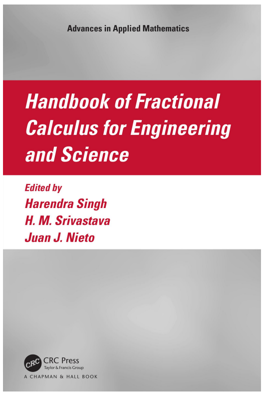

Handbook of Fractional Calculus for Engineering and Science

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New Fractional Integrals and Derivatives Results for the Generalized Mathieu-Type and Alternating Mathieu-Type Series

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*New Fractional Integrals and Derivatives Results for the Generalized Mathieu-Type and Alternating Mathieu-Type Series**

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12.1 Introduction

Several authors have recently proved the utility of fractional calculus operators in a variety of contexts. For more than four decades, the subject of fractional calculus has grown in prominence and appeal, owing to its

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demonstrated applications in a wide range of seemingly unrelated domains of science and engineering. Fractional calculus operators of any arbitrary real or complex order, such as Marichev–Saigo–Maeda (MSM), Saigo's, Riemann–Liouville (RL), and Erdélyi–Kober (EK), are the most often exploited tools in the theory and applications of fractional integrals and derivatives [1, 2]. Various recent developments in this area can be found in [3–11].

Let $\mathbb{C}$ denote the sets of complex numbers, $\mathbb{R}^+$ be the set of positive real numbers and $\mathbb{N}$ be the set of positive integers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let μ, μ', ξ, ξ', $\varpi \in \mathbb{C}$ and $x > 0$, then for $\Re(\varpi) > 0$, the left- and right-hand-sided Marichev–Saigo–Maeda (MSM) fractional integration operators are defined by

$$\left(\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi} f\right)(x) = \frac{x^{-\mu}}{\Gamma(\varpi)} \int_0^x (x-t)^{\varpi-1} t^{-\mu'} \times F_3\left(\mu,\mu',\xi,\xi';\varpi;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,\mathrm{d}t \tag{12.1}$$

and

$$\left(\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi} f\right)(x) = \frac{x^{-\mu'}}{\Gamma(\varpi)} \int_x^\infty (t-x)^{\varpi-1} t^{-\mu} \times F_3\left(\mu,\mu',\xi,\xi';\varpi;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,\mathrm{d}t, \tag{12.2}$$

respectively, $\Gamma(.)$ being the Euler Gamma function [12]. The corresponding left- and right-hand-sided Marichev–Saigo–Maeda (MSM) fractional differentiation operators have the respective forms

$$\begin{aligned}\left(D_{0+}^{\mu,\mu',\xi,\xi',\varpi} f\right)(x) &= \left(\mathcal{I}_{0+}^{-\mu',-\mu,-\xi',-\xi,-\varpi} f\right)(x)\\ &= \left(\frac{d}{dx}\right)^n \left(\mathcal{I}_{0+}^{-\mu',-\mu,-\xi'+n,-\xi,-\varpi+n} f\right)(x) \quad (n = [Re(\varpi)]+1)\\ &= \frac{1}{\Gamma(n-\varpi)}\left(\frac{d}{dx}\right)^n x^{\mu'} \int_0^x (x-t)^{n-\eta-1} t^{\varrho}\\ &\quad \times F_3\left(-\mu',-\mu,n-\xi',-\xi;n-\varpi;1-\frac{t}{x},1-\frac{x}{t}\right) f(t)\,\mathrm{d}t\end{aligned} \tag{12.3}$$

and

$$\begin{aligned}\left(\mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi} f\right)(x) &= \left(\mathcal{I}_{-}^{-\mu',-\mu,-\xi',-\xi,-\varpi} f\right)(x)\\ &= \left(-\frac{d}{dx}\right)^n \left(\mathcal{I}_{-}^{-\mu',-\mu,-\xi',-\xi'+n,-\varpi+n} f\right)(x) \quad (n = [Re(\varpi)]+1)\\ &= \frac{1}{\Gamma(n-\varpi)}\left(-\frac{d}{dx}\right)^n x^{\mu'} \int_x^\infty (t-x)^{n-\varpi-1} t^{\varrho'}\\ &\quad \times F_3\left(-\mu',-\mu,\xi',n-\xi;n-\varpi;1-\frac{x}{t},1-\frac{t}{x}\right) f(t)\,\mathrm{d}t,\end{aligned} \tag{12.4}$$

where $[\Re(\varpi)]$ is the integral part of the $\Re(\varpi)$ and $F_3(.)$ denotes the third Appell's hypergeometric function of two variables [13]:

$$F_3[\mu_1,\mu_2,\xi_1,\xi_2;\varpi;x_1,x_2]=\sum_{m_1,m_2=0}^{\infty}\frac{(\mu)_{m_1}(\mu)_{m_2}(\xi_1)_{m_1}(\xi_2)_{m_2}}{(\varpi)_{m_1+m_2}}\frac{x_1^{m_1}}{m_1!}\frac{x_2^{m_2}}{m_2!}\quad(\max\{\Re(x_1),\Re(x_2)\}<1), \tag{12.5}$$

which reduces to Gauss' hypergeometric function [13]

$$\begin{aligned}{}_2F_1[\mu_1,\xi_1;\varpi;x_1]&=F_3[\mu_1,\mu_2,\xi_1,\xi_2;\varpi;x_1,0]\\&=F_3[\mu_1,0,\xi_1,\xi_2;\varpi;x_1,x_2]\\&=F_3[\mu_1,\mu_2,\xi_1,0;\varpi;x_1,x_2].\end{aligned} \tag{12.6}$$

These left- and right-hand-sided Marichev–Saigo–Maeda (MSM) fractional integral (12.1) and (12.2), and differentiation operators (12.3) and (12.4), reduce to the left- and right-hand-sided Saigo fractional integral and differential operators involving the hypergeometric function ${}_2F_1$, respectively, by substituting the specific values to the parameters as $\mu = \mu + \xi$, $\mu' = \xi' = 0$, $\xi = -\eta$ and $\varpi = \mu$:

$$\left(\mathcal{I}_{0+}^{\mu,\xi,\eta}f\right)(x)=\frac{x^{-\mu-\xi}}{\Gamma(\mu)}\int_0^x(x-t)^{\mu-1}{}_2F_1\left(\mu+\xi,-\eta;\mu;1-\frac{t}{x}\right)f(t)\,\mathrm{d}t, \tag{12.7}$$

$$\left(\mathcal{I}_{-}^{\mu,\xi,\eta}f\right)(x)=\frac{1}{\Gamma(\mu)}\int_x^\infty(t-x)^{\mu-1}t^{-\mu-\xi}{}_2F_1\left(\mu+\xi,-\eta;\mu;1-\frac{x}{t}\right)f(t)\,\mathrm{d}t, \tag{12.8}$$

$$\begin{aligned}\left(\mathcal{D}_{0+}^{\mu,\xi,\eta}f\right)(x)&=\left(\mathcal{I}_{0+}^{-\mu,-\xi,\mu+\eta}f\right)(x)\\&=\left(\frac{d}{dx}\right)^n\left(\mathcal{I}_{0+}^{-\mu+n,-\xi-n,\mu+\eta-n}f\right)(x)\quad(n=[\Re(\mu)]+1),\end{aligned} \tag{12.9}$$

and

$$\begin{aligned}\left(\mathcal{D}_{-}^{\mu,\xi,\eta}f\right)(x)&=\left(\mathcal{I}_{-}^{-\mu,-\xi,\mu+\eta}f\right)(x)\\&=(-1)^n\left(\frac{d}{dx}\right)^n\left(\mathcal{I}_{-}^{-\mu+n,-\xi-n,\mu+\eta}f\right)(x)\quad(n=[\Re(\mu)]+1).\end{aligned} \tag{12.10}$$

Again these left- and right-hand-sided Saigo fractional integral (12.7) and (12.8), and differentiation operators (12.9) and (12.10) reduce to the left- and right-hand-sided Riemann–Liouville (RL) fractional integral and differential operators by setting $\xi = -\mu$ (see, e.g., [14–17]):

$$\left(\mathcal{I}_{0+}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} f(t)\,\mathrm{d}t, \tag{12.11}$$

$$\left(\mathcal{I}_{-}^{\mu} f\right)(x) = \frac{1}{\Gamma(\mu)} \int_x^\infty (t-x)^{\mu-1} f(t)\,\mathrm{d}t, \tag{12.12}$$

and

$$\begin{aligned}\left(\mathcal{D}_{0+}^{\mu} f\right)(x) &= \left(\frac{d}{dx}\right)^n \left(\mathcal{I}_{0+}^{n-\mu} f\right)(x) \qquad (n = [\Re(\mu)]+1)\\ &= \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mu)} \int_0^x (x-t)^{n-\mu-1} f(t)\,\mathrm{d}t\end{aligned} \tag{12.13}$$

$$\begin{aligned}\left(\mathcal{D}_{-}^{\mu} f\right)(x) &= (-1)^n \left(\frac{d}{dx}\right)^n \left(\mathcal{I}_{-}^{n-\mu} f\right)(x) \qquad (n = [\Re(\mu)]+1)\\ &= (-1)^n \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mu)} \int_x^\infty (t-x)^{n-\mu-1} f(t)\,\mathrm{d}t.\end{aligned} \tag{12.14}$$

