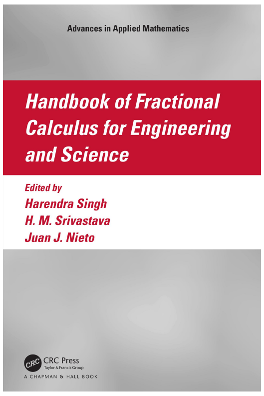

Handbook of Fractional Calculus for Engineering and Science

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Analysis of a Nonlinear System Arising in a Helium-Burning Network with Mittag–Leffler Law

Publication details
https://test.routledgehandbooks.com/doi/10.1201/9781003263517-2
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Published online on: 17 Feb 2022

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Analysis of a Nonlinear System Arising in a Helium-Burning Network with Mittag–Leffler Law

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2.1 Introduction

Differential equations are used to investigate real-world phenomena in order to study their basic properties, capture their behaviors, and predict future consequences. Most natural phenomena are nonlinear and complex in nature, and hence a reliable investigative tool is appropriate. However, scientists and mathematicians have pointed out the limitations of classical calculus to exemplify real-world problems, which can be examined with the help of generalized calculus concepts with fractional order known as fractional calculus (FC). Although this notion has attracted attention from many scholars recently, in fact it originated in Newton's time. Some of the most fascinating leaps in science and engineering began with FC. Due to its favorable characteristics, including the memory effect, non-locality, history, and

DOI: 10.1201/9781003263517-2

heredity, it has become the most influential tool in the examination and designation of complex and nonlinear mechanisms over the last few decades. The rapid evolution of computer software and mathematical algorithms has enabled many researchers to study the theory and fundamentals of FC with its associated notions and applications [1–16].

In most stars, energy is derived from the process of transformation of hydrogen into helium. The rapid temperature increase leads to hydrogen fusion. The stellar core contracts gravitationally due to the temperature which is high enough to start the process of helium burning. There are two types of helium burning, explosive and hydrostatic. Kinetic equations are used to describe the nucleo-synthesis of the elements in the stars. The following equation describes this phenomenon, with the change in the density number $\mathcal{N}_i$ of the species i over time t [17]:

$$\frac{d\mathcal{N}_i}{dt} = -\sum_j \mathcal{N}_i \mathcal{N}_j \langle \sigma v \rangle_{ij} + \sum_{k,l \neq i} \mathcal{N}_k \mathcal{N}_l \langle \sigma v \rangle_{kl},$$

where $\langle \sigma v \rangle pq$ denotes the interaction reaction involving species p and q. For a number density $\mathcal{N}_i$ and Avogadro's number $\mathcal{N}_A$ of the species i expressed in abundance $\mathcal{X}_i$ as $\mathcal{N}_i = \frac{\rho \mathcal{N}_A \mathcal{X}_i}{\mathcal{A}_i}$ where the mass density of gas ρ and $\mathcal{A}_i$ is the unit mass [18].

A triple-alpha collision occurring at high temperature (100 million degrees) in stellar helium core produces C^{12}. Later, C^{12} with He^4 produces O^{16} and continues the same processes [19]. These chemical processes are represented as follows:

$$3He^4 \to C^{12} + \gamma + 7.281Mev$$

$$C^{12} + He^4 \to O^{16} + \gamma + 7.15Mev$$

$$O^{16} + He^4 \to Ne^{20} + \gamma + 4.75Mev$$

The author in [20] used these chemical reactions to produce a mathematical model with a system of four equations which described the helium-burning phase as follows [21]:

$$\begin{aligned}
\frac{dx}{dt} &= -3\beta x^3 - \mu xz - \delta xy, \\
\frac{dy}{dt} &= \beta x^3 - \delta xy, \\
\frac{dz}{dt} &= \delta xy - \mu xz, \\
\frac{dr}{dt} &= \mu xz.
\end{aligned} \tag{2.1}$$

Here, x represents helium, y symbolizes carbon, z designates oxygen, and per unit mass of stellar material, r denotes a neon number of atoms. The rate of each reaction is defined by β, δ and μ.

Many scholars have produced definitions of differential and integral operators with fractional order, for example, Riemann, Caputo, Atangana, and Baleanu. However, each definition has its own limitations. Recently, many complex and vital models of these operators have been systematically and effectively studied to confirm their efficiency and effectiveness, illustrated by the appropriated operator. Most of the FC systems studied are explored with the Caputo operator. Many authors have modified these notions to accommodate exponential kernel and non-singularity. The Caputo–Fabrizio [22] and Atangana–Baleanu (AB) [23] are the most familiar and widely used fractional operators. More specifically, the AB operator using a generalized Mittag–Leffler function has been developed. The behavior and consequences of complex nonlinear phenomena, particularly in connection with super diffusive, random walk, discrete and statistical studies, the chaotic nature of dynamic systems, and many others, can be captured by the concept of fractional calculus [24–43].

Here, we consider the system (2.1) with a novel operator as follows:

$$\begin{aligned}
{}_{a}^{ABC}D_t^{\alpha} x(t) &= -3\beta x^3 - \delta xy - \mu xz, \\
{}_{a}^{ABC}D_t^{\alpha} y(t) &= \beta x^3 - \delta xy, \\
{}_{a}^{ABC}D_t^{\alpha} z(t) &= \delta xy - \mu xz, \\
{}_{a}^{ABC}D_t^{\alpha} r(t) &= \mu xz,
\end{aligned} \tag{2.2}$$

where α is *AB* fractional order.

Recently, numerous unconventional algorithms and schemes have been established to study differential systems having integer and arbitrary order. The homotopy analysis method (HAM) developed by Liao Shijun [44, 45] has been effectively used to investigate solutions to nonlinear models arising in engineering science without linearization, assuming any physical parameters and perturbation. However, as this system is heavy on time and computer memory, researchers have suggested combining HAM with previous transform schemes.

Here, we use q-HATM to assess the solution and analyze system behaviors of the four equations demonstrating the chemical reaction involved in the helium-burning network. The method used was made familiar by Singh et al. [46] using Laplace transform (LT) with the q-HAM concept. Due to the consistency and efficacy of this method, many researchers have been using it extensively to find solutions for numerous types of nonlinear models in various areas [47–54]. The method reduces computational work and time taken, while maintaining greater accuracy than other established techniques.

The most interesting current problem in astrophysics is nucleo-synthesis in the extremely hot, dense region at the center of a star. Helium and hydrogen burning and the CNO cycle are the essential phases in nuclear reactors. This phenomenon has been described and modelled using differential equations, and later analyzed with many analytical and numerical techniques [55–57]. The present study uses a novel technique with new fractional derivatives to apprehend the nature of the system.

2.2 Preliminaries

First, we recall the basic notions of FC and LT [23, 58–62].

