following vibration string equation in the domain $D = \{0 < q < I, 0 < p < \aleph\}$:

$$\frac{\partial^2 w(p,q)}{\partial q^2} = \frac{\partial^2 w(p,q)}{\partial p^2} + a\,{}^{C}D_q^{\beta} w(p,q) + b\,{}^{R}D_p^{\alpha} w(p,q), \tag{1.2}$$

with boundary conditions

$$w(0,q) = w(\aleph,q) = 0, \tag{1.3}$$

and initial conditions

$$\begin{aligned} w(p,0) &= \varphi(p), \\ w_q(p,0) &= \psi(p), \end{aligned} \tag{1.4}$$

where D_q^β -denotes the Caputo's temporal derivative with respect to the variable q of order $\beta(1<\beta<2)$, D_p^α is the Liouville's spatial derivative with respect to the variable p of order $\alpha(1<\alpha<2)$, i.e.,

$$D_q^\beta w(p,q) = \frac{1}{\Gamma(2-\beta)}\int_0^q (q-\tau)^{1-\beta} w''(p,\tau)d\tau,$$

$$D_p^\alpha w(p,q) = \frac{1}{\Gamma(2-\alpha)}\frac{d^2}{dp^2}\int_0^p (p-\xi)^{1-\alpha} w(\xi,q)d\xi.$$

The connection between derivatives of fractional order for both Caputo and Liouville operators can be expressed as

$$^{Rl}D_q^\gamma g(q) = ^{C} D_q^\gamma g(q) + \sum_{k=0}^{n-1}\frac{g^{(k)}(0)q^{k-\gamma}}{\Gamma(k+1-\gamma)}. \tag{1.5}$$

Analytical methods have the advantage of explaining the fundamentals of mechanical engineering problems and their physical connotations, making it possible to analyze a variety of physical and mechanical engineering problems and taking less time than the numerical method. However, scholars have found that obtaining exact solutions to PDEs is extremely complicated. Thus, many numerical methods such as the finite difference method, homotopy perturbation methods, finite element method, Galerkin method, Adomian decomposition method, and spectral method, have been developed to obtain the numerical solutions for FPDEs; see [18, 20, 21, 35–38] for examples. Our main goal in this work is to create a high-order numerical method for Eq. (1.2) and perform the corresponding numerical analysis for the method suggested, which is extended in [3, 18, 36, 39, 40]. Up to now, for FPDEs with nonlinear terms, many linearized schemes have been constructed. For diffusion problem in the time-derivative term, Li and Xu [41] established a time–space spectral system. By combining the compact difference method for spatial discretization and L_1 approximation for temporal discretization, a finite difference scheme was derived in [42]. In 2020, Guo et al. [43] developed a novel linearized finite difference/spectral-Galerkin scheme in order to solve the three-dimensional distributed-order space–time fractional nonlinear reaction-diffusion-wave equation, and proved the stability and convergence of the suggested scheme with numerical results. In 2019, Huang et al. [44] considered a two-dimensional nonlinear super-diffusion problem in the time-derivative term and proposed two conservative linearized ADI schemes to get the approximate solution of the model. Furthermore, a proposed scheme was proved that is uniquely solvable and convergent with $\mathcal{O}\left(\tau + h_x^2 + h_y^2 + \tau^{\alpha}\right)$ order in L_2 norm with mesh size h and time step τ. For the space–time fractional telegraph equation, Zhao and Li [45] proposed a linearized fractional difference/finite element approximation, and the suggested scheme proved to be unconditionally stable and accurate in both time and space using the energy method and mathematical induction. In [46], Sun and Wu introduced a finite difference method by adding two additional parameters to turn the original equation into a low-order equation system enabling error analysis. In [47], the fractional-order delay model was analyzed by Harendra Singh using the Chebyshev polynomials method. He and Pan [48] proposed an ADI scheme for Ginzburg-Landau FD equations, and the proposed method was proved to be unconditionally stable using the energy method and mathematical induction. To construct a compact difference method for fractional-order diffusion-wave equations, the equivalent integrodifferential equations and product trapezoidal law were used by Chen and Li [49]. Harendra Singh [50] proposed a computational method for solving the fractional-order advection-dispersion problem using the Jacobi collocation method. In 2016, Wang et al. [51] studied finite difference methods for both temporal and spatial FDs for differential equations. To improve the efficiency, they also proposed a preconditioner for the implementation of the scheme. They also developed a compact ADI finite difference method for studying the two-dimensional time fractional diffusion-wave equation in [52]. A reliable algorithm for giving the numerical solution of the Lane-Emden nonlinear equations and analyzing the error of the proposed scheme was studied in [53].

1.4 Notations and Preliminaries

This section describes several definitions and lemmas that are used in subsequent sections of this chapter.

Lemma 1.4.1

[54] If $w(q) \in C^2([0,I])$, then

$$ {}_0J_q w(q_{j+\frac{1}{2}}) = \frac{1}{2}\left[{}_0J_q w(q_{j+1}) + {}_0J_q w(q_j)\right] + \mathcal{O}(\tau^2), $$

where ${}_0J_q$ is the integral operator of the first order, and $q_{j+\frac{1}{2}} = q_j + \frac{1}{2}\tau$ with a size step τ; $q_j = j\tau$, moreover if $w(q) \in C^3([0,I])$; $0 < \theta < 1$,

$$ w'(q_{j+\frac{1}{2}}) = \frac{w(q_{j+1}) - w(q_j)}{\tau} + \mathcal{O}(\tau^2), $$

and

$$ {}_0^{RI}D_q^{\theta} w(q_{j+\frac{1}{2}}) = \frac{1}{2}\left[{}_0^{RI}D_q^{\theta} w(q_{j+1}) + {}_0^{RI}D_q^{\theta} h(q_j)\right] + \mathcal{O}(\tau^2). $$

Lemma 1.4.2

(see [55]) assume $w(p) \in C^4([p_{i-1}, p_{i+1}])$, let $\zeta(r) = w^{(4)}(p_i + rh) + w^{(4)}(p_i - rh)$, then

$$ \delta_p^2 w(p_i) = \frac{w(p_{i+1}) - 2w(p_i) + w(p_{i-1})}{h^2}. $$

Lemma 1.4.3

(see [56]) Assume ϖ_j is the weights of the generating function $(3/2 - 2z + z^2/2)^{-1}$, that is expressed as, $\varpi_j = 1 - 3^{-(j+1)}$. If $w(q) \in C^2([0,I])$ and $w(0) = w'(0) = 0$, for the first-order integral, we get the second-order approximation as follows

$$ \left|{}_0J_q w(q_{j+1}) - \tau \sum_{j=0}^{m+1} \varpi_{m+1-j} w(q_j)\right| \le C \max_{0 \le q \le q_{j+1}} |w''(q)|\, \tau^2, \tag{1.6} $$

where C represents a general constant the value of which varies from one line to another.

