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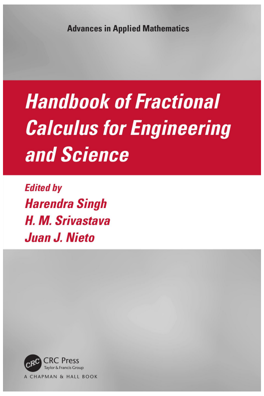

Handbook of Fractional Calculus for Engineering and Science

Harendra Singh, H. M. Srivastava, Juan J. Nieto

An Efficient Numerical Algorithm for Fractional Differential Equations

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7

An Efficient Numerical Algorithm for Fractional Differential Equations

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CONTENTS

7.1 Introduction

A fractional derivative is a generalization of integer-order derivatives that consists of non-integer order, and it can be a rational, irrational, or even complex-valued number. Fractional differential equations have received considerable attention from mathematicians, physicists, and engineers, and have been widely used in many interdisciplinary applications, such as diffusion process, visco-elasticity, electro-chemistry, biological systems, and control theory [1–4]. In [5], the analytical fractional sub-equation method

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is used to solve the historical Burgers–Huxley equation of fractional order. The Burgers–Huxley equation is widely used in many applications related to metallurgy, biology, chemistry, engineering, and mathematics. Several research papers have investigated the theory and solutions of fractional differential equations (see [6–9]).

It is well known that the integer-order differential operator is a local operator, whereas the fractional-order differential operator is non-local in the sense that the next state of the system depends not only upon its current state but also upon all of its preceding states. In the last decade, many authors have made notable contributions to both theory and application of fractional differential equations in areas as diverse as finance [10–12], physics [13–17], control theory [18], and hydrology [19]. Several published papers have shown the equivalence between transport equations using fractional-order derivatives and some heavy-tailed motions, thus extending the predictive capability of models built on the stochastic process of Brownian motion [20–21]. The motion can be heavy-tailed, implying extremely long-term correlation and fractional derivatives in time and/or space.

Finding the analytical solution of a fractional differential equation is not straightforward in general, due to the singular kernel appearing in the fractional derivative. Thus, it is necessary to develop accurate and efficient methods to solve fractional differential equations.

Until the early 1990s, no analytical method for such equations was available, even for linear fractional differential equations. In the 1990s powerful analytical methods, such as the homotopy analysis method (HAM) and variational iteration method (VIM), were proposed by Liao [22] and He [23] respectively. Several authors contributed research papers to find the numerical solution of initial value problems (IVPs) in fractional differential equations; ongoing research and relevant progress has been reported in the literature [24–27]. Other numerical methods proposed to find the numerical solution of FDEs include the spline collocation method [28], fractional Euler method, and modified trapezoidal rule using the generalized Taylor series expansion [29–30], fractional Adams method [31], Haar wavelet method [32], and operational matrix method [33]. Several other numerical algorithms to solve FDE are referenced in [34–39].

The family of RK formulae is one of the most widely used methods to find the numerical solution of IVPs in ordinary differential equations arising in various fields of applied mathematics and computational physics. An excellent book by J.C. Butcher [40] covers the development of Runge–Kutta methods and their applications. Several types of Runge–Kutta methods have been developed on the basis of stability properties and truncation error bounds. In the last two decades, several modifications to existing classical RK methods in the direction of new high-order more accurate RK methods have been envisaged. A detailed review of RK methods by Kalogiratou et al. [41] includes a short history of several modified RK methods.

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The Runge–Kutta method initially derived to find the numerical solution of IVPs in first-order differential equations is given by

$$\frac{du}{dt} = f(t, y),\ u(t_0) = u_0,\ t \in [t_0, b]. \tag{7.1}$$

where $u = u(t)$ is unknown solution function and f: $R^2 \to R$, and it is assumed that f satisfies the Lipschitz condition so that a unique solution of the IVP (7.1) exists.