Definition 1

In the Caputo sense for $f \in H^1(a,b)$, the AB fractional derivative is presented with normalization function $\mathcal{B}[\alpha]\left(\mathcal{B}[0]=\mathcal{B}[1]=1\right)$, as

$$ {}_{a}^{ABC}D_t^{\alpha}\left(f(t)\right)=\frac{\mathcal{B}[\alpha]}{1-\alpha}\int_a^t f'(\vartheta)E_{\alpha}\left[-\alpha\frac{(t-\vartheta)^{\alpha}}{1-\alpha}\right]d\vartheta. \tag{2.3}$$

Definition 2

The AB fractional derivative in Riemann–Liouville (RL) for $f \in H^1(a,b)$ is presented as

$$ {}_{a}^{ABR}D_t^{\alpha}\left(f(t)\right)=\frac{\mathcal{B}[\alpha]}{1-\alpha}\frac{d}{dt}\int_a^t f(\vartheta)E_{\alpha}\left[\alpha\frac{(t-\vartheta)^{\alpha}}{\alpha-1}\right]d\vartheta. \tag{2.4}$$

Definition 3

The fractional AB integral is presented as

$$ {}_{a}^{AB}I_t^{\alpha}\left(f(t)\right)=\frac{1-\alpha}{\mathcal{B}[\alpha]}f(t)+\frac{\alpha}{\mathcal{B}[\alpha]\Gamma(\alpha)}\int_a^t f(\vartheta)(t-\vartheta)^{\alpha-1}d\vartheta. \tag{2.5}$$

Definition 4

The LT of $f(t)$ with fractional AB derivative is

$$ L\left[{}_{0}^{ABC}D_t^{\alpha}\left(f(t)\right)\right](s)=\frac{\mathcal{B}[\alpha]}{1-\alpha}\frac{s^{\alpha}L\left[f(t)\right]-s^{\alpha-1}f(0)}{s^{\alpha}+\left(\alpha/(1-\alpha)\right)}. \tag{2.6}$$

Theorem 1

For the RL and AB derivatives, the Lipschitz conditions hold the following results [23]

$$\left\| {}^{ABR}_{a}D^{\alpha}_{t} f_1(t) - {}^{ABR}_{a}D^{\alpha}_{t} f_2(t) \right\| < K_2 \left\| f_1(x) - f_2(x) \right\|, \tag{2.7}$$

and

$$\left\| {}^{ABC}_{a}D^{\alpha}_{t} f_1(t) - {}^{ABC}_{a}D^{\alpha}_{t} f_2(t) \right\| < K_1 \left\| f_1(x) - f_2(x) \right\|. \tag{2.8}$$

Theorem 2

The fractional differential equation ${}^{ABC}_{a}D^{\alpha}_{t} f_1(t) = s(t)$ has a unique solution as follows [23]

$$f(t) = \frac{1-\alpha}{\mathcal{B}[\alpha]} s(t) + \frac{\alpha}{\mathcal{B}[\alpha]\Gamma(\alpha)} \int_0^t s(\varsigma)(t-\varsigma)^{\alpha-1} d\varsigma. \tag{2.9}$$

2.3 q-HATM Solution Procedure

Here, we use the equation to demonstrate the rule of the method with initial condition [63, 64]

$${}^{ABC}_{a}D^{\alpha}_{t} v(x,t) + \mathcal{N} v(x,t) + \mathcal{R} v(x,t) = f(x,t), \quad 0 < \alpha \le 1, \tag{2.10}$$

and

$$v(x,0) = \mathcal{G}(x). \tag{2.11}$$

By using LT on Eq. (2.10), then

$$\mathcal{L}[v(x,t)] - \frac{\mathcal{G}(x)}{s} + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)\left\{\mathcal{L}[\mathcal{R}v(x,t)] + \mathcal{L}[\mathcal{N} v(x,t)] - \mathcal{L}[f(x,t)]\right\} = 0. \tag{2.12}$$

For $\varphi(x,t;q)$, $\mathcal{N}$ is defined as

$$\mathcal{N}[\varphi(x,t;q)] = \mathcal{L}[\varphi(x,t;q)] - \frac{\mathcal{G}(x)}{s} + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right) \left\{\mathcal{L}[\mathcal{R}\varphi(x,t;q)] + L[\mathcal{N}\varphi(x,t;q)] - L[f(x,t)]\right\} \tag{2.13}$$

where $q \in \left[0, \frac{1}{n}\right]$. Now, the homotopy is given by [45, 46]

$$(1-nq)\mathcal{L}\left[\varphi(x,t;q)-v_0(x,t)\right]=\hbar q\mathcal{N}\left[\varphi(x,t;q)\right] \tag{2.14}$$

where $q \in \left[0, \frac{1}{n}\right]$. Then, the homotopy is given as

$$\varphi(x,t;0)=v_0(x,t),\varphi\left(x,t;\frac{1}{n}\right)=v(x,t) \tag{2.15}$$

Using Taylor's theorem, we have

$$\varphi(x,t;q)=v_0(x,t)+\sum_{m=1}^{\infty}v_m(x,t)q^m, \tag{2.16}$$

where

$$v_m(x,t)=\frac{1}{m!}\frac{\partial^m\varphi(x,t;q)}{\partial q^m}\bigg|_{q=0}. \tag{2.17}$$

For the appropriate value of $v_0(x,t)$, Eq. (2.13) unites at $q=\frac{1}{n}, n$ and $\hbar$. Therefore

$$v(x,t)=v_0(x,t)+\sum_{m=1}^{\infty}v_m(x,t)\left(\frac{1}{n}\right)^m. \tag{2.18}$$

Then, we obtain

$$\mathcal{L}\left[v_m(x,t)-\mathsf{k}_m v_{m-1}(x,t)\right]=\hbar\Re_m(\vec{v}_{m-1}) \tag{2.19}$$

where

$$\vec{v}_m=\left\{v_0(x,t),v_1(x,t),\ldots,v_m(x,t)\right\} \tag{2.20}$$

Eq. (2.19) simplifies, after making use of inverse LT, to

$$v_m(x,t)=\mathsf{k}_m v_{m-1}(x,t)+\hbar\mathcal{L}^{-1}\left[\Re_m(\vec{v}_{m-1})\right] \tag{2.21}$$

where

$$\Re_m(\vec{v}_{m-1})=L\left[v_{m-1}(x,t)\right]-\left(1-\frac{\mathsf{k}_m}{n}\right)\left(\frac{\mathcal{G}(x)}{s}+\frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left[f(x,t)\right]\right)$$
$$+\frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left[\mathcal{H}_{m-1}+Rv_{m-1}\right] \tag{2.22}$$

and

$$\mathsf{k}_m = \begin{cases} 0, m \le 1, \\ n, m > 1. \end{cases} \tag{2.23}$$

In Eq. (2.22), $\mathcal{H}_m$ is a homotopy polynomial which is

$$\mathcal{H}_m = \frac{1}{m!}\left[\frac{\partial^m \varphi(x,t;q)}{\partial q^m}\right]_{q=0} \text{and}\, \varphi(x,t;q) = \varphi_0 + q\varphi_1 + q^2\varphi_2 + \ldots. \tag{2.24}$$

By using Eqs. (2.21) and (2.22), we have

$$v_m(x,t) = (\mathsf{k}_m + \hbar)v_{m-1}(x,t) - \left(1 - \frac{\mathsf{k}_m}{n}\right)\mathcal{L}^{-1}\left(\frac{\mathcal{G}(x)}{s} + \frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right)L\left[f(x,t)\right]\right)$$
$$+\hbar\mathcal{L}^{-1}\left\{\frac{1}{\mathcal{B}[\alpha]}\left(1 - \alpha + \frac{\alpha}{s^\alpha}\right)L\left[Rv_{m-1} + \mathcal{H}_{m-1}\right]\right\}. \tag{2.25}$$

Using q-HATM, the series solution is

$$v(x,t) = v_0(x,t) + \sum_{m=1}^{\infty} v_m(x,t). \tag{2.26}$$

2.4 Solution for Projected System

The reliability of the projected scheme for ODEs, and PDEs with integer and fractional order, has been proved by researchers. For instance, authors in [49] presented a system of equations exemplifying India's experience of the Covid-19 pandemic, fractional Liénard's equation with convergence is analyzed in [52], a numerical simulation for fractional order-coupled Burgers' equations is presented in [46], and the nonlinear chaotic model is analyzed in [26] within the framework of the Mittag–Leffler law. The novelty of the present method is that it includes parameters that enhance accuracy of the result obtained with a limited number of series terms. It also offers a simple algorithm to solve nonlinear PDEs without converting the nonlinear to linear and PDE to ODE. The homotopy parameter can help to show the reliability and efficiency of the algorithm with curves.