Further, these left- and right-hand-sided Saigo fractional integral (12.7) and (12.8), and differentiation operators (12.9) and (12.10), reduce to the left- and right-hand-sided Erdélyi–Kober (EK) fractional integral and differential operators by setting $\xi = 0$ (see, e.g., [14, 16–18]):

$$\left(\mathcal{I}_{\eta,\mu}^{+} f\right)(x) = \frac{x^{-\mu-\eta}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} t^{\eta} f(t)\,\mathrm{d}t, \tag{12.15}$$

$$\left(\mathcal{K}_{\eta,\mu}^{-} f\right)(x) \equiv \frac{x^{\eta}}{\Gamma(\mu)} \int_x^\infty (t-x)^{\mu-1} t^{-\mu-\eta} f(t)\,\mathrm{d}t, \tag{12.16}$$

$$\begin{aligned}\left(\mathcal{D}_{\eta,\mu}^{+} f\right)(x) &= \left(\frac{d}{dx}\right)^n \left(\mathcal{I}_{0+}^{-\mu+n,-\mu,\mu+\eta-n} f\right)(x) \quad (n = [\Re(\mu)]+1)\\ &= x^{-\eta} \left(\frac{d}{dx}\right)^n \frac{1}{\Gamma(n-\mu)} \int_0^x t^{\mu+\eta} (x-t)^{n-\mu-1} f(t)\,\mathrm{d}t,\end{aligned} \tag{12.17}$$

and

$$\begin{aligned}\left(\mathcal{D}_{\eta,\mu}^{-}f\right)(x) &= (-1)^n\left(\frac{d}{dx}\right)^n\left(\mathcal{I}_{-}^{-\mu+n,-\mu,\mu+\eta}f\right)(x) \quad (n=[\Re(\mu)]+1) \\ &= x^{\eta+\mu}\left(\frac{d}{dx}\right)^n\frac{1}{\Gamma(n-\mu)}\int_x^\infty t^{-\eta}(t-x)^{n-\mu-1}f(t)\,\mathrm{d}t.\end{aligned} \tag{12.18}$$

In 1890, Émile Leonard Mathieu [19] investigated the functional series $S(r)$ of the form

$$S(r)=\sum_{m\geq 1}\frac{2m}{(m^2+r^2)^2}, \qquad r>0 \tag{12.19}$$

popularly known as the Mathieu series. Further, Pogány et al. [20] introduced the alternating Mathieu functional series $\tilde{S}(r)$:

$$\tilde{S}(r)=\sum_{m\geq 1}(-1)^{m-1}\frac{2m}{(m^2+r^2)^2}, \qquad r>0. \tag{12.20}$$

Closed integral forms for $S(r)$ and $\tilde{S}(r)$ are presented as (see, e.g., [20, 21]):

$$S(r)=\frac{1}{r}\int_0^\infty\frac{x\sin(rx)}{\mathrm{e}^x-1}\mathrm{d}x \tag{12.21}$$

and

$$\tilde{S}(r)=\frac{1}{r}\int_0^\infty\frac{x\sin(rx)}{\mathrm{e}^x+1}\mathrm{d}x. \tag{12.22}$$

Moreover by making use of fractional power λ in (12.19) and (12.20), the generalized Mathieu-type functional series $S_\lambda(r)$ and its alternating version $\tilde{S}_\lambda(r)$ studied in [22, p. 2, Eq. (12.6)] (see also [23, p. 181]) are defined by:

$$S_\lambda(r)=\sum_{m\geq 1}\frac{2m}{(m^2+r^2)^\lambda} \qquad (r>0, \lambda>1), \tag{12.23}$$

and

$$\tilde{S}_\lambda(r)=\sum_{m\geq 1}(-1)^{m-1}\frac{2m}{(m^2+r^2)^\lambda} \qquad (r>0, \lambda>1). \tag{12.24}$$

In the mathematical literature, these series have received a good deal of attention (see [20, 22, 24]).

Recently, Saxena et al. [25] studied several integral transforms, in particular Mellin, Laplace, Euler, and Hankel transforms for the generalized fractional-order Mathieu-type functional series

$$S_\lambda(r;z) = \sum_{m\geq 1} \frac{2mz^{m-1}}{(m^2+r^2)^\lambda}, \qquad (\lambda > 1, r \in \mathbb{R}, |z| < 1), \tag{12.25}$$

and written inform of a Mellin–Barnes-type contour integral for $|\arg(-z)| < \pi$ as follows [25]:

$$S_\lambda(r;z) = \frac{1}{\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}(-z)^{-s}, \tag{12.26}$$

where $\mathcal{C} = \mathcal{C}_{(i\kappa;\infty)}$ is the contour of integration (loop located in a horizontal strip) beginning from the point $\kappa - i\infty$ and ending at the point $\kappa + i\infty$, where $\kappa \in \mathbb{R} = (+\infty, -\infty)$ such that all the poles of the Gamma function $\Gamma(2-s)$ are separated from the poles of the Gamma function $\Gamma(s)$ with the usual indentations and assuming that the poles of the integrand are simple and integral converges.

They also studied alternating generalized Mathieu-type functional series as:

$$\tilde{S}_\lambda(r;z) = \sum_{m\geq 1}(-1)^{m-1} \frac{2mz^{m-1}}{(m^2+r^2)^\lambda}, \qquad (\lambda > 1, r \in \mathbb{R}, |z| < 1), \tag{12.27}$$

and in the form of Mellin–Barnes-type contour integral

$$\tilde{S}_\lambda(r;z) = \frac{1}{\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda} z^{-s} \quad |\arg(z)| < \pi. \tag{12.28}$$

Several other investigations, extensions and generalizations of the Mathieu series with its alternating variants can be found in [23, 26–33]. More recently a new theory for multi-parameter Mathieu-type and alternating Mathieu-type functional series has been studied by Parmar et al. [34].

Our aim in this chapter is to derive fractional calculus results for the generalized Mathieu-type and alternating Mathieu-type functional series by making extensive use of Marichev–Saigo–Maeda operator tools (12.1), (12.2), (12.3), and (12.4) in terms of $\bar{H}$-function. Also, particular cases for Saigo's, Riemann–Liouville (RL) and Erdélyi–Kober (EK) fractional integral and differentiation operators are established. Moreover, from the special case and application point of view, all the results are also deduced in terms of Fox's *H*-function [35]. Furthermore, we also observe that all the results derived here can also be represented in terms of *I*-function, and could be potentially useful in the areas of mathematics for engineering and mathematical physics.

12.2 Results of Fractional Integration of the Mathieu Series in Terms of $\bar{H}$-Function

In 1987, Inayat-Hussain [36] investigated a number of interesting properties and characteristics of the hypergeometric functions of several variables in an attempt to evaluate in two different ways some Feynman-type integrals that occur in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions. More importantly, while demonstrating the usage of these Feynman-type integrals, Inayat-Hussain [37] was led to a unique generalization of Charles Fox's well-known *H*-function (1897–1977) [35]. The polylogarithm of a complex order and the precise partition function of the Gaussian model in statistical mechanics are examples of particular cases of this novel $\bar{H}$-function of Inayat-Hussain [37]. Indeed, it is defined as follows in terms of a Mellin–Barnes-type contour integral (see for details [38, 39]):

$$\overline{H}_{p,q}^{m,n}[z] = \overline{H}_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_r, E_r; \mu_j)_{\overline{1,n}}, \cdots, (e_r, E_r)_{\overline{n+1,p}} \\ (f_r, F_r)_{\overline{1,m}}, \cdots, (f_r, F_r; \xi_j)_{\overline{m+1,q}} \end{matrix} \right. \right] = \frac{1}{2\pi i}\int_{£} \chi_{p,q}^{m,n}(s)\, z^s \, ds \qquad (\forall z, z \neq 0) \tag{12.29}$$

where the Mellin–Barnes-type integral, taken over the path $£ = £_{(i\kappa;\infty)}$ beginning from $\kappa - i\infty$ and ending at the point $\kappa + i\,\infty$ ($\kappa \in \mathbb{R}$), with suitable indentations to avoid poles of the factors $\Gamma(f_r - F_r s)$ from poles of the factors $\{\Gamma(1 - e_r + E_r s)\}\mu^r$ and $\chi_{p,q}^{m,n}(s)$ is given by

$$\chi_{p,q}^{m,n}(s) = \frac{E(s)\overline{F}(s)}{E'(s)\overline{F'}(s)}, \tag{12.30}$$

$$E(s) = \prod_{r=1}^{m}\Gamma(f_r - F_r s), \;\; \overline{F}(s) = \prod_{r=1}^{n}\{\Gamma(1 - e_r + E_r s)\}^{\mu_r}, \tag{12.31}$$

$$\overline{E}'(s) = \prod_{r=m+1}^{q} \{\Gamma(1 - f_r + F_r s)\}^{\xi_r}, \quad F'(s) = \prod_{r=n+1}^{p} \Gamma(e_r - E_r s), \tag{12.32}$$

with $e_r \in \mathbb{C}$ $(r = 1, \cdots, p)$, $f_r \in \mathbb{C}$ $(r = 1, \cdots, q)$, $E_r \in \mathbb{R}^+$ $(r = 1, \cdots, p)$ and $F_r \in \mathbb{R}^+$ $(r = 1, \cdots, q)$, and the exponents μ_r $(r = 1, \cdots, n)$ and ξ_r $(r = m+1, \cdots, q)$ can assume non-integer values. The conditions for the absolute convergence of the Mellin–Barnes-type contour integral which defines analytic function for $|\arg(z)| < \frac{\pi}{2} \wedge$ in (12.29) is given by Buschman and Srivastava [38, p. 4708], where

$$\wedge = \sum_{r=1}^{m} F_r + \sum_{r=1}^{n} |\mu_j| E_r - \sum_{r=m+1}^{q} |\xi_r| F_r - \sum_{r=n+1}^{p} E_r > 0.$$

Before starting our main results, we provide image formulas of the power function $t^{\varrho-1}$ for the fractional calculus operators

$$\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}, \ \mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi},$$
$$\mathcal{I}_{0+}^{\mu,\xi,\eta}, \ \mathcal{I}_{-}^{\mu,\xi,\eta},$$
$$I_{0+}^{\mu}, \ I_{-}^{\mu}, \ \mathcal{I}_{\eta,\mu}^{+}, \ \mathcal{K}_{\eta,\mu}^{-},$$

which will be useful in deriving our main results.