In the case of differential equations of integer order, the traditional RK family of formulae fails to solve IVPs having special characteristics in the solution, such as periodicity, energy conservation, oscillation, phase conservation, etc. To solve IVPs with periodic and oscillatory solutions, a first theoretical foundation was proposed by Gautschi [42] and Lyche [43] with a technique called functional fitting. The study of the exponentially fitted RK method is a modern development and is widely useful to solve equations with specific solution behavior. Much research has focused on construction of functionally fitted numerical methods. IVPs with exponential solutions can be solved exactly using an exponential fitting approach. Berghe et al. [44–46], Paternoster [47], and Simos [48] were the first to combine exponential fitting with existing RK methods, thus developing various types of exponentially fitted RK methods. Simos's approach differs from Berghe's in the sense that Berghe uses an additional parameter in the internal stages, which certainly improves the rate of convergence of the exponentially fitted Runge–Kutta method (ef-RKM), as shown in the numerical experiments given in Berghe et al. [44–46]. The principle of the fitting is based on the annihilation of the linear integral operator associated with the internal stages and external stage, provided the existing RK method has been converted into a functional form using Albrecht's approach [49]. Ixaru et al. [50] discovered an algorithm to compute the weights of exponentially fitted multi-step algorithms for ODEs. The development of exponential fitting of various numerical integrators is covered in an excellent monograph (see [51]). Ozawa [52] made the functional fitting of RKM with variable coefficients.

The main objective of the present paper is to construct a fractional Runge–Kutta method (FRKM) along with its exponential fitting to find the numerical solution of fractional linear and nonlinear differential equations. The fitting approach is based on the assumption of annihilation of internal and external integral operators associated with FRKM. Here, we assume that the internal and external integral operators associated with FRKM annihilate the set functions $\{e^{\pm\nu t}\}$ with unknown frequency $\nu \in \mathrm{R}$ (or $\nu \in i\mathrm{R}$). The optimum values of frequency ν are obtained by minimizing the term of local truncation error. Several exponential fitting techniques can be found in an excellent book by Ixaru and Berghe [51]. This theory can be generalized to solve the problem in fractional differential equations.

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Here, we first construct an algorithm for solving fractional differential equations based on the Runge–Kutta method. We consider the following time-fractional differential equation:

$$\frac{d^{\alpha}u(t)}{dt^{\alpha}} = f(t, u(t)), \quad 0 < \alpha \le 1. \tag{7.2}$$

subject to the initial conditions:

$$u(0) = u_0, t \in [0, T] \tag{7.3}$$

The fractional derivatives are considered in the Caputo sense because many numerical solutions of the FDE derived from a fractional derivative based on the Riemann–Liouville or the Grünwald–Letnikov definitions [1, 53–54] may cause mass balance error as shown by Zhang et al. [55]. Several research papers on the development of interesting numerical algorithms to solve FDE based on Caputo derivatives can be seen in [56–58].

7.2 Fractional Calculus

In this section, we give some basic definitions and properties of fractional calculus [1, 36, 38, 53].

Definition 7.2.1

A real function $f(x)$, $x > 0$, is said to be in a space C_{μ}, $\mu \in R$ if there exists a real number $p(>\mu)$ such that $f(x) = x^p f_1(x)$ where $f_1(x) \in C[0, \infty)$, and it is said to be in the space C_{μ}^{m} if $f^{(m)} \in C_{\mu}$, $m \in N$.

Definition 7.2.2

The Riemann–Liouville fractional integral operator of order $\alpha \ge 0$, of a function $f \in C_{\mu}$, $\mu \ge -1$, is defined as

$$J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)\, dt, \ \alpha > 0, x > 0, \tag{7.4}$$

$$J^0 = f(x). \tag{7.5}$$

We need the following properties of the operator J^{α}:

$$J^{\alpha}(x-a)^{\gamma} = \frac{\Gamma(\gamma+1)}{\Gamma(1+\gamma+\alpha)}(x-a)^{\gamma+\alpha}$$

$$J^{\alpha}J^{\beta}f(x)=J^{\alpha+\beta}f(x)$$

$$J^{\alpha}J^{\beta}f(x)=J^{\beta}J^{\alpha}f(x).$$

Definition 7.2.3

For m to be the smallest integer that exceeds α, the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_{\alpha}^{t}u(x,t)=\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\begin{cases}\frac{1}{\Gamma(m-\alpha)}\int_{0}^{t}(t-\tau)^{m-\alpha-1}\frac{\partial^{m}u(x,\tau)}{\partial\tau^{m}}d\tau, & for \quad m-1<\alpha<m\\ \frac{\partial^{m}u(x,t)}{\partial t^{m}}, & for \quad \alpha=m\in N\end{cases}. \tag{7.6}$$