The results from this scheme can be compared with the system cited in Eq. (2.2):

$$\begin{aligned} {}_{a}^{ABC}D_{t}^{\alpha}x(t) &= -3\beta x^{3} - \delta xy - \mu xz, \\ {}_{a}^{ABC}D_{t}^{\alpha}y(t) &= \beta x^{3} - \delta xy, \\ {}_{a}^{ABC}D_{t}^{\alpha}z(t) &= \delta xy - \mu xz, \\ {}_{a}^{ABC}D_{t}^{\alpha}r(t) &= \mu xz, \end{aligned} \tag{2.27}$$

with initial conditions

$$x(0) = x_{0}(t), y(0) = y_{0}(t), z(0) = z_{0}(t), r(0) = r_{0}(t) \tag{2.28}$$

Taking *LT* on Eq. (2.27) and then we have, using Eq. (2.28)

$$\begin{aligned} L[x(t)] &= \frac{1}{s}(x_{0}(t)) + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{S^{\alpha}}\right)L\{3\beta x^{3}+\delta xy+\mu xz\}, \\ L[y(t)] &= \frac{1}{s}(y_{0}(t)) - \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{S^{\alpha}}\right)L\{\beta x^{3}-\delta xy\}, \\ L[z(t)] &= \frac{1}{s}(z_{0}(t)) - \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{S^{\alpha}}\right)L\{-\mu xz+\delta xy\}, \\ L[r(t)] &= \frac{1}{s}(r_{0}(t)) - \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{S^{\alpha}}\right)L\{\mu xz\}. \end{aligned} \tag{2.29}$$

The non-linear operator *N* is defined as

$$\begin{aligned} &N^{1}[\varphi_{1},\varphi_{2},\varphi_{3},\varphi_{4}] \\ &\quad = L[\varphi_{1}(t;q)] - \frac{1}{s}(x_{0}(t)) + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{3\beta\varphi_{1}^{3}+\delta\varphi_{1}\varphi_{2}+\mu\varphi_{1}\varphi_{3}\}, \\ &N^{2}[\varphi_{1},\varphi_{2},\varphi_{3},\varphi_{4}] \\ &\quad = L[\varphi_{2}(t;q)] - \frac{1}{s}(y_{0}(t)) + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{3\beta\varphi_{1}^{3}+\delta\varphi_{1}\varphi_{2}\}, \\ &N^{3}[\varphi_{1},\varphi_{2},\varphi_{3},\varphi_{4}] \\ &\quad = L[\varphi_{3}(t;q)] - \frac{1}{s}(z_{0}(t)) + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{\delta\varphi_{1}\varphi_{2}+\mu\varphi_{1}\varphi_{3}\}, \\ &N^{4}[\varphi_{1},\varphi_{2},\varphi_{3},\varphi_{4}] \\ &\quad = L[\varphi_{4}(t;q)] - \frac{1}{s}(r_{0}(t)) + \frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^{\alpha}}\right)L\{\mu\varphi_{1}\varphi_{3}\}. \end{aligned} \tag{2.30}$$

The *m-th* order deformation equation at $\mathcal{H}(t)=1$ is given by the projected scheme by

$$
\begin{aligned}
L\left[x_m(t)-\mathrm{k}_m x_{m-1}(t)\right] &= \hbar \Re_{1,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right],\\
L\left[y_m(t)-\mathrm{k}_m y_{m-1}(t)\right] &= \hbar \Re_{2,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right],\\
L\left[z_m(t)-\mathrm{k}_m z_{m-1}(t)\right] &= \hbar \Re_{3,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right],\\
L\left[r_m(t)-\mathrm{k}_m r_{m-1}(t)\right] &= \hbar \Re_{4,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right],
\end{aligned}
\tag{2.31}
$$

where

$$
\begin{aligned}
&\Re_{1,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right] = L\left[x_{m-1}(t)\right]-\left(1-\frac{\mathrm{k}_m}{n}\right)\left\{\frac{1}{s}\left(x_0(t)\right)\right\}\\
&\quad+\frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{3\beta\sum_{i=0}^{m-1}\sum_{j=0}^{i}x_j x_{i-j}x_{m-1-i}+\delta\sum_{j=0}^{i}x_i y_{m-1-i}+\mu\sum_{j=0}^{i}x_i z_{m-1-i}\right\}\\
&\Re_{2,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right] = L\left[y_{m-1}(t)\right]-\left(1-\frac{\mathrm{k}_m}{n}\right)\left\{\frac{1}{s}\left(y_0(t)\right)\right\}\\
&\quad-\frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{\beta\sum_{i=0}^{m-1}\sum_{j=0}^{i}x_j x_{i-j}x_{m-1-i}-\delta\sum_{j=0}^{i}x_i y_{m-1-i}\right\},\\
&\Re_{3,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right] = L\left[z_{m-1}(t)\right]-\left(1-\frac{\mathrm{k}_m}{n}\right)\left\{\frac{1}{s}\left(z_0(t)\right)\right\}\\
&\quad-\frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{\delta\sum_{j=0}^{i}x_i y_{m-1-i}-\mu\sum_{j=0}^{i}x_i z_{m-1-i}\right\},\\
&\Re_{4,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right] = L\left[r_{m-1}(t)\right]-\left(1-\frac{\mathrm{k}_m}{n}\right)\left\{\frac{1}{s}\left(r_0(t)\right)\right\}\\
&\quad-\frac{1}{\mathcal{B}[\alpha]}\left(1-\alpha+\frac{\alpha}{s^\alpha}\right)L\left\{\mu\sum_{j=0}^{i}x_i z_{m-1-i}\right\}.
\end{aligned}
\tag{2.32}
$$

By employing inverse *LT* on Eq. (2.31), we get

$$
\begin{aligned}
x_m(t) &= \mathrm{k}_m x_{m-1}(t)+\hbar L^{-1}\left\{\Re_{1,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right]\right\},\\
y_m(t) &= \mathrm{k}_m y_{m-1}(t)+\hbar L^{-1}\left\{\Re_{2,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right]\right\},\\
z_m(t) &= \mathrm{k}_m z_{m-1}(t)+\hbar L^{-1}\left\{\Re_{3,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right]\right\},\\
r_m(t) &= \mathrm{k}_m r_{m-1}(t)+\hbar L^{-1}\left\{\Re_{4,m}\left[\vec{x}_{m-1},\vec{y}_{m-1},\vec{z}_{m-1},\vec{r}_{m-1}\right]\right\}.
\end{aligned}
\tag{2.33}
$$

By using $x_0(t) = 1$, $y_0(t) = z_0(t) = r_0(t) = 0$ and then solving the foregoing equations, we can obtain the terms of

$$\begin{aligned} x(t) &= x_0(t) + \sum_{m-1}^{\infty} x_m(t)\left(\frac{1}{n}\right)^m, \\ y(t) &= y_0(t) + \sum_{m-1}^{\infty} y_m(t)\left(\frac{1}{n}\right)^m, \\ z(t) &= z_0(t) + \sum_{m-1}^{\infty} z_m(t)\left(\frac{1}{n}\right)^m, \\ r(t) &= r_0(t) + \sum_{m-1}^{\infty} r_m(t)\left(\frac{1}{n}\right)^m. \end{aligned} \tag{2.34}$$