Lemma 1.4.4

(see [57]) Suppose $w(q) \in L^1(R)$, the approximation of the Riemann-Liouville operator using the shifted and weighted Grünwald difference formula holds

$$ {}_0^{RI}D_q^{\theta} w(q_{j+1}) = \tau^{-\theta} \sum_{j=0}^{m+1} \lambda_j^{(\theta)} w(q_{m+1-j}) + \mathcal{O}(\tau^2), \quad 0 < \theta < 1, \tag{1.7} $$

where

$$\lambda_0^{(\theta)} = \frac{2+\theta}{2}\rho_0^{(\theta)}, \quad \lambda_j^{(\theta)} = \frac{2+\theta}{2}\rho_j^{(\theta)} - \frac{\theta}{2}\rho_{j-1}^{(\theta)} \quad ; j \geq 1,$$

$$\rho_j^{(\theta)} = (-1)^j \binom{\theta}{j} \text{for } j \geq 0.$$

Theorem 1.4.5

Suppose $\left\{\lambda_s^{\theta}\right\}_{s=0}^{\infty}$ *and* $\left\{\varpi_s\right\}_{s=0}^{\infty}$ *are the weights mentioned above in Lemma 1.4.3 and* 1.4.4, *respectively. So, for each integer value k as well as any real vector* $(W_1, W_2, \ldots, Ws)^T \in R^s$, *the inequalities*

$$\sum_{m=0}^{s-1}\left(\sum_{k=0}^{m}\lambda_k^{\theta} W_{m+1-k}\right)W_{m+1} < 0,$$

and

$$\sum_{m=0}^{s-1}\left(\sum_{k=0}^{m}\varpi_k W_{m+1-k}\right)W_{m+1} < 0,$$

hold.

Proof. The proof of the second inequality can be found in [58]. Therefore, we prove here only the first inequality. To prove that the above quadratic form is negative, we need only prove that the following G symmetric Toeplitz matrix is negatively defined

$$G = \begin{pmatrix} \lambda_1^{(\theta)} & \lambda_0^{(\theta)} & & & \\ \lambda_2^{(\theta)} & \lambda_1^{(\theta)} & \lambda_0^{(\theta)} & & \\ \vdots & \lambda_2^{(\theta)} & \lambda_1^{(\theta)} & \ddots & \\ \lambda_{n-2}^{(\theta)} & \cdots & \ddots & \ddots & \lambda_0^{(\theta)} \\ \lambda_{n-1}^{(\theta)} & \lambda_{n-2}^{(\theta)} & \cdots & \lambda_2^{(\theta)} & \lambda_1^{(\theta)} \end{pmatrix},$$

let $S = \frac{G+G^T}{2}$ be the symmetric part of the matrix G, where the generating functions of G and G^T, can be expressed respectively in the following form

$$g_G(p) = \sum_{s=0}^{\infty}\lambda_s^{(\theta)} e^{i(s-1)p}, \quad g_{G^T}(p) = \sum_{s=0}^{\infty}\lambda_s^{(\theta)} e^{-i(s-1)p},$$

therefore, the generating function of S is written as $g(\theta,p) = \dfrac{g_G(p) + g_{G^T}(p)}{2}$, which is a continuous and periodic real-value function in the interval $[-\pi, \pi]$, such that $g_G(p)$ and $g_G{}^T(p)$ are conjugated, and using the coefficients of $\lambda_s^{(\theta)}$ given by Lemma 1.4.4, then

$$
\begin{aligned}
g(\theta,p) &= \frac{1}{2}\left(\sum_{s=0}^{\infty}\lambda_s^{(\theta)} e^{i(s-1)p} + \sum_{s=0}^{\infty}\lambda_s^{(\theta)} e^{-i(s-1)p}\right) \\
&= \frac{1}{2}\left(\frac{2+\theta}{2} e^{-ip}\sum_{s=0}^{\infty}\rho_s^{(\theta)} e^{isp} - \frac{\theta}{2}\sum_{s=0}^{\infty}\rho_s^{(\theta)} e^{isp} + \frac{2+\theta}{2} e^{ip}\sum_{s=0}^{\infty}\rho_s^{(\theta)} e^{-isp} - \frac{\theta}{2}\sum_{s=0}^{\infty}\rho_s^{(\theta)} e^{-isp}\right) \\
&= \frac{2+\theta}{4}\left(e^{-ip}\left(1-e^{ip}\right)^{\theta} + e^{ip}\left(1-e^{-ip}\right)^{\theta}\right) - \frac{\theta}{4}\left(\left(1-e^{ip}\right)^{\theta} + \left(1-e^{-ip}\right)^{\theta}\right).
\end{aligned}
$$

Now, we will prove for $1 < \alpha < 2$ the function $g(\theta,p) \le 0$, since $g(\theta,p)$ is a continuous real-value function and even, hence, we only assume its principal value in $[0,\pi]$, which leads to

$$
e^{i\gamma} - e^{i\eta} = 2i\sin\left(\frac{\gamma-\eta}{2}\right)e^{i\frac{(\gamma+\eta)}{2}},
$$

then

$$
g(\theta,p) = \left(2\sin\left(\frac{p}{2}\right)\right)^{\theta}\left(\frac{2+\theta}{2}\cos\left(\frac{\theta}{2}(p-\pi)-p\right) - \frac{\theta}{2}\cos\left(\frac{\theta}{2}(p-\pi)\right)\right),
$$

so, $g(\theta,p)$ decreases with respect to θ, that's mean that $(g(\theta,p) \le 0)$.