Definition 7.2.4

The fractional derivative of $f(x)$ in the Caputo sense is defined as

$$D^{\alpha}f(x)=J^{m-\alpha}D^{m}f(x)=\frac{1}{\Gamma(m-\alpha)}\int_{0}^{x}(x-t)^{m-\alpha-1}f^{m}(t)dt \tag{7.7}$$

for $m-1<\alpha\le m, m\in N, x>0, f\in C_{-1}^{m}$.
From (7.7), we get the following

$$D^{\alpha}(x-a)^{\gamma}=\begin{cases}\frac{\Gamma(\gamma+1)}{\Gamma(1+\gamma-\alpha)}(x-a)^{\gamma-\alpha}, & for \quad \alpha\le\gamma\\ 0, & for \quad \alpha>\gamma\end{cases} \tag{7.8}$$

Lemma 7.2.1

If $m-1<\alpha\le m$, $m\in N$ and $f\in C_{-1}^{m}$ and $\mu\ge -1$, then

$$D^{\alpha}J^{\alpha}f(x)=f(x), \tag{7.9}$$

$$J^{\alpha}D^{\alpha}f(x)=f(x)-\sum_{k=0}^{m-1}f^{(k)}(0^{+})\frac{(x-a)^{k}}{k!}, x>0. \tag{7.10}$$

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Definition 7.2.5

Mittag–Leffler function
The Mittag-Leffler function is the generalization of the exponential function. It was first defined as a single-parameter function and is defined in the form of a series as [1, 36]

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}, \alpha > 0, \alpha \in \mathbb{R}, z \in \mathbb{C}. \tag{7.11}$$

Later on it is generalized to a two-parameter function as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \alpha, \beta > 0, \alpha, \beta \in \mathbb{R}, z \in \mathbb{C}. \tag{7.12}$$

7.2.1 Fractional Taylor's Series Formula [59]

Let $f(x)$ be a real-valued function whose fractional derivatives of order up to $(n + 1)$ exist and are continuous in $[a,b]$, i. e., $D_a^{k\alpha} f(x) \in C[a,b]$, for $k = 0, 1, 2, \cdots, (n + 1)$, then

$$f(x) = \sum_{i=0}^{n} \frac{(x - x_0)^{i\alpha} \left(D_a^{i\alpha} f\right)(x_0)}{\Gamma(i\alpha + 1)} + \frac{(x - x_0)^{(n+1)\alpha} \left(D_a^{(n+1)\alpha} f\right)(\xi)}{\Gamma((n+1)\alpha + 1)}, \; x_0 \le \xi \le x, \forall x \in [a,b]. \tag{7.13}$$

where $0 < \alpha \le 1$, $D_a^{n\alpha} = D_a^{\alpha}. D_a^{\alpha}. D_a^{\alpha} \ldots D_a^{\alpha}$ (n - times).

If $\alpha = 1$ then the generalized Taylor's formula (7.13) reduces to classic Taylor's formula.

$$f(x) = \sum_{i=0}^{n} \frac{(x - x_0)^{i} \left(D_a^{i} f\right)(x_0)}{i!} + \frac{(x - x_0)^{(n+1)} \left(D_a^{(n+1)} f\right)(\xi)}{(n+1)!}, \; x_0 \le \xi \le x, \forall x \in [a,b]. \tag{7.14}$$

7.3 Derivation of the Method

In this section, we first derive the fractional Runge–Kutta method (FRKM) for the numerical solution of IVPs of fractional-order ordinary differential equations (FODEs). The derivation is similar to that of the Runge–Kutta method for integer-order differential equations. Here we use the generalized Taylor's series formula instead of the classical formula. We also construct

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the exponential fitting (ef) of the FRKM. The motivation behind the fitting is that the ef version of FRKM can be used to solve FDEs having exponential/trigonometric solutions more efficiently. Where frequencies tend to zero, the exponentially fitted FRK method becomes the standard FRK method.

7.3.1 Fractional Runge–Kutta Method (FRKM)

In this subsection, we give the formulation of the FRK method proposed previously by several researchers [36]. In order to introduce FRKM for nonlinear FDE (7.2), we take $t_n = t_0 + nh$, $n = 0, 1, \cdots N - 1$ with $h = b - a/N$.