2.5 Existence of Solutions

This section examines the existence and uniqueness of the solutions. We consider Eq. (2.27) as

$$\begin{cases} \mathcal{G}_1(t,x) = {}_0^{ABC}D_t^{\alpha}[x], \\ \mathcal{G}_1(t,y) = {}_0^{ABC}D_t^{\alpha}[y], \\ \mathcal{G}_1(t,z) = {}_0^{ABC}D_t^{\alpha}[z], \\ \mathcal{G}_1(t,r) = {}_0^{ABC}D_t^{\alpha}[r]. \end{cases} \tag{2.35}$$

From Eq. (2.35) and Theorem 2 we have

$$\begin{cases} x(t) - x(0) = \dfrac{1}{\mathcal{B}(\alpha)}\left[(1-\alpha)\mathcal{G}_1(t,x) + \dfrac{\alpha}{\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_1(\zeta,x)(t-\zeta)^{\alpha-1}\,d\zeta\right], \\ y(t) - y(0) = \dfrac{1}{\mathcal{B}(\alpha)}\left[(1-\alpha)\mathcal{G}_2(t,y) + \dfrac{\alpha}{\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_2(\zeta,y)(t-\zeta)^{\alpha-1}\,d\zeta\right], \\ z(t) - z(0) = \dfrac{1}{\mathcal{B}(\alpha)}\left[(1-\alpha)\mathcal{G}_3(t,z) + \dfrac{\alpha}{\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_3(\zeta,z)(t-\zeta)^{\alpha-1}\,d\zeta\right], \\ r(t) - r(0) = \dfrac{1}{\mathcal{B}(\alpha)}\left[(1-\alpha)\mathcal{G}_4(t,r) + \dfrac{\alpha}{\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_4(\zeta,r)(t-\zeta)^{\alpha-1}\,d\zeta\right]. \end{cases} \tag{2.36}$$

Theorem 3

The kernel $\mathcal{G}_1$ admits the Lipschitz condition and contraction if $0 \le \left(2\delta^2 + \frac{1}{2}\lambda_2\delta(a+b) + \tau_2(2+\lambda_2\delta) + \xi_1\right) < 1$ holds.

Proof. To prove the theorem, consider x and x_1 two functions, then

$$\begin{aligned}\left\|\mathcal{G}_1(t,x) - \mathcal{G}_1(t,x_1)\right\| &= \left\|3\beta\left[x^3(t) - x^3(t_1)\right] + \delta y(t)\left[x(t) - x(t_1)\right] - \mu z(t)\left[x(t) - x(t_1)\right]\right\| \\ &= \left\|\left(3\beta\left(x^2(t) + x^2(t_1) + x(t)x(t_1)\right) + \delta y(t) - \mu z(t)\right)\left[x(t) - x(t_1)\right]\right\| \\ &\le \left\|3\beta\left(x^2(t) + x^2(t_1) + x(t)x(t_1)\right) + \delta y(t) - \mu z(t)\right\|\left\|x(t) - x(t_1)\right\| \\ &\le \left(3\beta\left(a^2 + b^2 + ab\right) + \delta\lambda_2 - \mu\lambda_3\right)\left\|x(t) - x(t_1)\right\| \\ &\le \eta_1\left\|x(t) - x(t_1)\right\|,\end{aligned} \tag{2.37}$$

where $\|y(t)\| \le \lambda_2$ and $\|z(t)\| \le \lambda_3$ are bounded functions and also $\|x(t)\| \le a$ and $\|x(t_1)\| \le b$. Putting $\eta_1 = 3\beta(a^2 + b^2 + ab) + \delta\lambda_2 - \mu\lambda_3$ in Eq. (2.37), one can get

$$\left\|\mathcal{G}_1(t,x) - \mathcal{G}_1(t,x_1)\right\| \le \eta_1\left\|x(t) - x(t_1)\right\|. \tag{2.38}$$

Eq. (2.38) confirms the Lipschitz condition is attained for $\mathcal{G}_1$. If $0 \le (3\beta(a^2 + b^2 + ab) + \delta\lambda_2 - \mu\lambda_3) < 1$, then it leads to contraction. Also, we can prove

$$\begin{aligned}\left\|\mathcal{G}_2(t,y) - \mathcal{G}_2(t,y_1)\right\| &\le \eta_2\left\|y(t) - y(t_1)\right\|, \\ \left\|\mathcal{G}_3(t,z) - \mathcal{G}_3(t,z_1)\right\| &\le \eta_3\left\|z(t) - z(t_1)\right\|, \\ \left\|\mathcal{G}_4(t,r) - \mathcal{G}_4(t,r_1)\right\| &\le \eta_4\left\|r(t) - r(t_1)\right\|.\end{aligned} \tag{2.39}$$

With the help of the foregoing equations, Eq. (2.36) gives

$$\begin{cases} x_n(t) = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}_1(t,x_{n-1}) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_1(\zeta,x_{n-1})(t-\zeta)^{\alpha-1}\,d\zeta, \\ y_n(t) = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}_2(t,y_{n-1}) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_2(\zeta,y_{n-1})(t-\zeta)^{\alpha-1}\,d\zeta, \\ z_n(t) = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}_3(t,z_{n-1}) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_3(\zeta,z_{n-1})(t-\zeta)^{\alpha-1}\,d\zeta, \\ r_n(t) = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\mathcal{G}_4(t,r_{n-1}) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_4(\zeta,r_{n-1})(t-\zeta)^{\alpha-1}\,d\zeta. \end{cases} \tag{2.40}$$

The initial conditions are

$$x(0) = x_0(t), y(0) = y_0(t), z(0) = z_0(t), \text{ and } r(0) = r_0(t). \tag{2.41}$$

Between the successive difference terms, we have

$$\begin{cases} \phi_{1n}(t) = x_n(t) - x_{n-1}(t) \\ \quad = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(\mathcal{G}_1(t, x_{n-1}) - \mathcal{G}_1(t, x_{n-2})\right) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_1(\zeta, x_{n-1})(t-\zeta)^{\alpha-1}\, d\zeta, \\ \phi_{2n}(t) = y_n(t) - y_{n-1}(t) \\ \quad = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(\mathcal{G}_2(t, y_{n-1}) - \mathcal{G}_2(t, y_{n-2})\right) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_2(\zeta, y_{n-1})(t-\zeta)^{\alpha-1}\, d\zeta, \\ \phi_{3n}(t) = z_n(t) - z_{n-1}(t) \\ \quad = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(\mathcal{G}_3(t, z_{n-1}) - \mathcal{G}_3(t, z_{n-2})\right) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_3(\zeta, z_{n-1})(t-\zeta)^{\alpha-1}\, d\zeta, \\ \phi_{4n}(t) = r_n(t) - r_{n-1}(t) \\ \quad = \dfrac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(\mathcal{G}_4(t, r_{n-1}) - \mathcal{G}_4(t, r_{n-2})\right) + \dfrac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\displaystyle\int_0^t \mathcal{G}_4(\zeta, r_{n-1})(t-\zeta)^{\alpha-1}\, d\zeta. \end{cases} \tag{2.42}$$

Notice that

$$\begin{cases} x_n(t) = \displaystyle\sum_{i=1}^{n} \phi_{1i}(t), \\ y_n(t) = \displaystyle\sum_{i=1}^{n} \phi_{2i}(t), \\ z_n(t) = \displaystyle\sum_{i=1}^{n} \phi_{3i}(t), \\ r_n(t) = \displaystyle\sum_{i=1}^{n} \phi_{4i}(t). \end{cases} \tag{2.43}$$