Lemma 2.1 *Let* $\varrho, \mu, \mu', \xi, \xi', \varpi, \varrho \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) *If* ¾$(\varpi) > 0$ *and* $\Re(\varrho) > \max\{0, \Re(\mu + \mu' + \xi - \varpi), \Re(\mu' - \xi')\}$, then

$$\left(\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi} t^{\varrho-1}\right)(x) = \frac{\Gamma(\varrho)\Gamma(\varrho+\varpi-\mu-\mu'-\xi)\Gamma(\varrho+\xi'-\mu')}{\Gamma(\varrho+\xi')\Gamma(\varrho+\varpi-\mu-\mu')\Gamma(\varrho+\varpi-\mu'-\xi)} x^{\varrho+\varpi-\mu-\mu'-1} \tag{12.33}$$

(b) If $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(-\xi), \Re(\mu + \mu' - \varpi), \Re(\mu + \xi' - \varpi)\}$, then

$$\left(\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi} t^{\varrho-1}\right)(x) = \frac{\Gamma(1-\varrho-\xi)\Gamma(1-\varrho-\varpi+\mu+\mu')\Gamma(1-\varrho-\varpi+\mu+\xi')}{\Gamma(1-\varrho)\Gamma(1-\varrho-\varpi+\mu+\mu'+\xi')\Gamma(1-\varrho+\mu-\xi)} x^{\varrho+\varpi-\mu-\mu'-1}. \tag{12.34}$$

Lemma 2.2 Let ϱ, μ, ξ, η, $\varrho \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) If $\Re(\mu) > 0$ and $\Re(\varrho) > \max[0, \Re(\xi - \eta)]$, then

$$(\mathcal{I}_{0+}^{\mu,\xi,\eta} t^{\varrho-1})(x) = \frac{\Gamma(\varrho)\Gamma(\varrho+\eta-\xi)}{\Gamma(\varrho-\xi)\Gamma(\varrho+\mu+\eta)} x^{\varrho-\xi-1} \tag{12.35}$$

(b) If $\Re(\mu) > 0$ and $\Re(\varrho) < 1 + \min[\Re(\xi), \Re(\eta)]$, then

$$(\mathcal{I}_{-}^{\mu,\xi,\eta} t^{\varrho-1})(x) = \frac{\Gamma(1-\varrho+\xi)\Gamma(1-\varrho+\eta)}{\Gamma(1-\varrho)\Gamma(1-\varrho+\mu+\xi+\eta)} x^{\varrho-\xi-1}. \tag{12.36}$$

Lemma 2.3 Let $\varrho, \mu, \xi, \varrho \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) If $\Re(\mu) > 0$ and $\Re(\varrho) > 0$, then

$$(\mathcal{I}_{0+}^{\mu} t^{\varrho-1})(x) = \frac{\Gamma(\varrho)}{\Gamma(\varrho+\mu)} x^{\varrho+\mu-1}. \tag{12.37}$$

(b) If $0 < \Re(\mu) < 1 - \Re(\varrho)$, then

$$(\mathcal{I}_{-}^{\mu} t^{\varrho-1})(x) = \frac{\Gamma(1-\mu-\varrho)}{\Gamma(1-\varrho)} x^{\varrho+\mu-1}. \tag{12.38}$$

Lemma 2.4 Let $\varrho, \mu, \eta, \varrho \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) If $\Re(\mu) > 0$, $\Re(\varrho) > -\Re(\eta)$, then

$$(\mathcal{I}_{\eta,\mu}^{+} t^{\rho-1})(x) = \frac{\Gamma(\varrho+\eta)}{\Gamma(\varrho+\mu+\eta)} x^{\varrho-1}. \tag{12.39}$$

(b) If $\Re(\varrho) < 1 + \Re(\varrho)$, then

$$(\mathcal{K}_{\eta,\mu}^{-} t^{\varrho-1})(x) = \frac{\Gamma(1-\varrho+\eta)}{\Gamma(1-\varrho+\mu+\eta)} x^{\varrho-1}. \tag{12.40}$$

In this section, we present two composition formulas for Marichev–Saigo–Maeda (MSM) fractional integration operators (12.1) and (12.2) involving the generalized Mathieu-type functional series $S_\lambda(r,z)$ (12.26) in terms of the $\bar{H}$-function (12.29).

Theorem 2.1 Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) + \min\{0, \Re(\varpi - \mu - \mu' - \xi), \Re(-\xi' - \mu')\} > 0$. Then the following assertion for the MSM fractional operator $\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r,t^\varrho)$ (12.26) exists and holds true:

$$\left(\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\varpi-\mu-\mu'-1}$$

$$\times \overline{H}_{6,6}^{1,6}\left[-x^\rho \left| \begin{matrix} (1-\varrho,\rho;1),(1-\varrho-\varpi+\mu+\mu'+\xi,\rho;1),(1-\varrho-\xi'+\mu',\rho;1),(-1,1;1),(\pm ir,1;\lambda) \\ (0,1),(\pm ir-1,1;\lambda),(1-\varrho-\xi',\rho;1),(1-\rho-\varpi+\mu+\mu',\rho;1),(1-\varrho-\varpi+\mu'+\xi,\rho;1) \end{matrix} \right.\right]. \quad (12.41)$$

Proof. By making use of the operator (12.1) and expressing the definition of the generalized Mathieu-type functional series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26) and then interchanging the order of integration and applying the relation (12.33), we have

$$\left(\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)$$

$$= \frac{1}{\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\, t^{\varrho-\rho s-1}\right)(x)\, ds$$

$$= x^{\varrho+\varpi-\mu-\mu'-1}\frac{1}{\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}$$

$$\times \frac{\Gamma(\varrho-\rho s)\Gamma(\varrho+\varpi-\mu-\mu'-\xi-\rho s)\Gamma(\varrho+\xi'-\mu'-\rho s)}{\Gamma(\varrho+\xi'-\rho s)\Gamma(\varrho+\varpi-\mu-\mu'-\rho s)\Gamma(\varrho+\varpi-\mu'-\xi-\rho s)}(-x)^{-\rho s}\, ds$$

which, interpreted with the help of the definition (12.29), yields the required result (12.41).

Theorem 2.2 Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(-\xi), \Re(\mu + \mu' - \varpi), \Re(\mu + \xi' - \varpi)\}$. Then the following assertion for the MSM fractional operator $\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r,t^\varrho)$ (12.26) exists and holds true:

$$\left(\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\varpi-\mu-\mu'-1}$$

$$\times \overline{H}_{6,6}^{4,3}\left[-x^\rho \left| \begin{matrix} (-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho),(1-\varrho-\varpi+\mu+\mu'+\xi',\rho),(1-\varrho+\mu-\xi,\rho) \\ (0,1),(1-\varrho-\xi,\rho),(1-\varrho-\varpi+\mu+\mu',\rho),(1-\varrho-\varpi+\mu+\xi',\rho),(\pm ir-1,1;\lambda) \end{matrix} \right.\right]. \quad (12.42)$$

Proof. By making use of the operator (12.2) and expressing the definition of the generalized Mathieu-type functional series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26) and then changing the order of integrals and applying the relation (12.34), we have

$$\begin{aligned}
&\left(\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\rho-1}S_\lambda(r,t^\rho)\right\}\right)(x)\\
&= \frac{1}{\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}t^{\varrho-\rho s-1}\right)(x)\,ds\\
&= x^{\varrho+\varpi-\mu-\mu'-1}\frac{1}{\pi i}\int_{\mathcal{C}}\frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}\\
&\times\frac{\Gamma(1-\varrho-\xi+\rho s)\Gamma(1-\varrho-\varpi+\mu+\mu'+\rho s)\Gamma(1-\varrho-\varpi+\mu+\xi'+\rho s)}{\Gamma(1-\varrho+\rho s)\Gamma(1-\varrho-\varpi+\mu+\mu'+\xi'+\rho s)\Gamma(1-\varrho+\mu-\xi+\rho s)}(-x)^{-\rho s}\,ds
\end{aligned}$$

which, interpreted with the help of the definition (12.29), yields the required result (12.42).