We denote u_n as the numerical approximation of the exact solution $u(t_n)$. A general s-stage FRKM, to solve the fractional IVP (7.2)–(7.3) is given by

$$u_{n+1} = u_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \sum_{i=1}^{s} b_i k_i, \tag{7.15}$$

where

$$k_i = u_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \sum_{j=1}^{s} a_{ij} f\left(t_n + c_j h^\alpha, k_j\right),\ i = 1, 2, \cdots, s\ \left(n = 0, 1, \cdots N-1\right). \tag{7.16}$$

The difference equation (7.15) is called the external (objective) stage, whereas (7.16) is the internal stage of the FRKM. The two-stage explicit FRKM may be rewritten (in functional form) as

$$u_{n+1} = u_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \sum_{i=1}^{2} b_i f\left(t_n + c_i h^\alpha, U_i\right) \tag{7.17}$$

And

$$U_i = u_n + \frac{h^\alpha}{\Gamma(1+\alpha)} \sum_{j=1}^{i-1} a_{ij} f\left(t_n + c_j h^\alpha, U_j\right),\ i = 1, 2 \tag{7.18}$$

with $a_{1j} = 0$, $j = 1, 2$ (for explicit two-stage FRKM).

Applying the generalized Taylor's formula (7.13)–(7.14) in (7.17) and (7.18) we have the following order conditions:

$$b_1 + b_2 = \frac{1}{\Gamma(\alpha+1)}$$

$$b_2 c_2 = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}$$

$$b_2 a_{21} = \frac{\Gamma(\alpha+1)}{\Gamma(2\alpha+1)}.$$

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Solving the above order conditions, we get $c_2 = a_{21}$ and
$b_2 = \frac{\Gamma(\alpha+1)}{c_2\Gamma(2\alpha+1)}, b_1 = \frac{1}{\Gamma(\alpha+1)} - \frac{\Gamma(\alpha+1)}{c_2\Gamma(2\alpha+1)}$ with free parameter as c_2 (node).

For $\alpha = 1$, FRKM is converted into the classical two-stage Runge–Kutta method.

7.3.2 Exponentially Fitted Fractional Runge–Kutta Method (ef-FRKM)

In this subsection, we construct the exponential fitting of FRKM given in (7.15) to (7.16). Here, we assume that the ef-FRKM integrates exactly the exponential functions $e^{\pm\nu t}$ ($\nu \in$ R or $\nu \in i$R). The ef-FRKM is therefore best suited to solving fractional initial value problems (FIVPs) with exponential/periodic solutions. In view of the approach given in Albrecht [49] to form the fitting process, we first rewrite the FRKM in its equivalent functional form of equations (7.15)–(7.16) ($n = 0, 1, 2, \ldots N-1$) as:

$$u_{n+1} = u_n + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{i=1}^{s} b_i f\left(t_n + c_i h^\alpha, U_i\right) \tag{7.19}$$

$$U_i = u_n + \frac{h^\alpha}{\Gamma(\alpha+1)} \sum_{j=1}^{s} a_{ij} f\left(t_n + c_j h^\alpha, U_j\right),\ i = 1,2,\cdots,s\ \left(n = 0,1,\cdots N-1\right). \tag{7.20}$$

For the two-stage explicit ef-FRKM, we restrict $s = 2$ and $a_{1j} = 0$, $j = 1, 2$, $a_{21} = 0$.

The linear difference operator 'L' associated with the external stage (7.19) is defined as

$$L\left[u(t);h\right] \equiv u(t+h) - u(t) - \frac{h^\alpha}{\Gamma(1+\alpha)} \sum_{i=1}^{s} b_i f\left(t_n + c_i h^\alpha, U_i\right) \tag{7.21}$$

or

$$L\left[u(t);h\right] \equiv u(t+h) - u(t) - \frac{h^\alpha}{\Gamma(1+\alpha)} \sum_{i=1}^{s} b_i D^\alpha u\left(t + c_i h^\alpha\right) \tag{7.22}$$