1.5 Solving a FVEs by Fourier Method

Let us show how to solve Eq. (1.2) using the Fourier method. To apply the Fourier method, assume that

$$
w(p,q) = \Upsilon(p)\Psi(q), \tag{1.8}
$$

substitute from Eq. (1.8) in Eq. (1.2)

$$
\frac{\Psi''(q)}{\Psi(q)} - \frac{a}{\Psi(q)}\cdot D_q^{\beta}\Psi(q) = \frac{\Upsilon''(p)}{\Upsilon(p)} + \frac{b}{\Upsilon(p)}\cdot D_p^{\alpha}\Upsilon(p) = -\lambda. \tag{1.9}
$$

For the unknown function $\Upsilon(p)$, we obtain the ordinary linear differential equation

$$\begin{aligned} \Upsilon''(p) + bD_p^{\alpha}\Upsilon(p) + \lambda\Upsilon(p) = 0; \\ \Upsilon(0) = 0, \Upsilon(\aleph) = 0, \end{aligned} \tag{1.10}$$

the negative number λ is defined as the eigenvalue of Eq. (1.10) if and only if λ satisfies the function

$$\omega(\lambda) = \sum_{m=0}^{\infty}(-1)^m \sum_{j=0}^{m} \frac{\binom{m}{j} b^j \lambda^{m-j}}{\Gamma(2m+2-j\alpha)}. \tag{1.11}$$

In order to discuss the solution to the problem (1.10), let us integrate both sides of Eq. (1.10) for q from 0 to p,

$$\Upsilon'(p) + \frac{b}{\Gamma(1-\alpha)} \int_0^p \Upsilon(q)(p-q)^{-\alpha}\,dq = \lambda \int_0^p \Upsilon(q)dq + \Upsilon'(0), \tag{1.12}$$

for q from 0 to p, we can again integrate both sides of Eq. (1.12)

$$\Upsilon(p) + \int_0^p \frac{b}{\Gamma(2-\alpha)}(p-q)^{1-\alpha} - \lambda(p-q)\Upsilon(p)dq = p.\Upsilon'(0) + \Upsilon(0). \tag{1.13}$$

Table 1.1 lists the first seven eigenvalues that are found numerically using the computing language Wolfram Mathematica (Figures 1.2 and 1.3).

Then the eigenfunctions $\Upsilon_m(p)$ of the Eq.(5.3), have the form

$$\Upsilon_k(p) = \mathcal{A}\left[p + \sum_{m=1}^{\infty}(-1)^m \sum_{j=0}^{m} \frac{\binom{m}{j} b^j \lambda_k^{m-j}}{\Gamma(2m+2-j\alpha)} p^{2m+1-j\alpha} \right]; k = 1,2,\ldots,7. \tag{1.14}$$

TABLE 1.1

The First Seven Eigenvalues of the Boundary Value Problem (1.10) for $\aleph = 2$; $\alpha = 1.47$, $b = 1.8$

λ_1	λ_2	λ_3	λ_4	λ_5	λ_6	λ_7
4.983	17.479	35.949	60.421	90.511	126.105	175.501

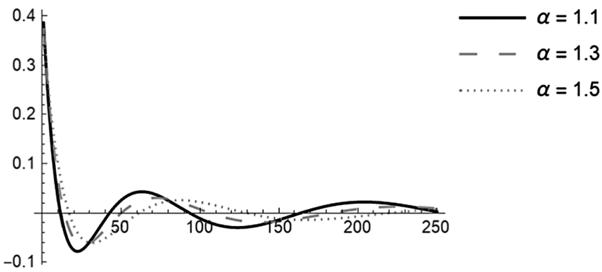

FIGURE 1.2
Eigenfunction $\omega(\lambda)$ of Eq. (1.11) at $b = 1.8$, corresponding to $\alpha = 1.1, 1.3, 1.5$.

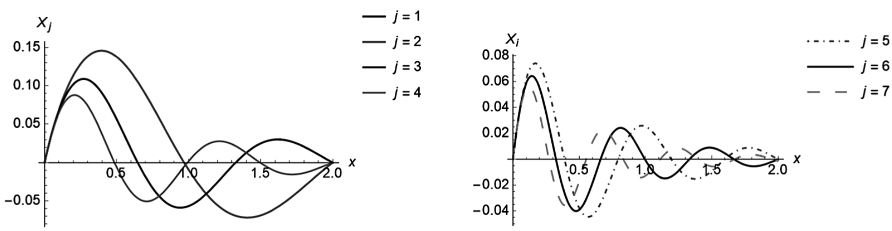

FIGURE 1.3
Eigenfunctions $\Upsilon j(p)$; $j = 1, \ldots, 7$, of Eq. (1.14) corresponding to $b = 1.8$, $\alpha = 1.47$.