We also deduce the fractional integral formulas for the Saigo's by substituting the specific values to the parameters as $\mu = \mu + \xi$, $\mu' = \xi' = 0$, $\xi = -\eta$ and $\varpi = \mu$, which are asserted by Corollary 2.1 and Corollary 2.2 below.

Corollary 2.1.

Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, ξ, η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) > 0$ and $\Re(\varrho) + \min[0, \Re(\eta-\xi)] > 0$. Then the following assertion for the Saigo's fractional operator $\mathcal{I}_{0+}^{\mu,\xi,\eta}$ of the $S_\lambda(r,t^\rho)$ (12.26) exists and holds true:

$$\begin{aligned}
&\left(\mathcal{I}_{0+}^{\mu,\xi,\eta}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)\\
&= 2x^{\varrho-\xi-1}\overline{H}_{5,5}^{1,5}\left[-x^\rho\,\middle|\,\begin{matrix}(1-\varrho,\rho;1),(1-\varrho+\xi-\eta,\rho;1),(-1,1;1),(\pm ir,1;\lambda)\\(0,1),(\pm ir-1,1;\lambda),(1-\varrho+\xi,\rho;1),(1-\varrho-\mu-\eta,\rho;1)\end{matrix}\right].
\end{aligned}\tag{12.43}$$

Corollary 2.2. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, ξ, η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) > 0$ and $\Re(\varrho) < 1 + \min[\Re(\xi), \Re(\eta)]$. Then the following assertion for the Saigo's fractional operator $\mathcal{I}_{-}^{\mu,\xi,\eta}$ of the $S_\lambda(r,t^\rho)$ (12.26) exists and holds true:

$$\begin{aligned}
&\left(\mathcal{I}_{-}^{\mu,\xi,\eta}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)\\
&= 2x^{\varrho-\xi-1}\overline{H}_{5,5}^{3,3}\left[-x^\rho\,\middle|\,\begin{matrix}(-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho),(1-\varrho+\mu+\xi+\eta,\rho)\\(0,1),(1-\varrho+\xi,\rho),(1-\varrho+\eta,\rho),(\pm ir-1,1;\lambda)\end{matrix}\right].
\end{aligned}\tag{12.44}$$

If we let $\xi = -\mu$ and $\xi = 0$ respectively, in Corollary 2.1 and Corollary 2.2, we deduce the following fractional integral formulas for the Riemann–Liouville (RL) and Erdélyi–Kober (EK), which are asserted by Corollary 2.3, Corollary 2.4, Corollary 2.5, and Corollary 2.6.

Corollary 2.3. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, $\in \mathbb{C}$ and satisfy the conditions $\Re(\mu) > 0$ and $\Re(\varrho) + \min[0, \Re(\eta)] > 0$. Then the following assertion for the RL fractional operator $\mathcal{I}_{0+}^{\mu}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(\mathcal{I}_{0+}^{\mu}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\mu-1}\overline{H}_{4,4}^{1,4}\left[-x^\rho \left| \begin{matrix} (1-\varrho,\rho;1),(-1,1;1),(\pm ir,1;\lambda) \\ (0,1),(\pm ir-1,1;\lambda),(1-\varrho-\mu,\rho;1) \end{matrix} \right.\right]. \tag{12.45}$$

Corollary 2.4. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, $\in \mathbb{C}$ and satisfy the conditions $0 < \Re(\mu) < 1 - \Re(\varrho)$. Then the following assertion for the RL fractional operator $\mathcal{I}_{-}^{\mu}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(\mathcal{I}_{-}^{\mu}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\mu-1}\overline{H}_{4,4}^{2,3}\left[-x^\rho \left| \begin{matrix} (-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho) \\ (0,1),(1-\varrho-\mu,\rho),(\pm ir-1,1;\lambda) \end{matrix} \right.\right]. \tag{12.46}$$

Corollary 2.5. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) > 0$ and $\Re(\varrho) + \min[0, \Re(\eta)] > 0$. Then the following assertion for the EK fractional operator $\mathcal{I}_{\eta,\mu}^{+}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(\mathcal{I}_{\eta,\mu}^{+}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho-1}\overline{H}_{4,4}^{1,4}\left[-x^\rho \left| \begin{matrix} (1-\varrho-\eta,\rho;1),(-1,1;1),(\pm ir,1;\lambda) \\ (0,1),(\pm ir-1,1;\lambda),(1-\varrho-\mu-\eta,\rho;1) \end{matrix} \right.\right]. \tag{12.47}$$

Corollary 2.6. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) > 0$ and $\Re(\varrho) < 1 + \Re(\eta)$. Then the following assertion for the EK fractional operator $\mathcal{K}_{\eta,\mu}^{-}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(\mathcal{K}_{\eta,\mu}^{-}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho-1}\overline{H}_{4,4}^{2,3}\left[-x^\rho \left| \begin{matrix} (-1,1;1),(\pm ir,1;\lambda),(1-\varrho+\mu+\eta,\rho) \\ (0,1),(1-\varrho+\eta,\rho),(\pm ir-1,1;\lambda) \end{matrix} \right.\right]. \tag{12.48}$$

12.3 Results of Fractional Differentiation of the Mathieu Series in Terms of $\bar{H}$-Function

In this section, we provide image formulas of the power function $t^{\rho-1}$ for the fractional calculus operators

$$\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi},\ \mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi},$$
$$\mathcal{D}_{0+}^{\mu,\xi,\eta},\ \mathcal{D}_{-}^{\mu,\xi,\eta},$$
$$\mathcal{D}_{0+}^{\mu},\ \mathcal{D}_{-}^{\mu},\ \mathcal{D}_{\eta,\mu}^{+},\ D_{\eta,\mu}^{-},$$

which will be useful in deriving our main results.

Lemma 3.1 Let ϱ, μ, μ', ξ, ξ', ϖ, $\varrho \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) If $\Re(\varpi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\varpi - \mu - \mu' + \xi'), \Re(\xi - \mu)\}$, then

$$\left(\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi} t^{\varrho-1}\right)(x) = \frac{\Gamma(\varrho)\Gamma(\varrho-\varpi+\mu+\mu'+\xi')\Gamma(\varrho-\xi+\mu)}{\Gamma(\varrho-\xi)\Gamma(\varrho-\varpi+\mu+\mu')\Gamma(\varrho-\varpi+\mu+\xi')} x^{\varrho-\varpi+\mu+\mu'-1} \quad (12.49)$$

(b) If $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(\xi'), \Re(\varpi - \mu - \mu', \Re(\eta - \mu' - \xi)\}$, then

$$\left(\mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi} t^{\varrho-1}\right)(x) = \frac{\Gamma(1-\varrho-\xi')\Gamma(1-\varrho+\varpi-\mu-\mu')\Gamma(1-\varrho+\varpi-\mu'-\xi)}{\Gamma(1-\varrho)\Gamma(1-\varrho+\varpi-\mu-\mu'-\nu)\Gamma(1-\varrho-\mu'-\xi')} x^{\varrho-\varpi+\mu+\mu'-1}. \quad (12.50)$$

Lemma 3.2 Let $\varrho, \mu, \xi, \eta \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) If $\Re(\mu) > 0$ and $\Re(\varrho) > -\min[0, \Re(\mu + \xi + \eta)]$, then

$$(\mathcal{D}_{0+}^{\mu,\xi,\eta} t^{\varrho-1})(x) = \frac{\Gamma(\varrho)\Gamma(\varrho+\mu+\xi+\eta)}{\Gamma(\varrho+\xi)\Gamma(\varrho+\eta)} x^{\varrho+\xi-1} \quad (12.51)$$

(b) If $\Re(\mu) > 0$, $\Re(\varrho) < 1 + \min[\Re(-\xi - n), \Re(\mu + \eta)]$ and $n = [\Re(\mu)] + 1$, then

$$(\mathcal{D}_{-}^{\mu,\xi,\eta} t^{\varrho-1})(x) = \frac{\Gamma(1-\varrho-\xi)\Gamma(1-\varrho+\mu+\eta)}{\Gamma(1-\varrho)\Gamma(1-\varrho+\eta-\xi)} x^{\varrho+\xi-1}. \quad (12.52)$$

Lemma 3.3 Let $\varrho, \mu \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) If $\Re(\mu) > 0$ and $\Re(\varrho) > 0$, then

$$(\mathcal{D}_{0+}^{\mu} t^{\varrho-1})(x) = \frac{\Gamma(\varrho)}{\Gamma(\varrho-\mu)} x^{\varrho-\mu-1}, \tag{12.53}$$

(b) If $\Re(\mu) > 0$, $\Re(\varrho) < 1 + \Re(\mu) - n)$ and $n = [\Re(\mu)] + 1$, then

$$(\mathcal{D}_{-}^{\mu} t^{\varrho-1})(x) = \frac{\Gamma(1-\varrho+\mu)}{\Gamma(1-\varrho)} x^{\varrho-\mu-1}, \tag{12.54}$$

Lemma 3.4 Let $\varrho, \mu, \eta \in \mathbb{C}$ and $x > 0$. Then there is the following relationship:

(a) If $\Re(\mu) > 0$ and $\Re(\varrho) > -\Re(\mu + \eta)$, then

$$(\mathcal{D}_{\eta,\mu}^{+} t^{\varrho-1})(x) = \frac{\Gamma(\varrho+\mu+\eta)}{\Gamma(\varrho+\eta)} x^{\varrho-1}. \tag{12.55}$$

(b) If $\Re(\mu) > 0$, $\Re(\varrho) < 1 + \Re(\mu + \eta) - n$ and $n = [\Re(\mu)] + 1$, then

$$(\mathcal{D}_{\eta,\mu}^{-} t^{\varrho-1})(x) = \frac{\Gamma(1-\varrho+\mu+\eta)}{\Gamma(1-\varrho-\eta)} x^{\varrho-1}. \tag{12.56}$$

In this section, we present two composition formulas for Marichev–Saigo–Maeda (MSM) fractional differentiation operators (12.3) and (12.4) involving the generalized Mathieu-type functional series $S_\lambda(r,z)$ (12.26) in terms of the $\bar{H}$-function (12.29).