And the linear operators 'L_i' associated with the internal stage (7.20) are defined as:

$$L_i\left[u(t);h\right] \equiv u\left(t + c_i h^\alpha\right) - u(t) - \frac{h^\alpha}{\Gamma(1+\alpha)} \sum_{j=1}^{s} b_i f\left(t_n + c_j h^\alpha, U_j\right) \tag{7.23}$$

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or,

$$L_i\left[u(t);h\right] \equiv u\left(t+c_ih^{\alpha}\right) - u(t) - \frac{h^{\alpha}}{\Gamma(1+\alpha)} \sum_{j=1}^{s} b_i D^{\alpha}\left(t_n + c_j h^{\alpha}\right). \tag{7.24}$$

The operators L and L_i satisfy

$$L\left[1;h\right] = L_i\left[1;h\right] \equiv 0 \tag{7.25}$$

The operators L and L_i also annihilate the function $y(x) = x$ which ensures the consistency condition of ef-FRKM as:

$$c_i = \sum_{j=1}^{i-1} a_{ij}$$

under the limit $\alpha \to 1$.

For the two-stage explicit ef-FRKM the condition becomes $a_{21} = -\dfrac{\left(h^{\alpha}c_2\right)^{\alpha}\alpha(-1+\alpha)\pi \cos ec(\pi\alpha)}{h^{\alpha}}$ and under the limit $\alpha \to 1$ it becomes $a_{21} = c_2$. That verifies the consistency condition for explicit two-stage RKM.

To compute the coefficients of ef-FRKM, we assume that the external and internal stage operators, given in equations (7.22) and (7.24) respectively, exactly integrate the two exponential functions $e^{\pm\nu t}$ ($\nu \in$ R or $\nu \in i$R).

So, taking $u(t) = e^{\pm\nu t}$ in equations (7.22) and (7.24), we can obtain the order conditions for exponential fitting of the two-stage explicit FRKM. Solving the order conditions, one can get the following coefficients for a two-stage explicit ef-FRKM under the limit $\alpha \to 1$

$$\begin{aligned}
b_1 &= \frac{-1+2c_2}{2c_2} - \frac{\left(1-4c_2+4c_2^2\right)\omega^2}{24c_2} + \frac{\left(-1+6c_2-10c_2^2+8c_2^4\right)\omega^4}{720c_2} + O(\omega)^5, \\
b_2 &= \frac{1}{2c_2} - \frac{\left(-1+2c_2^2\right)\omega^2}{24c_2} + \frac{h^4\left(1-5c_2^2+7c_2^4\right)\omega^4}{720c_2} + O(\omega)^5, \\
a_{21} &= c_2 - \frac{1}{2}\left(c_2^2\omega\right) + \frac{1}{6}c_2^3\omega^2 - \frac{1}{24}\left(c_2^4\omega^3\right) + \frac{1}{120}c_2^5\omega^4 + O(\omega)^5.
\end{aligned}$$

where $\omega = \nu h$ with unknown frequency ν.

From the above expressions, it is inferred that under the limit $\omega \to 0$, ef-FRKM is converted into the standard FRKM with the following values of the coefficients:

$$b_1 = \frac{-1+2c_2}{2c_2},\ b_2 = \frac{1}{2c_2},\ a_{21} = c_2.$$

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where node c_2 is the free parameter. The optimum value of node c_2 can be calculated by minimizing the truncation error (see Section 7.4).

7.4 Truncation Error

In order to find the local truncation error (LTE) for FRKM proposed in Section 7.3.1, we denote $u(t_{n+1})$ as an exact solution at the point t_{n+1}. We denote $e_{n+1} = u(t_{n+1}) - u_{n+1}$ as an error in the numerical solution u_{n+1}. We compute the LTE for a scalar IVP. In this computation, we use the generalized Taylor's series approximation for manipulation and simplification to find the expression of truncation error $e_{n+1} = u(t_{n+1}) - u_{n+1}$. The truncation error for two-stage explicit FRKM can be obtained in the following ways.