The eigenfunction system (1.10) is complete [59], but it is not orthogonal. As a consequence, therefore, we propose (according to [3]) the following conjugate system to obtain that the system is biorthogonal to Eq. (1.10)

$$\begin{aligned} &\tilde{\Upsilon}''(p) + bD_p^{\alpha}\tilde{\Upsilon}(p) + \lambda\tilde{\Upsilon}(p) = 0, \\ &\tilde{\Upsilon}(0) = \tilde{\Upsilon}(\aleph) = 0; \qquad \alpha \in (1,2), \end{aligned} \tag{1.15}$$

where, $\tilde{\Upsilon}_k(p)$ is the eigenfunctions of Eq. (1.15), then

$$\tilde{\Upsilon}_k(p) = \Upsilon_k(1-p), \tag{1.16}$$

then,

$$\tilde{\Upsilon}_k(p) = \sum_{m=0}^{\infty}(-1)^m \sum_{j=0}^{m} \frac{\binom{m}{j} b^j \lambda_k^{m-j}}{\Gamma(2m+2-j\alpha)}(1-p)^{2m+1-j\alpha}; k = 1,2,\ldots,7. \tag{1.17}$$

TABLE 1.2
The Inner Product of $\langle \Upsilon_k, \widetilde{\Upsilon}_m \rangle$ at $b = 1.8$ and $\alpha = 1.47$

$\langle \Upsilon_k, \widetilde{\Upsilon}_m \rangle$	Υ_1	Υ_2	Υ_3	Υ_4	Υ_5	Υ_6	Υ_7
$\widetilde{\Upsilon}_1$	0.05106	0	0	0	0	0	0
$\widetilde{\Upsilon}_2$	0	−0.0095	0	0	0	0	0
$\widetilde{\Upsilon}_3$	0	0	0.0031	0	0	0	0
$\widetilde{\Upsilon}_4$	0	0	0	−0.0013	0	0	0
$\widetilde{\Upsilon}_5$	0	0	0	0	0.0006	0	0
$\widetilde{\Upsilon}_6$	0	0	0	0	0	−0.0003	0
$\widetilde{\Upsilon}_7$	0	0	0	0	0	0	0.0002

The inner product $\langle \Upsilon_k(p), \widetilde{\Upsilon}_m(p) \rangle$, for the case $\aleph = 2$; $\alpha = 1.47$, $b = 1.8$ can be calculated approximately, that are given in Table 1.2

Next, we seek the general solution of the following Equation

$$\Psi''(q) - aD_q^\beta \Psi(q) + \lambda \Psi(q) = 0, \tag{1.18}$$

Let us take

$$\Psi''(q) = Y(q), \tag{1.19}$$

integrate Eq. (1.19) more than once from 0 to q

$$\Psi(q) = \int_0^q (q-s) Y(s) ds + \Psi'(0) q + \Psi(0), \tag{1.20}$$

solving Eq. (1.20), we obtain

$$Y(q) + \int_0^q \left[\frac{-a}{\Gamma(2-\beta)} (q-\zeta)^{1-\beta} + \lambda (q-\zeta) \right] Y(\zeta) d\zeta = -\lambda \mathcal{A} q - \lambda \mathcal{B}, \tag{1.21}$$

such that $\mathcal{A} = \Psi'(0), \mathcal{B} = \Psi(0)$. Then the solution of Eq. (1.21) takes the form

$$\Psi(q) = \mathcal{A}\left[q - \frac{\lambda q^3}{6} + \sum_{m=1}^{\infty}(-1)^{m+1}\sum_{k=0}^{m}\frac{\binom{m}{k}(-a)^k \lambda^{m+1-k}}{\Gamma(2m+4-k\beta)} q^{2m+3-k\beta}\right]; \qquad (1.22)$$

$$+\mathcal{B}\left[1 - \frac{\lambda q^2}{2} + \sum_{m=1}^{\infty}(-1)^{m+1}\sum_{k=0}^{m}\frac{\binom{m}{k}(-a)^k \lambda^{m+1-k}}{\Gamma(2m+3-k\beta)} q^{2m+2-k\beta}\right],$$

For simplicity, we can put

$$T(q) = q - \frac{\lambda q^3}{6} + \sum_{m=1}^{\infty}(-1)^{m+1}\sum_{k=0}^{m}\frac{\binom{m}{k}(-a)^k \lambda^{m+1-k}}{\Gamma(2m+4-k\beta)} q^{2m+3-k\beta}, \qquad (1.23)$$

and

$$\widetilde{T}(q) = 1 - \frac{\lambda q^2}{2} + \sum_{m=1}^{\infty}(-1)^{m+1}\sum_{k=0}^{m}\frac{\binom{m}{k}(-a)^k \lambda^{m+1-k}}{\Gamma(2m+3-k\beta)} q^{2m+2-k\beta}, \qquad (1.24)$$

then the solution of Eq. (1.18) can be rewritten as

$$\Psi(q) = \mathcal{A}.T(q) + \mathcal{B}.\widetilde{T}(q). \qquad (1.25)$$

Hence, the solution of Eq. (1.2) is written out in a standard form

$$w(p,q) = \sum_{k=1}^{\infty}\Upsilon_k(p)\Psi_k(q) = \sum_{k=1}^{\infty}\Upsilon_k(p)\left[\mathcal{A}_k.T_k(q) + \mathcal{B}_k.\widetilde{T}_k(q)\right]. \qquad (1.26)$$

We will use at $q = 0$, the following form of the initial condition (1.4) and (1.26)

$$w(p,0) = \sum_{k=1}^{\infty}\Upsilon_k(p)\left[\mathcal{A}_k.T_k(0) + \mathcal{B}_k.\widetilde{T}_k(0)\right] = \varphi(p),$$

Further, to use the initial condition (1.4), we differentiate both sides of (1.16) with respect to q at $q = 0$:

$$w_q(p,0) = \sum_{k=1}^{\infty} \Upsilon_k(p)\left[\mathcal{A}_k.T_k'(0) + \mathcal{B}_k.\widetilde{T}'_k(0)\right] = \psi(p). \tag{1.27}$$

Multiply both sides of each equation of Eq. (1.27) by $\widetilde{\Upsilon}_k$, therefore, from the orthogonality of the functions Υ_k and $\widetilde{\Upsilon}_k$ we obtain

$$\begin{aligned} \mathcal{A} &= \frac{\langle \varphi(p), \widetilde{\Upsilon}_k(p)\rangle}{\langle \Upsilon_k(p), \widetilde{\Upsilon}_k(p)\rangle}; \\ \mathcal{B}_k &= \frac{\langle \psi(p), \widetilde{\Upsilon}_k(p)\rangle}{\langle \Upsilon_k(p), \widetilde{\Upsilon}_k(p)\rangle}, \end{aligned} \tag{1.28}$$

which allows us to write out the solution of problem Eq. (1.2) in the following formula

$$w(p,q) = \sum_{k=1}^{\infty} \frac{\Upsilon_k(p)}{\langle \Upsilon_k(p), \widetilde{\Upsilon}_k(p)\rangle}\left[T_k(q)\langle \psi(p), \widetilde{\Upsilon}_k(p)\rangle + \widetilde{T}_k(q)\langle \varphi(p), \widetilde{\Upsilon}_k(p)\rangle\right]. \tag{1.29}$$