With the assistance of Eq. (2.38) and plugging the norm on Eq. (2.42), we get

$$\left\|\phi_{1n}(t)\right\| \le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_1\left\|\phi_{1(n-1)}(t)\right\| + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_1\int_0^t \left\|\phi_{1(n-1)}(\zeta)\right\| d\zeta. \tag{2.44}$$

Similarly, we have

$$\begin{aligned}\left\|\phi_{2n}(x,t)\right\| &\le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_2\left\|\phi_{2(n-1)}(t)\right\| + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_2\int_0^t\left\|\phi_{2(n-1)}(\zeta)\right\|d\zeta,\\ \left\|\phi_{3n}(x,t)\right\| &\le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_3\left\|\phi_{3(n-1)}(t)\right\| + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_3\int_0^t\left\|\phi_{3(n-1)}(\zeta)\right\|d\zeta,\\ \left\|\phi_{4n}(x,t)\right\| &\le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_4\left\|\phi_{4(n-1)}(t)\right\| + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_4\int_0^t\left\|\phi_{4(n-1)}(\zeta)\right\|d\zeta.\end{aligned} \tag{2.45}$$

Next, we demonstrate subsequent results with the aid of the above results.

Theorem 4

If we have specific t_0, then the solution for Eq. (2.27) will exist and be unique. Further,

$$\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_i + \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_i < 1.$$

for $i = 1, 2, 3, 4$.

Proof. Consider bounded functions x, y, z, and r admitting the Lipschitz condition. Then, using Eq. (2.43) and Eq. (2.45), one can get

$$\begin{aligned}\left\|\phi_{1i}(t)\right\| &\le \frac{\left\|x_n(0)\right\|}{\mathcal{B}(\alpha)}\left[\frac{\alpha}{\Gamma(\alpha)}\eta_1 + (1-\alpha)\eta_1\right]^n,\\ \left\|\phi_{2i}(t)\right\| &\le \frac{\left\|y_n(0)\right\|}{\mathcal{B}(\alpha)}\left[\frac{\alpha}{\Gamma(\alpha)}\eta_2 + (1-\alpha)\eta_2\right]^n,\\ \left\|\phi_{3i}(t)\right\| &\le \frac{\left\|z_n(0)\right\|}{\mathcal{B}(\alpha)}\left[\frac{\alpha}{\Gamma(\alpha)}\eta_3 + (1-\alpha)\eta_3\right]^n,\\ \left\|\phi_{4i}(t)\right\| &\le \frac{\left\|r_n(0)\right\|}{\mathcal{B}(\alpha)}\left[\frac{\alpha}{\Gamma(\alpha)}\eta_4 + (1-\alpha)\eta_4\right]^n.\end{aligned} \tag{2.46}$$

The existence and continuity of the results achieved are thus verified. To verify that Eq. (2.46) is a solution for Eq. (2.27), consider

$$\begin{aligned}x &= x(0) + x_n(t) - \mathcal{K}_{1n}(t),\\ y &= y(0) + y_n(t) - \mathcal{K}_{2n}(t),\\ z &= z(0) + z_n(t) - \mathcal{K}_{3n}(t),\\ r &= r(0) + r_n(t) - \mathcal{K}_{4n}(t).\end{aligned} \tag{2.47}$$

Let us consider

$$\begin{aligned}\|\mathcal{K}_{1n}(t)\| &= \left\|\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(\mathcal{G}_1(t,x)-\mathcal{G}_1(t,x_{n-1})\right)+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t (t-\zeta)^{\mu-1}\left(\mathcal{G}_1(\zeta,x)-\mathcal{G}_1(\zeta,x_{n-1})\right)d\zeta\right\| \\ &\le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\left\|\left(\mathcal{G}_1(t,x)-\mathcal{G}_1(t,x_{n-1})\right)\right\|+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t\left\|\left(\mathcal{G}_1(\zeta,x)-\mathcal{G}_1(\zeta,x_{n-1})\right)\right\|d\zeta \\ &\le \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_1\|x-x_{n-1}\|+\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_1\|x-x_{n-1}\|t.\end{aligned} \tag{2.48}$$

Similarly, at t_0 we have

$$\|\mathcal{K}_{1n}(t)\| \le \left(\frac{1-\alpha}{\mathcal{B}(\alpha)}+\frac{\alpha t_0}{\mathcal{B}(\alpha)\Gamma(\alpha)}\right)^{n+1}\eta_1^{n+1}M. \tag{2.49}$$

As n tends to ∞, then $\|\mathcal{K}_{1n}(t)\|$ approaches to 0 with respect to Eq. (2.49). Similarly, we can prove for $\|\mathcal{K}_{2n}(t)\|$, $\|\mathcal{K}_{3n}(x,t)\|$ and $\|\mathcal{K}_{4n}(x,t)\|$.

Next, we prove the uniqueness of the projected system result. Suppose, considering the set of other solutions $x^*(t)$, $y^*(t)$, $z^*(t)$ and $r^*(t)$, then

$$x(t)-x^*(t)=\frac{1}{\mathcal{B}(\alpha)}\left[(1-\alpha)\left(\mathcal{G}_1(t,x)-\mathcal{G}_1(t,x^*)\right)+\frac{\alpha}{\Gamma(\alpha)}\int_0^t\left(\mathcal{G}_1(\zeta,x)-\mathcal{G}_1(\zeta,x^*)\right)d\zeta\right]. \tag{2.50}$$

Now, Eq. (2.50) simplifies on plugging the norm

$$\begin{aligned}\|x(t)-x^*(t)\| &= \left\|\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\int_0^t\left(\mathcal{G}_1(\zeta,x)-\mathcal{G}_1(\zeta,x^*)\right)d\zeta+\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\left(\mathcal{G}_1(t,x)-\mathcal{G}_1(t,x^*)\right)\right\| \\ &\le \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_1 t\|x(t)-x^*(t)\|+\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_1\|x(t)-x^*(t)\|.\end{aligned} \tag{2.51}$$

On simplification

$$\|x(t)-x^*(t)\|\left(1-\frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_1 t-\frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_1\right)\le 0. \tag{2.52}$$

Clearly $x(t) - x^*(t) = 0$, if

$$\left(1 - \frac{\alpha}{\mathcal{B}(\alpha)\Gamma(\alpha)}\eta_1 t - \frac{(1-\alpha)}{\mathcal{B}(\alpha)}\eta_1\right) \geq 0. \tag{2.53}$$

Therefore, Eq. (2.53) proves the theorem.

Theorem 5

Suppose xn, yn, zn, rn, x, y, z, and r presented in the Banach space $\left(\mathfrak{B}[0,T], \|\cdot\|\right)$. If $0 < \lambda i < 1$, for $i = 1, 2, 3, 4$, then Eq. (2.26) converges to Eq. (2.10).