By generalized Taylor's series (7.13), we may express the exact solution $u(t_{n+1})$ as

$$u(t_{n+1}) = u(t_n + h) = u(t_n) + \frac{h^{\alpha}}{\Gamma(\alpha+1)} D_a^{\alpha} u(t_n) + \frac{h^{2\alpha}}{\Gamma(2\alpha+1)} D_a^{2\alpha} u(t_n) + \frac{h^{3\alpha}}{\Gamma(3\alpha+1)} D_a^{3\alpha} u(t_n) + \cdots \tag{7.26}$$

Following the approach given in [36], we use the values of various fractional derivatives as:

where

$$\begin{aligned} D_a^{\alpha} u(t_n) &= f(t_n, u_n), \\ D_a^{2\alpha} u(t_n) &= f_t(t_n, u_n) + f_y(t_n, u_n) D_a^{\alpha} u(t_n) = f_t(t_n, u_n) + f_y(t_n, u_n) f(t_n, u_n), \\ D_a^{3\alpha} u(t_n) &= f_{tt}(t_n, u_n) + 2 D_a^{\alpha} u(t_n) f_{ty}(t_n, u_n) + \left(D_a^{\alpha} u(t_n)\right)^2 f_{yy}(t_n, u_n) + f_t(t_n, u_n) f_y(t_n, u_n) \\ &\quad + D_a^{\alpha} u(t_n) \left(f_y(t_n, u_n)\right)^2. \end{aligned}$$

On the other hand, the numerical solution u_{n+1} by the two-stage explicit FRKM is given in Section 7.3.1, represented by the equation (7.15), so the truncation error expression is given by:

$$e_{n+1} = h^{3\alpha} \left[\left(\frac{c_2}{2\Gamma(2\alpha+1)} - \frac{1}{\Gamma(3\alpha+1)} \right) D^{3\alpha} u(t_n) - \frac{c_2}{2\Gamma(2\alpha+1)} \begin{pmatrix} f_t(t_n, u_n) f_y(t_n, u_n) \\ + D^{\alpha} u(t_n) \left(f_y(t_n, u_n)\right)^2 \end{pmatrix} \right]. \tag{7.27}$$

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From the expression of TE (7.27), the optimum value of the free parameter node c_2 can be obtained by minimizing it, and the optimum value of c_2 is given by

$c_2 = 2\Gamma(2\alpha + 1)/\Gamma(3\alpha + 1)$, and under the limit $\alpha \to 1$, $c_2 = 2/3$ (the optimum case of the classic two-stage Runge–Kutta method).

7.5 Numerical Experiments

In this section, we consider the two initial value problems in fractional-order differential equations and apply the FRKM and ef-RKM developed in Sections 7.3.1 and 7.3.2. The results obtained here are further compared in Tables 7.1 and 7.2, and the numerical solution profiles are depicted in Figures 7.1 and 7.2.

Example 1

First, we consider a fractional logistic growth model governed by the following fractional model [60–61]

$$\frac{d^\alpha N(t)}{dt^\alpha} = r^\alpha \frac{N(t)}{M}\left(1 - \frac{N(t)}{M}\right), \quad N(0) = N_0;\ 0 < \alpha \le 1. \tag{7.28}$$

where N_0 is the initial density of the population, r is the growth rate of the population, and M is maximum carrying capacity. In the asymptotic

TABLE 7.1

Numerical results of Example 1 for $\alpha = 1$ with step size $h = 0.001$.

t	N_{exact}	N_{apprx}.(by ef-FRKM)	Absolute Error $\lvert N_{exact} - N_{approx.}\rvert$
0.0	0.750000	0.750000	0.0
0.1	0.768278	0.768278	5.1618 (−10)
0.2	0.785601	0.785601	9.0720 (−10)
0.3	0.801964	0.801964	1.19296 (−9)
0.4	0.817367	0.817367	1.39157 (−9)
0.5	0.831824	0.831824	1.51917 (−9)
0.6	0.845353	0.845353	1.58989 (−9)
0.7	0.857980	0.857980	1.61587 (−9)
0.8	0.869734	0.869734	1.60737 (−9)
0.9	0.880651	0.880651	1.57296 (−9)
1.0	0.890768	0.890768	1.51967 (−9)

Notation: $a(-b)$ means $a \times 10^{-b}$

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TABLE 7.2

Comparison of absolute error for Example 2 for $\alpha = 1$ with varying step sizes h.