1.6 Solving FVEs Numerically by Finite Difference Scheme

Considering Eq. (1.2) with condition (1.3), we find that if we assume for Eq. (1.2) an equivalent form, the precision of the discrete approximations can be improved. First, let us take the fractional integral operator ${}_0J_q$ in Eq. (1.2), then

$$\frac{\partial w(p,q)}{\partial q} - aD_q^{\theta} w(p,q) = J_q \frac{\partial^2 w(p,q)}{\partial p^2} + bJ_q D_p^{\alpha} w(p,q), \tag{1.30}$$

where $0 < \theta = \beta - 1 < 1$ and $J_q D_q^{\beta} w(p,q) = D_q^{\beta-1} w(p,q)$.

In order to discretize Eq. (1.30), we introduce the temporal step size $\tau = I/\mathbb{N}$ with a non-zero integer $\mathbb{N}$, and $q_n = n\tau$; $n = 0, 1, \ldots, \mathbb{N}$, and define a grid function time $\Omega_\tau = \{q_n \mid n \geq 0\}$. For a spatial discretization, let $h = \aleph / \mathbb{M}$ with a non-zero integer $\mathbb{M}$, and $p_i = ih$; $0 \leq i \leq \aleph$, and define a grid function space $\Omega_h = \left\{p_i \mid \quad 0 \leq i \leq \mathbb{M}\right\}$. Suppose that on $\Omega_h \times \Omega_\tau$, there exists a grid function $W = \left\{w_i^n \mid \quad 0 \leq i \leq \mathbb{M}, n \geq 0\right\}$, such that for any $w, g \in W$,

$$w_i^{n+\frac{1}{2}} = \frac{1}{2}\left[w_i^{n+1} + w_i^n\right], \qquad \delta_q w_i^{n+\frac{1}{2}} = \frac{1}{\tau}\left[w_i^{n+1} - w_i^n\right],$$
$$\langle w^n, g^n\rangle = h\sum_{i=1}^{M-1} w_i g_i, \qquad \| w^n \|^2 = \langle w, w\rangle,$$
$$\| w^n \|_\infty = \max_{0\le i\le \mathbb{M}} | w_i^n |, \qquad \langle \delta_p^2 w, g\rangle = -\langle \delta_p w, \delta_p g\rangle.$$

Then using Eq. (1.5) and Lemma 1.4.1, a weighted Crank–Nicolson method for Eq. (1.30) at the point $(p_i, q_{n+\frac{1}{2}})$ formed as

$$\frac{w(p_i,q_{n+1}) - w(p_i,q_n)}{\tau} - \frac{a}{2}D_q^\theta\left(w(p_i,q_{n+1}) + w(p_i,q_n) - 2w(p_i,0)\right) =$$
$$\frac{1}{2}\left(J_q\frac{\partial^2 w(p_i,q_{n+1})}{\partial p^2} + J_q\frac{\partial^2 w(p_i,q_n)}{\partial p^2}\right) + \frac{b}{2}\left(J_q D_p^\alpha w(p_i,q_{n+1}) + J_q D_p^\alpha w(p_i,q_n)\right) + \mathcal{O}(\tau^2), \tag{1.31}$$

assume that $w(p,q) \in C_{p,q}^{5,3}([0,\aleph]\times[0,\Im])$, let $w_i^n = w(p_i, q_n)$, so we suggest the following Crank-Nicolson method, which based on Lemma 1.4.4 and Theorem 1.4.5. to Eq. (1.31), then

$$\frac{w_i^{n+1} - w_i^n}{\tau} - \frac{a}{2}\tau^{-\theta}\left(\sum_{k=0}^{n+1}\lambda_k^{(\theta)} w_i^{n+1-k} + \sum_{k=0}^{n}\lambda_k^{(\theta)} w_i^{n-k} - 2\lambda_0^{(\theta)}\varphi_i\right) =$$
$$\frac{\tau}{2}\left(\sum_{k=0}^{n+1}\omega_k\delta_p^2 w_i^{n+1-k} + \sum_{k=0}^{n}\omega_k\delta_p^2 w_i^{n-k}\right) + \frac{b}{2}\tau\left(\sum_{k=0}^{n+1}\omega_k\delta_p^\alpha w_i^{n+1-k} + \sum_{k=0}^{n}\omega_k\delta_p^\alpha w_i^{n-k}\right) + \mathcal{O}(\tau^2 + h^2), \tag{1.32}$$

since $w(i,0) = \varphi_i(p)$, and multiply τ in Eq. (1.32) then

$$w_i^{n+1} - \frac{a}{2}\tau^{1-\theta}\left(\sum_{k=0}^{n+1}\lambda_k^{(\theta)} w_i^{n+1-k} + \sum_{k=0}^{n}\lambda_k^{(\theta)} w_i^{n-k} - 2\lambda_0^{(\theta)}\varphi\right)$$
$$= w_i^n + \frac{\tau^2}{2}\left(\sum_{k=0}^{n+1}\omega_k\delta_p^2 w_i^{n+1-k} + \sum_{k=0}^{n}\omega_k\delta_p^2 w_i^{n-k}\right)$$
$$+ \frac{b}{2}\tau^2\left(\sum_{k=0}^{n+1}\omega_k\delta_p^\alpha w_i^{n+1-k} + \sum_{k=0}^{n}\omega_k\delta_p^\alpha w_i^{n-k}\right) + \mathcal{O}(\tau^3 + \tau h^2), \tag{1.33}$$

ignoring the truncation error term $\mathcal{O}\left(\tau^3 + \tau h^2\right)$ from Eq. (1.33) and replacing w_i^n by its numerical solution W_i^n, we obtain the following scheme for Eq. (1.33)

$$W_i^{n+1}-\frac{a}{2}\tau^{1-\theta}\left(\sum_{k=0}^{n+1}\lambda_k^{(\theta}W_i^{n+1-k}+\sum_{k=0}^{n}\lambda_k^{(\theta)}W_i^{n-k}-2\lambda_0^{(\theta)}\varphi\right)=$$
$$W_i^n+\frac{\tau^2}{2}\left(\sum_{k=0}^{n+1}\omega_k\delta_p^2W_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_p^2W_i^{n-k}\right)+\frac{b}{2}\tau^2\left(\sum_{k=0}^{n+1}\omega_k\delta_p^\alpha W_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_p^\alpha W_i^{n-k}\right). \tag{1.34}$$