Theorem 3.1 Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\rho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\varpi - \mu - \mu' + \xi'), \Re(\xi - \mu)\}$. Then the following assertion for the MSM fractional operator $\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho-\xi-1}$$
$$\times \overline{H}_{6,6}^{1,6}\left[-x^\rho \,\middle|\, \begin{matrix} (1-\varrho,\rho;1),(1-\varrho+\varpi-\mu-\mu'-\xi',\rho;1),(1-\varrho+\xi-\mu,\rho;1),(-1,1;1),(\pm ir,1;\lambda) \\ (0,1),(\pm ir-1,1;\lambda),(1-\varrho+\xi,\rho;1),(1-\varrho+\varpi-\mu-\mu',\rho;1),(1-\varrho+\varpi-\mu-\xi',\rho;1) \end{matrix}\right]. \tag{12.57}$$

Proof. By making use of the operator (12.3) and expressing the definition of the generalized Mathieu-type functional series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26) and then changing the order of integrals and applying the relation (12.49), we have for $x > 0$

$$\begin{aligned}&\left(\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)\\&=\frac{1}{\pi i}\int_C\frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(D_{0+}^{\mu,\mu',\xi,\xi',\varpi}t^{\varrho-\rho s-1}\right)(x)\,ds\\&=x^{\varrho-\xi-1}\frac{1}{\pi i}\int_C\frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}\\&\times\frac{\Gamma(\varrho-\rho s)\Gamma(\varrho-\varpi+\mu+\mu'+\xi'-\rho s)\Gamma(\varrho-\xi+\mu-\rho s)}{\Gamma(\varrho-\xi-\rho s)\Gamma(\varrho-\varpi+\mu+\mu'-\rho s)\Gamma(\varrho-\varpi+\mu+\xi'-\rho s)}(-x)^{-\rho s}\,ds\end{aligned}$$

which, interpreted with the help of the definition (12.29), yields the required derivative formula (12.57).

Theorem 3.2 Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(\xi'), \Re(\varpi - \mu - \mu', \Re(\eta - \mu' - \xi)\}$. Then the following assertion for the MSM fractional operator $\mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r,t^\rho)$ (12.26) exists and holds true:

$$\begin{aligned}&\left(\mathcal{D}_{-}^{\mu,\xi,\eta}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)=2x^{\varrho+\xi-1}\\&\times\overline{H}_{6,6}^{4,3}\left[-x^\rho\left|\begin{matrix}(-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho),(1-\varrho+\varpi-\mu-\mu'-\nu,\rho),(1-\varrho-\mu'-\xi',\rho)\\(0,1),(1-\varrho-\xi',\rho),(1-\varrho+\varpi-\mu-\mu',\rho),(1-\varrho+\varpi-\mu'-\xi,\rho),(\pm ir-1,1;\lambda)\end{matrix}\right.\right].\end{aligned} \tag{12.58}$$

Proof. By making use of the operator (12.4) and expressing the definition of the generalized Mathieu-type functional series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26) and then changing the order of integrals and applying the relation (12.50), we have for $x > 0$

$$\begin{aligned}&\left(\mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)\\&=\frac{1}{\pi i}\int_C\frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(\mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi}t^{\varrho-\rho s-1}\right)(x)\,ds\\&=x^{\varrho+\xi-1}\frac{1}{\pi i}\int_C\frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}\\&\quad\times\frac{\Gamma(1-\varrho-\xi'+\rho s)\Gamma(1-\varrho+\varpi-\mu-\mu'+\rho s)\Gamma(1-\varrho+\varpi-\mu'-\xi+\rho s)}{\Gamma(1-\varrho+\rho s)\Gamma(1-\varrho+\varpi-\mu-\mu'-\nu+\rho s)\Gamma(1-\varrho-\mu'-\xi'+\rho s)}(-x)^{-\rho s}\,ds\end{aligned}$$

which, interpreted with the help of the definition (12.29), yields the required formula (12.58).

We also establish the fractional integral formulas for the Saigo's by substituting the specific values to the parameters as $\mu = \mu + \xi$, $\mu' = \xi' = 0$, $\xi = -\eta$ and $\varpi = \mu$, which are asserted by Corollary 3.1 and Corollary 3.2 below.

Corollary 3.1. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \xi, \eta, \varpi \in \mathbb{C}$ with $\Re(\mu) > 0$, $\Re(\mu + \xi + \eta) \neq 0$ and satisfy the conditions $\Re(\varrho) + \min[0, \Re(\mu + \xi + \eta)] > 0$. Then the following assertion for the Saigo's fractional operator $\mathcal{D}_{0+}^{\mu,\xi,\eta}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(\mathcal{D}_{0+}^{\mu,\xi,\eta}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)$$
$$= 2x^{\varrho-\xi-1}\overline{H}_{5,5}^{1,5}\left[-x^\rho\left|\begin{matrix}(1-\varrho,\rho;1),(1-\varrho-\mu-\xi-\eta,\rho;1),(-1,1;1),(\pm ir,1;\lambda)\\(0,1),(\pm ir-1,1;\lambda),(1-\varrho-\xi,\rho;1),(1-\varrho-\eta,\rho;1)\end{matrix}\right.\right] \tag{12.59}$$

Corollary 3.2. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \xi, \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) \geq 0$ and $\Re(\varrho) < 1 + \min[\Re(-\xi - n), \Re(\mu + \eta)]$, $n = [\Re(\mu)] + 1$. Then the following assertion for the Saigo's fractional operator $\mathcal{D}_{-}^{\mu,\xi,\eta}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(\mathcal{D}_{-}^{\mu,\xi,\eta}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)$$
$$= 2x^{\varrho+\xi-1}\overline{H}_{5,5}^{3,3}\left[-x^\rho\left|\begin{matrix}(-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho),(1-\varrho+\eta-\xi,\rho)\\(0,1),(1-\varrho-\xi,\rho),(1-\varrho+\mu+\eta,\rho),(\pm ir-1,1;\lambda)\end{matrix}\right.\right] \tag{12.60}$$

If we let $\xi = -\mu$ and $\xi = 0$ respectively, in Corollary 3.1 and Corollary 3.2, we deduce the following fractional integral formulas for the Riemann–Liouville (RL) and Erdélyi–Kober (EK), which are asserted by Corollary 3.3, Corollary 3.4, Corollary 3.5, and Corollary 3.6 below.

Corollary 3.3. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) > 0$ and $\Re(\varrho) + \min[0, \Re(\eta)] > 0$. Then the following assertion for the RL fractional operator $\mathcal{D}_{0+}^{\mu}$ of the $S\lambda(r, t\varrho)$ (12.26) exists and holds true:

$$\left(\mathcal{D}_{0+}^{\mu}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\mu-1}\overline{H}_{4,4}^{1,4}\left[-x^\varrho\left|\begin{matrix}(1-\varrho,\rho;1),(-1,1;1),(\pm ir,1;\lambda)\\(0,1),(\pm ir-1,1;\lambda),(1-\varrho+\mu,\rho;1)\end{matrix}\right.\right]. \tag{12.61}$$

Corollary 3.4. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) \geq 0$ and $\Re(\varrho) < \Re(\mu) - [\Re(\mu)]$. Then the following assertion for the RL fractional operator D_{-}^{μ} of the $S\lambda(r, t\varrho)$ (12.26) exists and holds true:

$$\left(\mathcal{D}_{-}^{\mu}\left\{t^{\varrho-1}S_{\lambda}(r,t^{\rho})\right\}\right)(x) = x^{\varrho-\mu-1}\overline{H}_{4,4}^{2,3}\left[-x^{\varrho}\left|\begin{matrix}(-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho)\\(0,1),(1-\varrho+\mu,\rho),(\pm ir-1,1;\lambda)\end{matrix}\right.\right]. \tag{12.62}$$