	Abs Error by ef-FRKM $\lvert N_{exact} - N_{approx.} \rvert$			
t	$h = 1/16$	$h = 1/32$	$h = 1/64$	$h = 1/128$
0.000	0.0	0.0	0.0	0.0
0.125	2.33 (−4)	5.79 (−4)	1.44 (−5)	3.59 (−6)
0.250	4.34 (−4)	1.08 (−4)	2.70 (−5)	6.75 (−6)
0.375	6.06 (−4)	1.52 (−4)	3.81 (−5)	9.53 (−6)
0.500	7.53 (−4)	1.90 (−4)	4.77 (−5)	1.19 (−5)
0.625	8.77 (−4)	2.23 (−4)	5.62 (−5)	1.14 (−5)
0.750	9.81 (−4)	2.51 (−4)	6.36 (−5)	1.60 (−5)
0.875	1.06 (−3)	2.75 (−4)	7.00 (−5)	1.76 (−5)
1.000	1.13 (−3)	2.96 (−4)	7.56 (−5)	1.91 (−5)

Notation: $a\,(-b)$ means $a \times 10^{-b}$

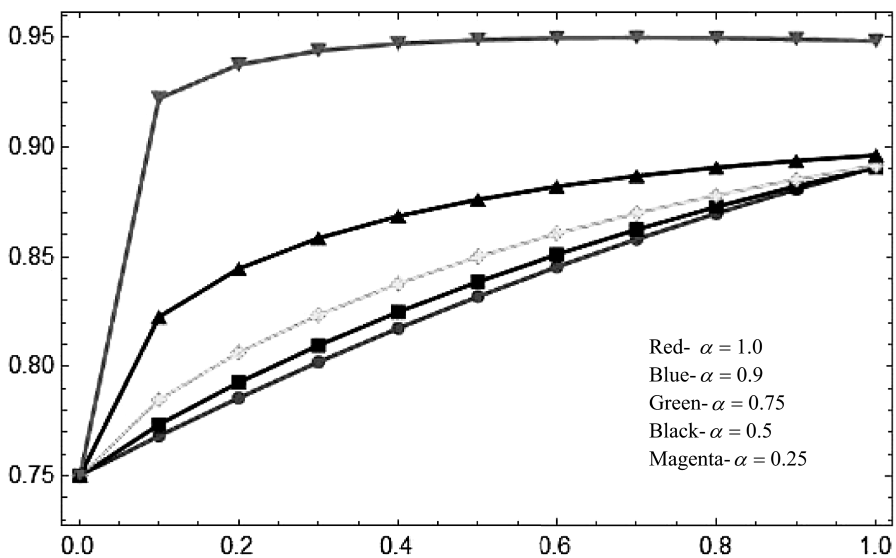

FIGURE 7.1
Comparison of numerical solutions to Example 1 for various values $\alpha = 1, 0.9, 0.75, 0.5, 0.25$ for $N(0) = 0.75$ with step size $h = 0.001$ by ef-FRKM

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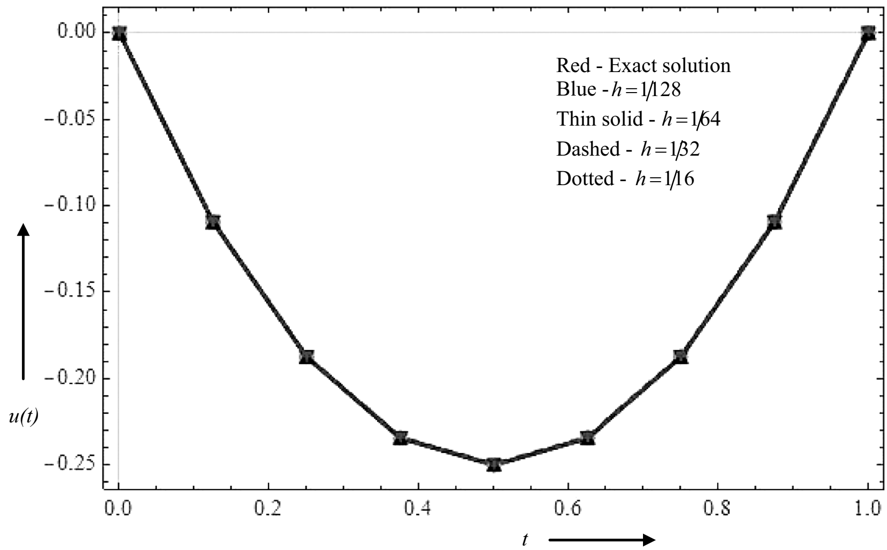