Proof: Let $\{\mathfrak{D}_n\}$ be the sequence and partial sum of Eq. (2.26). Now, we need to illustrate that $\{\mathcal{S}_n\}$ is Cauchy sequence in $\left(\mathfrak{B}[0,T], \|\cdot\|\right)$. Let

$$\begin{aligned} \| \mathfrak{D}_{n+1}(t) - \mathfrak{D}_n(t) \| &= \| x_{n+1}(t) \| \\ &\leq \lambda_1 \| x_n(t) \| \\ &\leq \lambda_1^2 \| x_{n-1}(t) \| \leq \ldots \leq \lambda_1^{n+1} \| x_0(t) \|. \end{aligned}$$

Now, for every $n, m \in N$ $(m \leq n)$, we have

$$\begin{aligned} \| \mathfrak{D}_n - \mathfrak{D}_m \| &= \| (\mathfrak{D}_n - \mathfrak{D}_{n-1}) + (\mathfrak{D}_{n-1} - \mathfrak{D}_{n-2}) + \ldots + (\mathfrak{D}_{m+1} - \mathfrak{D}_m) \| \\ &\leq \| \mathfrak{D}_n - \mathfrak{D}_{n-1} \| + \| \mathfrak{D}_{n-1} - \mathfrak{D}_{n-2} \| + \ldots + \| \mathfrak{D}_{m+1} - \mathfrak{D}_m \| \\ &\leq \left(\lambda_1^n + \lambda_1^{n-1} + \ldots + \lambda_1^{m+1}\right) \| x_0 \| \\ &\leq \lambda_1^{m+1}\left(\lambda_1^{n-m-1} + \lambda_1^{n-m-2} + \ldots + \lambda_1 + 1\right) \| x_0 \| \\ &\leq \lambda_1^{m+1}\left(\frac{1-\lambda_1^{n-m}}{1-\lambda_1}\right) \| x_0 \|. \end{aligned} \tag{2.54}$$

But $0 < \lambda_1 < 1$, therefore $\lim_{n,m\to\infty} \|\mathfrak{D}_n - \mathfrak{D}_m\| = 0$. Hence, $\{\mathfrak{D}_n\}$ is the Cauchy sequence.

2.6 Results and Discussion

This study has explored solutions for fractional-order differential systems illustrating chemical reactions in the helium-burning network, with the help of q-HATM in the framework of the Mittag–Leffler law. In this section, we establish the behavior of the results achieved. In Figure 2.1, the response of q-HATM results in distinct order. The considered method is associated with the homotopy parameter, which can help us to regulate the convergence providence of the result obtained. In association with this, we draw an $\hbar$-curve for $n = 1$ and 2 in Figure 2.2; similar behavior can be expected for the remaining

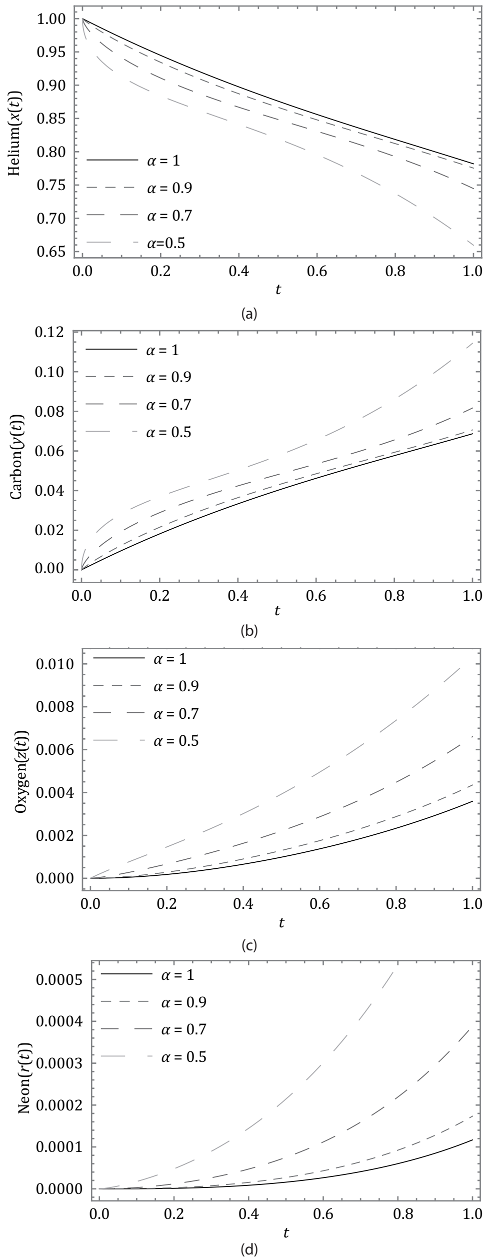

FIGURE 2.1
Nature of solution obtained for (a) helium ($x(t)$), (b) carbon ($y(t)$), (c) oxygen ($z(t)$) and (d) neon ($r(t)$) with different α at $\hbar = -1$ and $n = 1$.

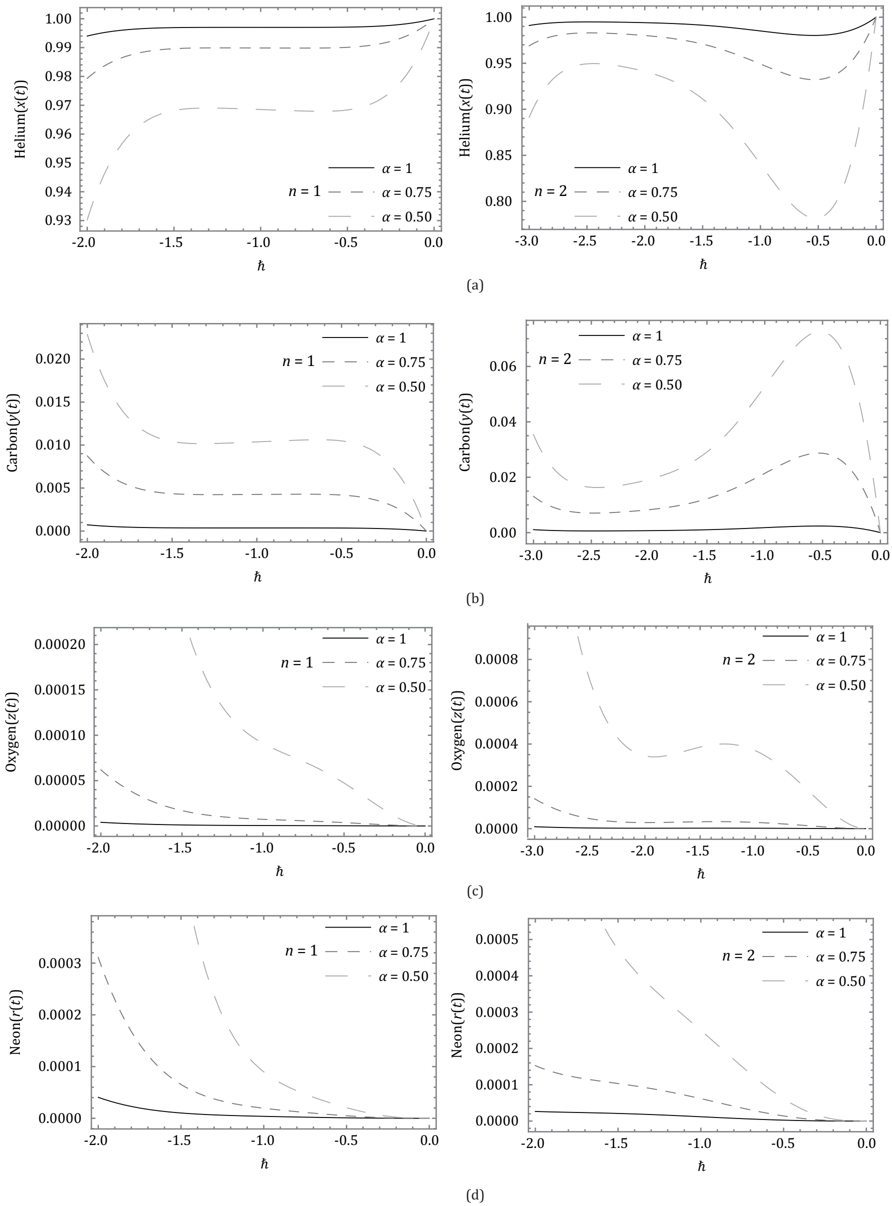

FIGURE 2.2
ħ-curves for (a) helium ($x(t)$), (b) carbon ($y(t)$), (c) oxygen ($z(t)$) and (d) neon ($r(t)$) with distinct α at $t = 0.01$ for $n = 1$ and 2.

cases. The solid and straight-line segment illustrates the convergence of the result. The $\hbar$-curve enables us to control and adjust the convergence domain of the solution obtained. For apt $\hbar$, the solution rapidly congregates to the exact result. In complex and nonlinear models, the slight discrepancy leads to close examination of the results and prediction of the response of the relevant phenomena in a systematic and better way. Further, from all the captured figures, we can observe that the projected algorithm is very effective in investigating the coupled fractional system.