Corollary 3.5. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) \geq 0$ and $\Re(\varrho) > -\Re(\mu + \eta)$. Then the following assertion for the EK fractional operator $\mathcal{D}_{\eta,\mu}^{+}$ of the $S\lambda(r, t\varrho)$ (12.26) exists and holds true:

$$\left(\mathcal{D}_{\eta,\mu}^{+}\left\{t^{\varrho-1}S_{\lambda}(r,t^{\rho})\right\}\right)(x) = 2x^{\varrho-1}\overline{H}_{4,4}^{1,4}\left[-x^{\rho}\left|\begin{matrix}(1-\varrho-\mu-\eta,\rho;1),(-1,1;1),(\pm ir,1;\lambda)\\(0,1),(\pm ir-1,1;\lambda),(1-\varrho-\eta,\rho;1)\end{matrix}\right.\right]. \tag{12.63}$$

Corollary 3.6. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\mu) > 0$ and $\Re(\varrho) < 1 + \Re(\eta)$. Then the following assertion for the EK fractional operator $D_{\eta,\mu}^{-}$ of the $S\lambda(r, t\varrho)$ (12.26) exists and holds true:

$$\left(\mathcal{D}_{\eta,\mu}^{-}\left\{t^{\varrho-1}S_{\lambda}(r,t^{\rho})\right\}\right)(x) = x^{\varrho-1}\overline{H}_{4,4}^{2,3}\left[-x^{\rho}\left|\begin{matrix}(-1,1;1),(\pm ir,1;\lambda),(1-\varrho+\eta,\rho)\\(0,1),(1-\varrho+\mu+\eta,\rho),(\pm ir-1,1;\lambda)\end{matrix}\right.\right]. \tag{12.64}$$

12.4 Special Cases in Terms of Fox's *H*-Function

The H-function is defined by C. Fox [35] in his studies of symmetrical Fourier kernels as the Mellin–Barnes-type path integral (see, for details, [40]):

$$\begin{aligned} H_{p,q}^{m,n}(z) &= H_{p,q}^{m,n}\left[z\left|\begin{matrix}(e_r,E_r)\\(f_r,F_r)\end{matrix}\right.\right] = H_{p,q}^{m,n}\left[z\left|\begin{matrix}(e_1,E_1),\cdots,(e_r,E_r)\\(f_1,F_1),\cdots,(f_r,F_r)\end{matrix}\right.\right] \\ &= \frac{1}{2\pi i}\int_{\pounds}\psi_{p,q}^{m,n}(s)\,z^s\,ds \qquad (\forall z, z\neq 0) \end{aligned} \tag{12.65}$$

where

$$\psi_{p,q}^{m,n}(s) = \frac{E(s)F(s)}{E'(s)F'(s)}, \tag{12.66}$$

$$E(s) = \prod_{r=1}^{m} \Gamma(f_r - F_r s), \;\; B(s) = \prod_{rr=1}^{n} \Gamma(1 - e_r + E_r s), \tag{12.67}$$

$$E'(s) = \prod_{r=m+1}^{q} \Gamma(1 - f_r + F_r s), \;\; F'(s) = \prod_{r=n+1}^{p} \Gamma(e_r - E_r s), \tag{12.68}$$

with $e_r \in \mathbb{C}$ $(r = 1,\cdots,p)$, $f_r \in \mathbb{C}$ $(r = 1,\cdots,q)$, $E_r \in \mathbb{R}^+$ $(r = 1,\cdots,p)$ and $F_r \in \mathbb{R}^+$ $(r = 1,\cdots,q)$. and £ is a suitable contour of the Mellin–Barnes type separating the poles of $\Gamma(f_r + F_r s)$ $(r = 1,\cdots,m)$ from those of $\Gamma(1 - e_r - E_r s)$ $(r = 1,\cdots,n)$ with the usual indentations. An empty product, when it occurs, is taken to be interpreted as 1, the integers $m,\ n,\ p,\ q$ satisfy the inequalities $0 \le m \le q$ and $0 \le n \le p$. The Mellin–Barnes-type contour integration representing in the H-function converges absolutely and defines an analytic function for $|\arg(z)| < \frac{\pi}{2}\Omega$, where

$$\Omega = \sum_{r=1}^{m} F_r - \sum_{r=m+1}^{q} F_r + \sum_{r=1}^{n} E_r - \sum_{r=n+1}^{p} E_r > 0.$$

Our results derived in previous sections are in terms of $\bar{H}$-function (12.29). An important special case of $\bar{H}$-function when integral powers assume positive integer values yields Fox's H-function (12.65). Also, if we put $\lambda = 2$ in all the results in Sections 12.2 and 12.3, we obtain the results for the Mathieu-type functional series

$$S(r;z) = \sum_{m\ge1} \frac{2mz^{m-1}}{(m^2 + r^2)^2}, \qquad (r \in \mathbb{R}, |z| < 1). \tag{12.69}$$

We present only the main results for the Marichev–Saigo–Maeda (MSM) fractional operators as Corollaries 4.1 to 4.4 without proof.

Corollary 4.1. Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) + \min\{0, \Re(\varpi - \mu - \mu' - \xi), \Re(-\xi' - \mu')\} > 0$. Then the following assertion for the MSM fractional operator $\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S(r,t^{\rho})$ (12.69) exists and holds true:

$$\left(\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S(r,t^{\rho})\right\}\right)(x)=2x^{\varrho+\varpi-\mu-\mu'-1}$$
$$\times H_{8,8}^{1,8}\left[-x^{\rho}\left|\begin{matrix}(1-\varrho,\rho),(1-\varrho-\varpi+\mu+\mu'+\xi,\rho),(1-\varrho-\xi'+\mu',\rho),(-1,1),(\pm ir,1),(\pm ir,1)\\(0,1),(\pm ir-1,1),(\pm ir-1,1),(1-\varrho-\xi',\rho),(1-\varrho-\varpi+\mu+\mu',\rho),(1-\varrho-\varpi+\mu'+\xi,\rho)\end{matrix}\right.\right]. \tag{12.70}$$

Corollary 4.2. Let $\lambda - 1$, $r \in \mathbb{R}^{+}$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(-\xi), \Re(\mu + \mu' - \varpi), \Re(\mu + \xi' - \varpi)\}$. Then the following assertion for the MSM fractional operator $\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S(r,t^{\rho})$ (12.69) exists and holds true:

$$\left(\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S(r,t^{\rho})\right\}\right)(x)=2x^{\varrho+\varpi-\mu-\mu'-1}$$
$$\times H_{8,8}^{4,5}\left[-x^{\rho}\left|\begin{matrix}(-1,1),(\pm ir,1),(\pm ir,1),(1-\varrho,\rho),(1-\varrho-\varpi+\mu+\mu'+\xi',\rho),(1-\varrho+\mu-\xi,\rho)\\(0,1),(1-\varrho-\xi,\rho),(1-\varrho-\varpi+\mu+\mu',\rho),(1-\varrho-\varpi+\mu+\xi',\rho),(\pm ir-1,1),(\pm ir-1,1)\end{matrix}\right.\right]. \tag{12.71}$$

Corollary 4.3. Let $\lambda - 1$, $r \in \mathbb{R}^{+}$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\varpi - \mu - \mu' + \xi'), \Re(\xi - \mu)\}$. Then the following assertion for the MSM fractional operator $\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S(r,t^{\rho})$ (12.69) exists and holds true:

$$\left(\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S(r,t^{\rho})\right\}\right)(x)=2x^{\varrho-\xi-1}$$
$$\times H_{8,8}^{1,8}\left[-x^{\rho}\left|\begin{matrix}(1-\varrho,\rho),(1-\varrho+\varpi-\mu-\mu'-\xi',\rho),(1-\varrho+\xi-\mu,\rho),(-1,1),(\pm ir,1),(\pm ir,1)\\(0,1),(\pm ir-1,1),(\pm ir-1,1),(1-\varrho+\xi,\rho),(1-\varrho+\varpi-\mu-\mu',\rho),(1-\varrho+\varpi-\mu-\xi',\rho)\end{matrix}\right.\right]. \tag{12.72}$$

Corollary 4.4. Let $\lambda - 1$, $r \in \mathbb{R}^{+}$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(\xi'), \Re(\varpi - \mu - \mu', \Re(\eta - \mu' - \xi)\}$. Then the following assertion for the MSM fractional operator $\mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S(r,t^{\rho})$ (12.69) exists and holds true:

$$\left(\mathcal{D}_{-}^{\mu,\xi,\eta}\left\{t^{\varrho-1}S(r,t^{\rho})\right\}\right)(x)=2x^{\varrho+\xi-1}$$
$$\times H_{8,8}^{4,5}\left[-x^{\rho}\left|\begin{matrix}(-1,1),(\pm ir,1),(\pm ir,1),(1-\varrho,\rho),(1-\varrho+\varpi-\mu-\mu'-\nu,\rho),(1-\varrho-\mu'-\xi',\rho)\\(0,1),(1-\varrho-\xi',\rho),(1-\varrho+\varpi-\mu-\mu',\rho),(1-\varrho+\varpi-\mu'-\xi,\rho),(\pm ir-1,1),(\pm ir-1,1)\end{matrix}\right.\right]. \tag{12.73}$$

Interested readers can derive the other results for Saigo's, Riemann–Liouville (RL) and Erdélyi–Kober (EK) operators.