FIGURE 7.2
Comparison of solution profiles to Example 2 obtained by ef-FRKM for α = 1 with various step sizes h

limit the normalized population N_0/M approaches to unity. The closed-form exact solution of FIVP (7.28) is given by [60]

$$N(t) = \frac{MN_0}{N_0 + (M-N)e^{-rt}}. \tag{7.29}$$

We have solved the fractional logistic growth model (7.28) using both our developed methods – FRKM (given in Section 7.3.1) and ef-FRKM (developed in Section 7.3.2). Results are reported in Figures 7.1 and 7.2 and Table 7.1. We denote $e_N = |y(t) - y_N|$ as absolute endpoint error and $e_n = |y_{exact}(t_n) - y_n|$ as pointwise absolute error.

Example 2

Next, we consider a fractional IVP governed by the following fractional model

$$\frac{d^\alpha u(t)}{dt^\alpha} = -\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + t^2 - t - u(t), \;\; u(0)=0;\; 0<\alpha\le 1,\; t\in[0,1]. \tag{7.30}$$

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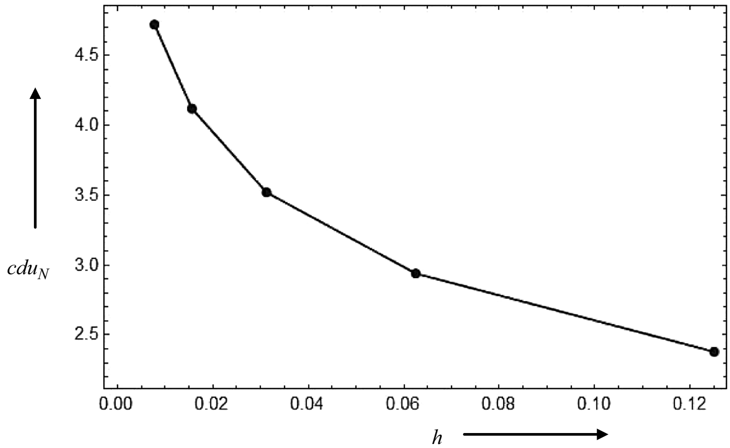

FIGURE 7.3
Comparison of correct digits of the solution (t = 1) versus step size h obtained by second-order ef-FRKM (Section 7.3.2) for Example 2 (α = 1).

The closed-form exact solution of FIVP (7.30) is $u(t) = t^2 - t$.
We denote $e_N = |u(t) - u_N|$ as absolute endpoint error and
$e_n = |u(t_n) - u_n|$ as pointwise absolute error.

Also $cdu_N = -\log_{10}(|u(t) - u_N|)$ is denoted as correct digits in endpoint solution (a crude estimate of the error). We have implemented the ef-FRKM (Section 7.3.2) to solve numerically the FDE (7.30) for the step sizes $h = 1/N$ with $N = 2^i$, i = 4, 5, 6, 7, and results are reported in Table 7.2. The numerical solution profiles for α = 1 with varying step sizes h are plotted in Figure 7.2. The comparison of the pointwise absolute error versus step size h is shown in Table 7.2. Figure 7.3 shows the relationship between step sizes h and correct digits in solution (logarithm of endpoint absolute error). From Figure 7.3, it is observed, we can achieve convergence by minimizing the step size h.

7.6 Conclusions

In this paper, we have successfully constructed a fractional two-stage Runge–Kutta method along with its exponential fitting for the numerical solution of IVPs in fractional differential equations. The proposed algorithms have tested two FIVPs that have physical importance. The ef-FRKM method exactly integrates two exponential functions: *exp* ($\pm vt$) with unknown frequency v. The computation of the local truncation error (LTE) is made and

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by minimizing this LTE the optimum value of free parameter c_2 can be calculated. Two fractional-order IVPs are solved using the methods given in Sections 7.3.1 and 7.3.2. Pointwise absolute errors are tabulated in Tables 7.1 and 7.2. From Tables 7.1–7.2 and Figures 7.1 and 7.2, it is observed that the errors incurred in two-stage explicit ef-FRKM (Section 7.3.2) are relatively small, ensuring the efficiency of the proposed method.

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