2.7 Conclusion

In this study, the q-HATM is effectively borrowed to achieve a solution for a fractional-order helium-burning network system. The existence as well as the uniqueness of the solution is established using fixed-point hypothesis. The borrowed fractional operator is demarcated with generalized Mittag–Leffler function which leads to non-local and non-singular kernel. The projected solution procedure assesses the solution for differential equations where conversion, discretization, or perturbation cannot be used. We considered five different examples to confirm the exactness as well as the reliability of the method. The plots confirm the effect of fractional order in real-world problems, rendering the results obtained with the algorithm the more motivating and fascinating. These types of models, which contribute to scholarship on nonlinear phenomena, represent a fresh and innovative approach to the study of real-world difficulties.

References

[1] J. Liouville, Mémoire surquelques questions de géometrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions, *J. Ecole. Polytech.*, 13 (1832) 1–69.

[2] G. F. B. Riemann, *Versuch einer allgemeinen Auffassung der Integration und Differentiation*, *Gesammelte Mathematische Werke*, Leipzig, (1896).

[3] M. Caputo, *Elasticita e Dissipazione*, Zanichelli, Bologna, (1969).

[4] K. S. Miller, B. Ross, *An Introduction to Fractional Calculus and Fractional Differential Equations*, A Wiley, New York, (1993).

[5] I. Podlubny, *Fractional Differential Equations*, Academic Press, New York, (1999).

[6] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, (2006).

[7] D. Baleanu, Z. B. Guvenc, J. A. Tenreiro Machado, *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer Dordrecht Heidelberg, London New York, (2010).

[8] C. Baishya, S. J. Achar, P. Veeresha, D. G. Prakasha, Dynamics of a fractional epidemiological model with disease infection in both the populations, *Chaos*, 31 (043130) (2021). DOI: 10.1063/5.0028905.

[9] S. W. Yao, E. Ilhan, P. Veeresha, H. M. Baskonus, A powerful iterative approach for quintic complex Ginzburg-Landau equation within the frame of fractional operator, *Fractals*, 25 (5) (2021). DOI: 10.1142/S0218348X21400235.

[10] D. Baleanu, G. C. Wu, S. D. Zeng, Chaos analysis and asymptotic stability of generalized Caputo fractional differential equations, *Chaos Soliton. Fractal.*, 102 (2017), 99–105.

[11] P. Veeresha, E. Ilhan, D. G. Prakasha, H. M. Baskonus, W. Gao, Regarding on the fractional mathematical model of Tumour invasion and metastasis, *Comp. Model. Eng. Sci.*, 127 (3) (2021), 1013–1036.

[12] H. M. Baskonus, T. A. Sulaiman, H. Bulut, On the new wave behavior to the Klein-Gordon-Zakharov equations in plasma physics, *Indian J. Phys.*, 93 (3) (2019), 393–399.

[13] D. G. Prakasha, P. Veeresha, H. M. Baskonus, Analysis of the dynamics of hepatitis E virus using the Atangana-Baleanu fractional derivative, *Eur. Phys. J. Plus*, 134 (241) (2019). DOI: 10.1140/epjp/i2019-12590-5.

[14] L. Akinyemi, S. N. Huseen, A powerful approach to study the new modified coupled Korteweg–de Vries system, *Math. Comput. Simul.*, 177 (2020), 556–567.

[15] P. Veeresha, D. G. Prakasha, Solution for fractional Kuramoto–Sivashinsky equation using novel computational technique. *Int. J. Appl. Comput. Math.*, 7 (33) (2021). DOI: 10.1007/s40819-021-00956-0.

[16] L. Akinyemi, A fractional analysis of Noyes–Field model for the nonlinear Belousov–Zhabotinsky reaction, *Comput. Appl. Math.*, 39 (2020), 1–34.

[17] V. Kourganoff, *Introduction to the Physics of Stellar Interiors*, D. Reidel Publishing Company, Dordrecht, (1973).

[18] H. J. Haubold, A. M. Mathai, On thermonuclear reaction rates, *Astrophysics Space Sci.*, 258 (1998), 185–199.

[19] M. I. Nouh, M. A. Sharaf, A. S. Saad, Symbolic analytical solutions for the abundances differential equations of the Helium burning phase, *Astron. Nachr.*, 324 (5) (2003), 432–436.

[20] D. D. Clayton, *Principles of Stellar Evolution and Nucleosynthesis*, University of Chicago Press, Chicago, (1983).

[21] M. I. Nouh, Computational method for a fractional model of the Helium burning network, *New Astron.*, 66 (2019), 40–44.

[22] M. Caputo, M. Fabrizio, A new definition of fractional derivative without singular kernel, *Prog. Fract. Differ. Appl.*, 1 (2) (2015), 73–85.

[23] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel theory and application to heat transfer model, *Therm. Sci.*, 20 (2016), 763–769.

[24] H. Singh, A. M. Wazwaz, Computational method for reaction diffusion-model arising in a spherical catalyst, *Int. J. Appl. Comput. Math.*, 7 (65) (2021). DOI: 10.1007/s40819-021-00993-9.

[25] H. Singh, H. M. Srivastava, D. Kumar, A reliable algorithm for the approximate solution of the nonlinear Lane-Emden type equations arising in astrophysics, *Numer. Methods Partial Differ. Equ.*, 34 (5) (2018), 1524–1555.

[26] P. Veeresha, D. G. Prakasha, A.-H. Abdel-Aty, H. Singh, E. E. Mahmoud, S. Kumar, An efficient approach for fractional nonlinear chaotic model with Mittag-Leffler law, *J. King Saud Univ. Sci.*, 33 (2) (2021). DOI: 10.1016/j.jksus.2021.101347.

[27] H. Singh, H. M. Srivastava, Numerical investigation of the fractional-order Liénard and duffing equations arising in oscillating circuit theory, *Front. Phys.*, 8 (120). DOI: 10.3389/fphy.2020.00120.

[28] C. Baishya, Dynamics of a fractional stage structured predator-prey model with prey refuge, *Indian J. Ecol.*, 47 (4) (2020), 1118–1124.

[29] L. Akinyemi, P. Veeresha, M. Senol, Numerical solutions for coupled nonlinear Schrodinger-Korteweg-de Vries and Maccari's systems of equations, *Mod. Phys. Lett. B*, (2021). DOI: 10.1142/s0217984921503395.

[30] H. Singh, Analysis for fractional dynamics of Ebola virus model, *Chaos Soliton. Fractal.*, 138 (109992) (2020). DOI: 10.1016/j.chaos.2020.109992.