12.5 Further Observations and Applications

In the investigations of the previous section, we established various fractional calculus results for the generalized Mathieu-type and alternating Mathieu-type functional series by extensive use of the Marichev–Saigo–Maeda, Saigo's, Riemann–Liouville and Erdélyi–Kober tools. The (presumably) new and (potentially) useful results are expressed in terms of the generalized H-function, that is the $\bar{H}$-function. Moreover, as a special case and from the applications point of view, all the results are obtained in terms of the Fox H-function. From the point of view of other applications, all the results obtained in the previous sections can be written in terms of the I-function introduced by Rathie [41], which contains several special cases as various special functions, for example, Inayat-Hussein $\bar{H}$-function, H-function, and G-function in the following form:

$$I_{p,q}^{m,n}(z) = I_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_r, E_r, \mu_r)_{\overline{1,p}} \\ (f_r, F_r, \xi_r)_{\overline{1,q}} \end{matrix} \right. \right] = I_{p,q}^{m,n}\left[z \left| \begin{matrix} (e_1, E_1, \mu_1), \cdots, (e_r, E_r, \mu_p) \\ (f_1, F_1, \xi_1), \cdots, (f_r, F_r, \xi_q) \end{matrix} \right. \right] \tag{12.74}$$
$$= \frac{1}{2\pi i} \int_{£} \phi_{p,q}^{m,n}(s)\, z^s\, ds \qquad (\forall z, z \neq 0)$$

for all $z \neq 0$, where

$$\phi_{p,q}^{m,n}(s) = \frac{\bar{E}(s)\bar{F}(s)}{\bar{E}'(s)\bar{F}'(s)}, \tag{12.75}$$

$$\bar{E}(s) = \prod_{r=1}^{m} \{\Gamma(f_r - F_r s)\}^{\xi_r}, \; \bar{F}(s) = \prod_{r=1}^{n} \{\Gamma(1 - e_r + E_r s)\}^{\mu_r}, \tag{12.76}$$

$$\bar{E}'(s) = \prod_{j=m+1}^{q} \{\Gamma(1 - f_r + F_r s)\}^{\xi_j}, \; \bar{F}'(s) = \prod_{j=n+1}^{p} \{\Gamma(e_r - E_r s)\}^{\mu_j}, \tag{12.77}$$

with $e_r \in \mathbb{C}$ $(j = 1, \cdots, p)$, $f_r \in \mathbb{C}$ $(j = 1, \cdots, q)$, $E_r \in \mathbb{R}^+$ $(j = 1, \cdots, p)$ and $F_r \in \mathbb{R}^+$ $(j = 1, \cdots, q)$. and the exponents μ_j $(j = 1, \cdots, n)$ and ξ_j $(j = m+1, \cdots, q)$ can take non-integer values. £ is a suitable contour of the Mellin–Barnes type separating the poles of $\{\Gamma(f_r - F_r s)\}_{\xi j}$ $(j = 1, \cdots, m)$ from those of $\Gamma(1 - e_r + E_r s)\}_{\mu j}$ $(j = 1, \cdots, n)$ with the usual indentations. An empty product is interpreted as 1, the integers m, n, p, q satisfy the inequalities $0 \le m \le q$ and $0 \le n \le p$. Also, the Mellin–Barnes contour integral representing the I-function converges absolutely and defines an analytic function for $|\arg(z)| < \frac{\pi}{2}\Delta$, where

$$\Delta = \sum_{j=1}^{m} |\xi_j| F_r + \sum_{j=1}^{n} |\mu_j| E_r - \sum_{j=m+1}^{q} |\xi_j| F_r - \sum_{j=n+1}^{p} |\mu_j| E_r > 0.$$

Example 5.1

Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\rho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, μ', ξ, ξ', η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) + \min\{0, \Re(\varpi - \mu - \mu' - \xi), \Re(-\xi' - \mu')\} > 0$. Then the following assertion for the MSM fractional operator $I_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(I_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\varpi-\mu-\mu'-1}$$
$$\times I_{6,6}^{1,6}\left[-x^\rho \left| \begin{matrix} (1-\varrho,\rho,1),(1-\varrho-\varpi+\mu+\mu'+\xi,\rho,1),(1-\varrho-\xi'+\mu',\rho,1),(-1,1,1),(\pm ir,1,\lambda) \\ (0,1,1),(\pm ir-1,1,\lambda),(1-\varrho-\xi',\rho,1),(1-\varrho-\varpi+\mu+\mu',\rho,1),(1-\varrho-\varpi+\mu'+\xi,\rho,1) \end{matrix} \right.\right]. \tag{12.78}$$

Proof. By making use of the operator (12.1) and expressing the definition of the generalized Mathieu-type functional series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26), and then changing the order of integrals and applying the relation (12.33), we have

$$\left(I_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)$$
$$= \frac{1}{\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(I_{0+}^{\mu,\mu',\xi,\xi',\varpi}t^{\varrho-\rho s-1}\right)(x)\,ds$$
$$= x^{\varrho+\varpi-\mu-\mu'-1}\frac{1}{\pi i}\int_{\mathcal{C}} \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}$$
$$\times \frac{\Gamma(\varrho-\rho s)\Gamma(\varrho+\varpi-\mu-\mu'-\xi-\rho s)\Gamma(\varrho+\xi'-\mu'-\rho s)}{\Gamma(\varrho+\xi'-\rho s)\Gamma(\varrho+\varpi-\mu-\mu'-\rho s)\Gamma(\varrho+\varpi-\mu'-\xi-\rho s)}(-x)^{-\rho s}\,ds$$

which, interpreted with the help of the definition (12.74), yields the required result (12.78).

Example 5.2

Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\rho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, μ', ξ, ξ', η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(-\xi), \Re(\mu + \mu' - \varpi), \Re(\mu + \xi' - \varpi)\}$. Then the following assertion for the MSM fractional operator $I_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(I_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\varpi-\mu-\mu'-1}$$
$$\times I_{6,6}^{4,3}\left[-x^\rho \left| \begin{matrix} (-1,1,1),(\pm ir,1,\lambda),(1-\varrho,\rho,1),(1-\varrho-\varpi+\mu+\mu'+\xi',\rho,1),(1-\varrho+\mu-\xi,\rho,1) \\ (0,1,1),(1-\varrho-\xi,\rho,1),(1-\varrho-\varpi+\mu+\mu',\rho,1),(1-\varrho-\varpi+\mu+\xi',\rho,1),(\pm ir-1,1,\lambda) \end{matrix} \right.\right]. \tag{12.79}$$

Proof. By making use of the operator (12.2) and expressing the definition of the generalized Mathieu-type functional series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26), and then changing the order of integrals and applying the relation (12.34), we have

$$\left(I_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)$$
$$=\frac{1}{\pi i}\int_C \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(I_{-}^{\mu,\mu',\xi,\xi',\varpi}t^{\varrho-\rho s-1}\right)(x)\,ds$$
$$=x^{\varrho+\varpi-\mu-\mu'-1}\frac{1}{\pi i}\int_C \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}$$
$$\times\frac{\Gamma(1-\varrho-\xi+\rho s)\Gamma(1-\varrho-\varpi+\mu+\mu'+\rho s)\Gamma(1-\varrho-\varpi+\mu+\xi'+\rho s)}{\Gamma(1-\varrho+\rho s)\Gamma(1-\varrho-\varpi+\mu+\mu'+\xi'+\rho s)\Gamma(1-\varrho+\mu-\xi+\rho s)}(-x)^{-\rho s}\,ds$$

which, interpreted with the help of the definition (12.74), yields the required result (12.79).

Example 5.3

Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, μ', ξ, ξ', η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\varpi-\mu-\mu'+\xi'), \Re(\xi-\mu)\}$. Then the following assertion for the MSM fractional operator $D_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r,t^\rho)$ (12.26) exists and holds true:

$$\left(D_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)=2x^{\varrho-\xi-1}$$
$$\times I_{6,6}^{1,6}\left[-x^\rho\left|\begin{matrix}(1-\varrho,\rho,1),(1-\varrho+\varpi-\mu-\mu'-\xi',\rho,1),(1-\varrho+\xi-\mu,\rho,1),(-1,1,1),(\pm ir,1,\lambda)\\(0,1,1),(\pm ir-1,1,1,\lambda),(1-\varrho+\xi,\rho,1),(1-\varrho+\varpi-\mu-\mu',\rho,1),(1-\varrho+\varpi-\mu-\xi',\rho,1)\end{matrix}\right.\right]. \tag{12.80}$$

Proof. By making use of the operator (12.3) and expressing the definition of the generalized Mathieu series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26), and then changing the order of integrals and applying the relation (12.49), we have for $x > 0$

$$\left(D_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)$$
$$=\frac{1}{\pi i}\int_C \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(D_{0+}^{\mu,\mu',\xi,\xi',\varpi}t^{\varrho-\rho s-1}\right)(x)\,ds$$
$$=x^{\varrho-\xi-1}\frac{1}{\pi i}\int_C \frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}$$
$$\times\frac{\Gamma(\varrho-\rho s)\Gamma(\varrho-\varpi+\mu+\mu'+\xi'-\rho s)\Gamma(\varrho-\xi+\mu-\rho s)}{\Gamma(\varrho-\xi-\rho s)\Gamma(\varrho-\varpi+\mu+\mu'-\rho s)\Gamma(\varrho-\varpi+\mu+\xi'-\rho s)}(-x)^{-\rho s}\,ds$$

which, interpreted with the help of the definition (12.74), yields the required derivative formula (12.80).