1.7 Stability and Convergence Analysis

This section discusses the stability and convergence of the suggested ADI method (1.33). Let a grid function $W=\left\{w_i^n \mid \quad 0\le i\le \mathbb{M}, n\ge 0; w_0=w_{\mathbb{M}}=0\right\}$ is a grid function on $\Omega_h\times\Omega_\tau$.

Lemma 1.7.1

(see[55]) For any w, $u\in W$, there exist linear difference operator to the operator δ_p^α that is denoted by $\delta_p^{\alpha/2}$, where

$$\langle\delta_p^\alpha w,u\rangle=\langle\delta_p^{\alpha/2}w,\delta_p^{\alpha/2}u\rangle,\qquad \langle\delta_p^2 w,u\rangle=-\langle\delta_p w,\delta_p u\rangle.$$

Theorem 1.7.2

Let $w(p,q)\in c_{p,q}^{6,3}\left([0,\aleph]\times[0,\Im]\right)$ and suppose $w(p,q)$ is the analytical solution of Eq. (1.2) and the numerical solution of scheme (1.33)–(1.34) is $W(p,q)$, which is defined as $\left\{W_i^n \mid 0\le i\le\mathbb{M}, 0\le j\le\mathbb{N}\right\}$. Then for every $n\tau\le T$, it achieves that

$$\| W^j-w^j\|\le\tilde{c}(\tau^2+h^2).$$

Proof. Subtracting Eq. (1.34) from (1.33) and denoting $e_i^j=w_i^j-W_i^j$, then we have

$$e_i^{m+1}-e_i^m-\frac{a}{2}\tau^{1-\theta}\left(\sum_{j=0}^{m+1}\lambda_j^{(\theta)}e_i^{m+1-j}+\sum_{j=0}^{m}\lambda_j^{(\theta)}e_i^{m-j}-2\lambda_0^{(\theta)}e_i^0\right)$$
$$=\frac{\tau^2}{2}\left(\sum_{j=0}^{m+1}\omega_j\delta_p^2e_i^{m+1-j}+\sum_{j=0}^{m}\omega_j\delta_p^2e_i^{m-j}\right)+\frac{b}{2}\tau^2\left(\sum_{j=0}^{m+1}\omega_j\delta_p^\alpha e_i^{m+1-j}+\sum_{j=0}^{m}\omega_j\delta_p^\alpha e_i^{m-j}\right)$$
$$+\mathcal{O}(\tau^3+\tau h^2). \tag{1.35}$$

Multiplying Eq. (1.35) by $h\left(e_i^{m+1}+e_i^m\right)$ summing over i from 0 to $\mathbb{M}-1$, we obtain that

$$\| e^{m+1} \|^2 - \| e^m \|^2 = \frac{a}{2}\tau^{1-\theta}\left(\sum_{j=0}^{m}\lambda_j^{(\theta)}\langle e^{m+1-j} + e^{m-j} - e^0, e^{m+1} + e^m\rangle + \lambda_{m+1}^{(\theta)}\langle e^0, e^{m+1} + e^m\rangle\right)$$
$$+\frac{\tau^2}{2}\left(\sum_{j=0}^{m}\omega_j\langle\delta_p^2(e^{m+1-j} + e^{m-j}), e^{m+1} + e^m\rangle + \omega_{m+1}\langle\delta_p^2 e^0, e^{m+1} + e^m\rangle\right)$$
$$+\frac{b}{2}\tau^2\left(\sum_{j=0}^{m+1}\omega_j\langle\delta_p^\alpha(e^{m+1-j} + e^{m-j}), e^{m+1} + e^m\rangle + \omega_{m+1}\langle\delta_p^\alpha e^0, e^{m+1} + e^m\rangle\right)$$
$$+\langle\mathcal{O}(\tau^3 + \tau h^2), e^{m+1} + e^m\rangle. \tag{1.36}$$

Since $e_i^0 = 0$ for $0 \le i \le \mathbb{M}$, and summing over m from 0 to $R - 1$, we obtain that

$$\| e^R \|^2 = \frac{a}{2}\tau^{1-\theta}\sum_{m=0}^{R-1}\sum_{j=0}^{m}\lambda_j^{(\theta)}\langle e^{m+1-j} + e^{m-j}, e^{m+1} + e^m\rangle$$
$$+\frac{\tau^2}{2}\sum_{m=0}^{R-1}\sum_{j=0}^{m}\omega_j\langle\delta_p^2\left(e^{m+1-j} + e^{m-j}\right), e^{m+1} + e^m\rangle$$
$$+\frac{b}{2}\tau^2\sum_{m=0}^{R-1}\sum_{j=0}^{m}\omega_j\langle\delta_p^\alpha\left(e^{m+1-j} + e^{m-j}\right), e^{m+1} \tag{1.37}$$
$$+ e^m\rangle + \sum_{m=0}^{R-1}\langle\mathcal{O}(\tau^3 + \tau h^2), e^{m+1} + e^m\rangle.$$

The use of Theorem 1.4.5 yields the following inequality, from which it can be inferred that the first three terms on the whole of the right side of Eq. (1.37) are negative.

$$\| e^R \|^2 \le \sum_{m=0}^{R-1}\langle\tilde{c}(\tau^3 + \tau h^2), e^{m+1} + e^m\rangle. \tag{1.38}$$

Suppose,

$$\| e^S \| = \max_{0 \le R \le N} \| e^R \|,$$

then we obtain

$$\| e^S \| \le \tilde{c}(\tau^2 + h^2). \tag{1.39}$$

Thus, the proof is complete.