[31] H. Singh, J. Singh, S. D. Purohit, D. Kumar, *Advanced Numerical Methods for Differential Equations: Applications in Science and Engineering*, CRC Press Taylor and Francis, Boca Raton, 2021.

[32] H. Singh, D. Kumar, *D. Baleanu, Methods of Mathematical Modelling*: *Fractional Differential Equations*, CRC Press Taylor and Francis, Boca Raton, 2019.

[33] G. Yel, C. Cattani, H. M. Baskonus, W. Gao, On the complex simulations with dark-bright to the Hirota-Maccari system, *J. Comput. Nonlinear Dynam.*, 16 (6) (2021). DOI: 10.1115/1.4050677.

[34] W. Gao, H. M. Baskonus, L. Shi, New investigation of bats-hosts-reservoir-people coronavirus model and application to 2019-nCoV system, *Adv. Differ. Equ.*, 391 (2020). DOI: 10.1186/s13662-020-02831-6.

[35] H. Singh, Jacobi collocation method for the fractional advection-dispersion equation arising in porous media, *Numer. Methods Partial Differ. Equ.*, (2020). DOI: 10.1002/num.22674.

[36] D. G. Prakasha, N. S. Malagi, P. Veeresha, New approach for fractional Schrödinger–Boussinesq equations with Mittag-Leffler kernel, *Math. Meth. Appl. Sci.*, 43 (2020), 9654–9670.

[37] H. Singh, Numerical simulation for fractional delay differential equations, *Int. J. Dynam. Control*, 9 (2021), 463–474.

[38] A. Kumar, E. Ilhan, A. Ciancio, G. Yel, H. M. Baskonus, Extractions of some new travelling wave solutions to the conformable Date-Jimbo-Kashiwara-Miwa equation, *AIMS Math.*, 6 (5), 4238–4264.

[39] M. Senol, L. Akinyemi, A. Ata, O. S. Iyiola, Approximate and generalized solutions of conformable type Coudrey–Dodd–Gibbon–Sawada–Kotera equation, *Int. J. Mod. Phys. B*, 35(2) (2021), 2150021.

[40] P. Veeresha, D. G. Prakasha, S. Kumar, A fractional model for propagation of classical optical solitons by using non-singular derivative, *Math. Meth. Appl. Sci.*, (2020). DOI: 10.1002/mma.6335.

[41] H. Singh, Analysis of drug treatment of the fractional HIV infection model of CD4+ T-cells, *Chaos Soliton. Fractal.* 146(110868) (2021). DOI: 10.1016/j.chaos.2021.110868.

[42] L. Akinyemi, M. Senol, S. N. Huseen, Modified homotopy methods for generalized fractional perturbed Zakharov–Kuznetsov equation in dusty plasma, *Adv. Diff. Equ.*, 45 (2021), 1–27.
[43] D. G. Prakasha, et al., An efficient computational technique for time-fractional Kaup-Kupershmidt equation, *Numer. Methods Partial Differ. Equ.*, 37 (2) (2021), 1299–1316.
[44] S. J. Liao, Homotopy analysis method and its applications in mathematics, *J. Basic Sci. Eng.*, 5(2) (1997), 111–125.
[45] S. J. Liao, Homotopy analysis method: a new analytic method for nonlinear problems, *Appl. Math. Mech.*, 19 (1998), 957–962.
[46] J. Singh, D. Kumar, R. Swroop, Numerical solution of time- and space-fractional coupled Burgers' equations via homotopy algorithm, *Alexandria Eng. J.*, 55(2) (2016) 1753–1763.
[47] H. M. Srivastava, D. Kumar, J. Singh, An efficient analytical technique for fractional model of vibration equation, *Appl. Math. Model.*, 45 (2017), 192–204.
[48] P. Veeresha, D. G. Prakasha, D. Baleanu, An efficient technique for fractional coupled system arisen in magneto-thermoelasticity with rotation using Mittag-Leffler kernel, *J. Comput. Nonlinear Dynam.*, 16 (1) (2021). DOI: 10.1115/1.4048577.
[49] M. S. Kiran, et al., A mathematical analysis of ongoing outbreak COVID-19 in India through nonsingular derivative, *Numer. Meth. Partial Diffe. Equa.*, (2020). DOI: 10.1002/num.22579
[50] P. Veeresha, E. Ilhan, H. M. Baskonus, Fractional approach for analysis of the model describing wind-influenced projectile motion, *Phys. Scr.*, 96 (2021). DOI: 10.1088/1402-4896/abf868.
[51] P. Veeresha, D. G. Prakasha, Novel approach for modified forms of Camassa–Holm and Degasperis–Procesi equations using fractional operator, *Commun. Theor. Phys.*, 72 (10) (2020). DOI: 10.1088/1572-9494/aba24b.
[52] D. Kumar, R. P. Agarwal, J. Singh, A modified numerical scheme and convergence analysis for fractional model of Lienard's equation, *J. Comput. Appl. Math.*, 399 (2018), 405–413.
[53] P. Veeresha, D. G. Prakasha, A reliable analytical technique for fractional Caudrey-Dodd-Gibbon equation with Mittag-Leffler kernel, *Nonlinear Eng.*, 9 (1) (2020), 319–328.
[54] P. Veeresha, D. G. Prakasha, H. M. Baskonus, New numerical surfaces to the mathematical model of cancer chemotherapy effect in Caputo fractional derivatives, *Chaos* 29 (013119) (2019). DOI: 10.1063/1.5074099.
[55] H. L. Duorah, R.S. Kushwaha, Helium-burning reaction products and the rate of energy generation, *Astrophysical J.*, 137 (1963), 566–571.
[56] R. K. Saxena, A.M. Mathai, H.J. Haubold, On fractional kinetic equations, *Astrophys. Space Sci.*, 282 (2002), 281–287.
[57] W. R. Hix, F.K. Thielemann, Computational methods for nucleosynthesis and nuclear energy generation, *J. Comput. Appl. Math.*, 109 (1) (1999), 321–351.
[58] J. Singh, D. Kumar, Z. Hammouch, A. Atangana, A fractional epidemiological model for computer viruses pertaining to a new fractional derivative, *Appl. Math. Comput.*, 316 (2018), 504–515.

[59] P. Veeresha, D. G. Prakasha, Z. Hammouch, *An efficient approach for the model of thrombin receptor activation mechanism with Mittag-Leffler function, The International Congress of the Moroccan Society of Applied Mathematics* (pp. 44–60). Springer, Cham.

[60] A. Atangana, B. T. Alkahtani, Analysis of the Keller-Segel model with a fractional derivative without singular kernel, *Entropy*, 17 (2015), 4439–4453.

[61] A. Atangana, B.T. Alkahtani, Analysis of non- homogenous heat model with new trend of derivative with fractional order, *Chaos Soliton. Fractal.*, 89 (2016), 566–571.

[62] A. Atangana, D. Baleanu, New fractional derivatives with non-local and non-singular kernel theory and application to heat transfer model, *Therm. Sci.*, 20 (2016), 763–769.

[63] D. Kumar, R. P. Agarwal, J. Singh, A modified numerical scheme and convergence analysis for fractional model of Lienard's equation, *J. Comput. Appl. Math.* 339 (2018), 405–413.

[64] P. Veeresha, D. G. Prakasha, J. Singh, D. Kumar, D. Baleanu, Fractional Klein-Gordon-Schrödinger equations with Mittag-Leffler memory, *Chinese J. Phy.*, 68 (2020), 65–78.