Example 5.4

Let $\lambda - 1$, $r \in \mathbb{R}^{+}$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let ϱ, μ, μ', ξ, ξ', η, $\varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(\xi'), \Re(\varpi - \mu - \mu', \Re(\eta - \mu' - \xi)\}$. Then the following assertion for the MSM fractional operator $D_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda(r, t^\rho)$ (12.26) exists and holds true:

$$\left(D_{-}^{\mu,\xi,\eta}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\xi-1}$$
$$\times I_{6,6}^{4,3}\left[-x^\rho\left|\begin{matrix}(-1,1,1),(\pm ir,1,\lambda),(1-\varrho,\rho,1),(1-\varrho+\varpi-\mu-\mu'-\nu,\rho,1),(1-\varrho-\mu'-\xi',\rho,1)\\(0,1,1),(1-\varrho-\xi',\rho.1),(1-\varrho+\varpi-\mu-\mu',\rho,1),(1-\varrho+\varpi-\mu'-\xi,\rho,1),(\pm ir-1,1,\lambda)\end{matrix}\right.\right]. \tag{12.81}$$

Proof. By making use of the operator (12.4) and expressing the definition of the generalized Mathieu-type functional series $S_\lambda(r,z)$ in the form of the contour integral representation (12.26), and then changing the order of integrals and applying the relation (12.50), we have for $x > 0$

$$\left(D_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}S_\lambda(r,t^\rho)\right\}\right)(x)$$
$$= \frac{1}{\pi i}\int_{C}\frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda(-1)^{-\rho s}}{\{\Gamma(2\pm ir-s)\}^\lambda}\left(D_{-}^{\mu,\mu',\xi,\xi',\varpi}t^{\varrho-\rho s-1}\right)(x)\,ds$$
$$= x^{\varrho+\xi-1}\frac{1}{\pi i}\int_{C}\frac{\Gamma(s)\Gamma(2-s)\{\Gamma(1\pm ir-s)\}^\lambda}{\{\Gamma(2\pm ir-s)\}^\lambda}$$
$$\times\frac{\Gamma(1-\varrho-\xi'+\rho s)\Gamma(1-\varrho+\varpi-\mu-\mu'+\rho s)\Gamma(1-\varrho+\varpi-\mu'-\xi+\rho s)}{\Gamma(1-\varrho+\rho s)\Gamma(1-\varrho+\varpi-\mu-\mu'-\nu+\rho s)\Gamma(1-\varrho-\mu'-\xi'+\rho s)}(-x)^{-\rho s}\,ds$$

which, interpreted with the help of the definition (12.74), yields the required formula (12.81).

Here, we have mentioned only the results for Marichev–Saigo–Maeda (MSM) fractional operators. Other results can be derived for Saigo's, Riemann–Liouville(RL) and Erdélyi–Kober (EK) operators, and these are left as an exercise for interested readers.

To conclude this section, we present some results for generalized alternating Mathieu-type functional series $\tilde{S}_\lambda(r;z)$ (12.28).

$$\tilde{S}_\lambda(r;z) = \sum_{n\ge 1}(-1)^{n-1}\frac{2nz^{n-1}}{(n^2+r^2)^\lambda}, \qquad (\lambda > 1, r\in\mathbb{R}, |z|<1), \tag{12.82}$$

Theorem 5.1 Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) + \min\{0, \Re(\varpi - \mu - \mu' - \xi), \Re(-\xi' - \mu')\} > 0$. Then the following assertion for the MSM fractional operator $\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda\left(r,t^\rho\right)$ (12.28) exists and holds true:

$$\left(\mathcal{I}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}\tilde{S}_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\varpi-\mu-\mu'-1} \times \overline{H}_{6,6}^{1,6}\left[x^\rho \middle| \begin{matrix} (1-\varrho,\rho;1),(1-\varrho-\varpi+\mu+\mu'+\xi,\rho;1),(1-\varrho-\xi'+\mu',\rho;1),(-1,1;1),(\pm ir,1;\lambda) \\ (0,1),(\pm ir-1,1;\lambda),(1-\varrho-\xi',\rho;1),(1-\varrho-\varpi+\mu+\mu',\rho;1),(1-\varrho-\varpi+\mu'+\xi,\rho;1) \end{matrix}\right]. \tag{12.83}$$

Theorem 5.2 Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(-\xi), \Re(\mu + \mu' - \varpi), \Re(\mu + \xi' - \varpi)\}$. Then the following assertion for the MSM fractional operator $\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda\left(r,t^\rho\right)$ (12.28) exists and holds true:

$$\left(\mathcal{I}_{-}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}\tilde{S}_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho+\varpi-\mu-\mu'-1} \times \overline{H}_{6,6}^{4,3}\left[x^\rho \middle| \begin{matrix} (-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho),(1-\varrho-\varpi+\mu+\mu'+\xi',\rho),(1-\varrho+\mu-\xi,\rho) \\ (0,1),(1-\varrho-\xi,\rho),(1-\varrho-\varpi+\mu+\mu',\rho),(1-\varrho-\varpi+\mu+\xi',\rho),(\pm ir-1,1;\lambda) \end{matrix}\right]. \tag{12.84}$$

Theorem 5.3 Let $\lambda - 1$, $r \in \mathbb{R}^+$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) > \max\{0, \Re(\varpi - \mu - \mu' + \xi'), \Re(\xi - \mu)\}$. Then the following assertion for the MSM fractional operator $\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_\lambda\left(r,t^\rho\right)$ (12.28) exists and holds true:

$$\left(\mathcal{D}_{0+}^{\mu,\mu',\xi,\xi',\varpi}\left\{t^{\varrho-1}\tilde{S}_\lambda(r,t^\rho)\right\}\right)(x) = 2x^{\varrho-\xi-1} \times \overline{H}_{6,6}^{1,6}\left[x^\rho \middle| \begin{matrix} (1-\varrho,\rho;1),(1-\varrho+\varpi-\mu-\mu'-\xi',\rho;1),(1-\varrho+\xi-\mu,\rho;1),(-1,1;1),(\pm ir,1;\lambda) \\ (0,1),(\pm ir-1,1;\lambda),(1-\varrho+\xi,\rho;1),(1-\varrho+\varpi-\mu-\mu',\rho;1),(1-\varrho+\varpi-\mu-\xi',\rho;1) \end{matrix}\right]. \tag{12.85}$$

Theorem 5.4 Let $\lambda - 1$, $r \in \mathbb{R}^{+}$, $\varrho > 0$ and let the contour $\mathcal{C}$ be taken as in (12.28). Let $\varrho, \mu, \mu', \xi, \xi', \eta, \varpi \in \mathbb{C}$ and satisfy the conditions $\Re(\varpi) > 0$ and $\Re(\varrho) < 1 + \min\{\Re(\xi'), \Re(\varpi - \mu - \mu', \Re(\eta - \mu' - \xi)\}$. Then the following assertion for the MSM fractional operator $\mathcal{D}_{-}^{\mu,\mu',\xi,\xi',\varpi}$ of the $S_{\lambda}\left(r,t^{\rho}\right)$ (12.28) exists and holds true:

$$\left(\mathcal{D}_{-}^{\mu,\xi,\eta}\left\{t^{\varrho-1}\tilde{S}_{\lambda}(r,t^{\rho})\right\}\right)(x) = 2x^{\varrho+\xi-1} \times \overline{H}_{6,6}^{4,3}\left[x^{\rho}\left|\begin{matrix}(-1,1;1),(\pm ir,1;\lambda),(1-\varrho,\rho),(1-\varrho+\varpi-\mu-\mu'-\nu,\rho),(1-\varrho-\mu'-\xi',\rho)\\(0,1),(1-\varrho-\xi',\rho),(1-\varrho+\varpi-\mu-\mu',\rho),(1-\varrho+\varpi-\mu'-\xi,\rho),(\pm ir-1,1;\lambda)\end{matrix}\right.\right]. \tag{12.86}$$

12.6 Concluding Remarks

In this chapter, we have established some new and interesting fractional calculus results for the generalized Mathieu-type functional series and alternating Mathieu-type functional series by extensive use of the Marichev–Saigo–Maeda, Saigo's, Riemann–Liouville and Erdélyi–Kober tools. The results have been expressed in terms of generalized hypergeometric functions such as Rathie's I-function, Inayat-Hussain's $\bar{H}$-function, and Fox's H-function.

It is hoped that our results will have potential uses in the areas of mathematics for engineering and mathematical physics. Applications related to engineering are under investigation and will form part of a subsequent paper on this subject.

A similar process can be applied to the result obtained when $\lambda = 2$ for alternating Mathieu-type functional series $\tilde{S}\left(r;z\right)$:

$$\tilde{S}(r;z) = \sum_{m\geq 1}(-1)^{m-1}\frac{2mz^{m-1}}{(m^2+r^2)^2}, \qquad (\lambda > 1, r \in \mathbb{R}, |z| < 1), \tag{12.87}$$

Interested readers may wish to further research these results.

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