Theorem 1.7.3

Consider the numerical solution of scheme Eqs (1.33)–(1.34) is stable in a grid function $\left\{W_i^n|, 0 \le i \le \mathbb{M}|, 0 \le n \le \mathbb{N}\right\}$, *which holds*

$$\| W^J \| \le \tilde{c}.$$

Proof. Multiplying Eq. (1.34) by $h\left(W_i^{m+1} + U_i^m\right)$ and summing from $1 \le i \le \mathbb{M} - 1$,

$$\begin{aligned}
&\| W^{m+1} \|^2 - \| W^m \|^2 \\
&= \frac{a}{2}\tau^{1-\theta}\left[\sum_{j=0}^{m}\lambda_j^{(\theta)}\langle W^{m+1-j} + W^{m-j}, W^{m+1} + W^m\rangle + \lambda_{m+1}^{(\theta)}\langle \varphi, W^{m+1} + W^m\rangle\right. \\
&\left. -2\lambda_0^{(\theta)}\langle \varphi, W^{m+1} + W^m\rangle\right] + \frac{\tau^2}{2}\sum_{j=0}^{m}\omega_j\langle \delta_p^2(W^{m+1-j} + W^{m-j}), W^{m+1} + W^m\rangle \\
&+ \frac{\tau^2}{2}\omega_{m+1}\langle \delta_p^2\varphi, W^{m+1} + W^m\rangle + \frac{b}{2}\tau^2\sum_{j=0}^{m+1}\omega_j\langle \delta_p^\alpha(W^{m+1-j} + W^{m-j}), W^{m+1} + W^m\rangle \\
&+ \frac{b}{2}\tau^2\omega_{m+1}\langle \delta_p^\alpha\varphi, W^{m+1} + W^m\rangle.
\end{aligned} \tag{1.40}$$

Without loss of generality, we take a homogeneous initial condition Eq. (1.4); $w_i^0 = 0$ and summing Eq. (1.40) over m from 0 to $J - 1$, we get

$$\begin{aligned}
\| W^J \|^2 - \| W^0 \|^2 &= \frac{a}{2}\tau^{1-\theta}\sum_{j=0}^{m}\lambda_j^{(\theta)}\langle W^{m+1-j} + W^{m-j}, W^{m+1} + W^m\rangle \\
&+ \frac{\tau^2}{2}\sum_{j=0}^{m}\omega_j\langle \delta_p^2(W^{m+1-j} + W^{m-j}), W^{m+1} + W^m\rangle \\
&+ \frac{b}{2}\tau^2\sum_{j=0}^{m+1}\omega_j\langle \delta_p^\alpha(W^{m+1-j} + W^{m-j}), W^{m+1} + W^m\rangle,
\end{aligned} \tag{1.41}$$

using Theorem 1.4.5 and Lemma 1.7.1, and by arguments similar in the proof of Theorem 1.7.2, we get

$$\| W^J \| \le \tilde{c}.$$

Theorem 1.7.4

Consider the numerical solution of scheme Eqs (1.33)–(1.34) is stable in a grid function $\left\{W_i^n|,0\le i\le \mathbb{M}|,0\le n\le \mathbb{N}\right\}$; $W_i^0\neq 0$, *which holds*

$$\| W^J \| \le \tilde{c}\left(\|\varphi\|^2 - \frac{a}{4}T^{1-\theta}\|\varphi\|^2 + \frac{T^2}{4}\|\delta_p^2\varphi\|^2 + \frac{b}{4}T^2\|\delta_p^\alpha\varphi\|^2 \right).$$

Proof. If we take initial conditions that are not equal to zero in (1.34) summation over m from 0 to $J-1$, and use Cauchy Schwartz inequality, Theorem 1.4.5, then we get

$$\begin{aligned} \|W^J\|^2 \le \|W^0\|^2 &+ \frac{a}{2}\tau^{1-\theta}\left[\sum_{m=0}^{J-1}\lambda_{m+1}^{(\theta)}\|\varphi\|\|W^{m+1}+W^m\| - 2\sum_{m=0}^{J-1}\lambda_0^{(\theta)}\|\varphi\|\|W^{m+1}+W^m\|\right] \\ &+ \frac{\tau^2}{2}\sum_{m=0}^{J-1}\omega_{m+1}\|\delta_p^2\|\|W^{m+1}+W^m)\| + \frac{b}{2}\tau^2\sum_{m=0}^{K-1}\omega_{m+1}\|\delta_p^\alpha\|\|W^{m+1}+W^m)\|, \end{aligned} \tag{1.42}$$

Using Young inequality, Eq. (1.42) yields

$$\begin{aligned} \|W^J\|^2 \le \|\varphi\|^2 &+ \frac{a}{4}\tau^{1-\theta}\sum_{m=0}^{J-1}\lambda_{m+1}^{(\theta)}\|\varphi\|^2 + \|\frac{a}{4}\tau^{1-\theta}\sum_{m=0}^{J-1}W^{m+1}+W^m\|^2 - \frac{a}{2}\tau^{1-\theta}\sum_{m=0}^{J-1}\lambda_0^{(\theta)}\|\varphi\|^2 \\ &- \frac{a}{2}\tau^{1-\theta}\sum_{m=0}^{J-1}W^{m+1}+W^m\|^2 + \frac{\tau^2}{4}\sum_{m=0}^{J-1}\omega_{m+1}\|\delta_p^2\varphi\|^2 + \frac{\tau^2}{4}\sum_{m=0}^{J-1}\|W^{m+1}+W^m\|^2 \\ &+ \frac{b}{4}\tau^2\sum_{m=0}^{J-1}\omega_{m+1}\|\delta_p^2\varphi\|^2 + \frac{b}{4}\tau^2\sum_{m=0}^{J-1}\|W^{m+1}+W^m\|^2 . \end{aligned} \tag{1.43}$$

Since for $\tau^{1-\theta}\sum_{m=0}^{J-1}\omega_{m+1}$ and $\tau^{1-\theta}\sum_{m=0}^{J-1}\lambda_{m+1}^{(\theta)}$ limited (bounded), and we can use Gronwall's inequality $\|W^{m+1}+W^m\|\to 0$, then Eq. (1.43) is written as

$$\begin{aligned} \|W^J\|^2 \le \tilde{c}\|\varphi\|^2 &+ \frac{a}{4}\tau^{1-\theta}\sum_{m=0}^{J-1}\lambda_{m+1}^{(\theta)}\|\varphi\|^2 - \frac{a}{2}\tau^{1-\theta}\sum_{m=0}^{J-1}\lambda_0^{(\theta)}\|\varphi\|^2 \\ &+ \frac{\tau^2}{4}\sum_{m=0}^{J-1}\omega_{m+1}\|\delta_p^2\varphi\|^2 + \frac{b}{4}\tau^2\sum_{m=0}^{J-1}\omega_{m+1}\|\delta_p^\alpha\varphi\|^2 . \end{aligned} \tag{1.44}$$

Let $\tau \sum_{m=0}^{N-1} = T$, thus

$$\| W^J \|^2 \leq \left(\| \varphi \|^2 - \frac{a}{4} T^{1-\theta} \| \varphi \|^2 + \frac{T^2}{4} \| \delta_p^2 \varphi \|^2 + \frac{b}{4} T^2 \| \delta_p^\alpha \varphi \|^2 \right). \tag{1.45}$$

1.8 Numerical Examples

This section presents numerical examples to demonstrate the computational performance and theoretical findings of our proposed methods.

Example 1.8.1

$$\begin{aligned} \frac{\partial^2 w(p,q)}{\partial q^2} &= \frac{\partial^2 w(p,q)}{\partial p^2} + aD_q^\beta w(p,q) + bD_p^\alpha w(p,q), \quad 0 < p < \aleph, 0 < q < \Im, \\ w(0,q) &= w(\aleph,q) = 0, \\ w(p,0) &= p(1-p), \\ w_q(p,0) &= p^3(p-1), \end{aligned} \tag{1.46}$$

when $a = -1.8$, $b = 0.5$, $I = 1$ and $\aleph = 1$ corresponding to $\alpha = 1.5$, $\beta = 1.47$. First, we note that the analytical solution $w(p,q)$ of Eq. (1.46) fulfill all the smoothness conditions needed by schemes (1.29) and (1.34). Then in Figure 1.4, let us take the step size $\tau = h = 1/20$ for plotting a curve at $I = 1$ of the analytical solutions and the numerical solutions for $\alpha = 1.5$, $\beta = 1.47$.

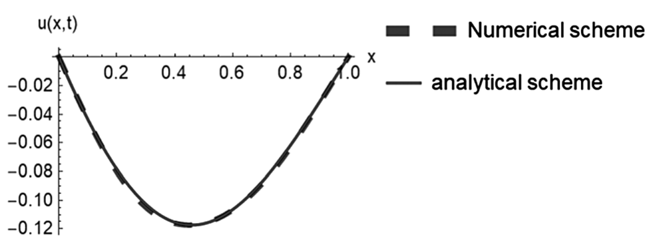

FIGURE 1.4
Analytical solution (1.29) and numerical solution (1.34) of ADI scheme for $\alpha = 1.5$, $\beta = 1.47$.

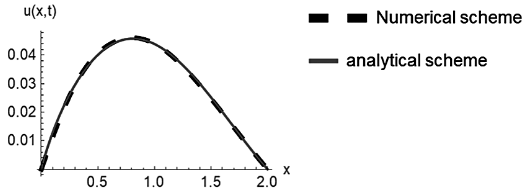

FIGURE 1.5
Analytical solution Eq. (1.29) and numerical solution Eq. (1.34) of ADI scheme for $\alpha = 1.47$, $\beta = 1.47$.

Example 1.8.2

$$\begin{aligned}
&\frac{\partial^2 w(p,q)}{\partial q^2} = \frac{\partial^2 w(p,q)}{\partial p^2} + aD_q^{\beta} w(p,q) + bD_p^{\alpha} w(p,q), \quad 0 < p < \aleph, 0 < q < \Im, \\
&w(0,q) = w(\aleph,q) = 0, \\
&w(p,0) = 0, \\
&w_q(p,0) = p(p-2),
\end{aligned} \tag{1.47}$$

when $a = -0.5$, $b = 1.8$, $I = 2$ and $\aleph = 2$ corresponding to $\alpha = 1.47$, $\beta = 1.47$.

In Figure 1.5, let us take step size $\tau = h = 2/16$ for plotting the curves at $I = 2$ of the approximate solution Eq. (1.34) and the analytical solution of the scheme Eq. (1.29) corresponding to $\beta = 1.47$ and $\alpha = 1.47$,

From Figures 1.4 and 1.5, we find that the analytical solution agrees well with our numerical solution.

1.9 Conclusion

This chapter has described and demonstrated FVEs with viscoelastic damping. Using the Fourier method, we deduced the analytical solution of the problem. The numerical scheme was constructed. The suggested ADI scheme demonstrated stability and convergence with second-order accuracy in space and time. We found that the analytical solution is in perfect agreement with our numerical solution. Both the analytical and numerical methods showed that the suggested methods are efficient for solving one-dimensional time–space FVEs and are also applicable to other FPDEs. For future work, the proposed computation technology will be compared to other compact methods for homogeneous and non-homogeneous models, and the model's applicability in dynamic systems and mechanical engineering will be studied.

Author Contributions

In this research paper, all authors contributed equally. The published version of the paper has been read and approved by all authors.

Conflicts of Interest

The authors declare no conflict of interest.

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