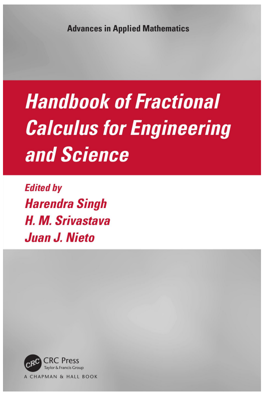

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Harendra Singh, H. M. Srivastava, Juan J. Nieto

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Kamlesh Jangid, Sapna Meena, Sanjay Bhatter, S. D. Purohit
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*Generalization of Fractional Kinetic Equations Containing Incomplete I-Functions**

Kamlesh Jangid
Rajasthan Technical University, Kota, India

Sapna Meena and Sanjay Bhatter
Malaviya National Institute of Technology, Jaipur, India

S. D. Purohit
Rajasthan Technical University, Kota, India

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8.1 Introduction and Mathematical Preliminaries

Mathematical computation is a valuable tool for gaining knowledge of physical processes. A mathematical model clearly enables researchers to create, optimize, and forecast the future success of various treatment schemes. Fractional-order mathematical models are a step forward from traditional models using fractional calculus. Fractional calculus is sometimes useful in the analysis of these models and their solutions, which are then applied to a wide range of scientific and engineering subjects. Diffusion, reaction-diffusion, fluid flow, turbulence, oscillation, electric networks, control systems, chemical physics, electrochemistry, epidemiology, and other essential

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applications may be shown. As illustrated in [1, 4, 11, 22–27, 30, 31] a variety of mathematical models relevant to real-world problems have already revealed several implications of fractional calculus. For a detailed description of fractional calculus operators, including their characteristics and implications in numerous fields, see [28, 29].

Fractional kinetic equations are widely used in astrophysics, control systems, and mathematical physics; therefore the solution of fractional kinetic equations has attracted the interest of many researchers. As a result, a large number of recent research articles (see [2, 3, 5, 7, 12, 14, 16, 19–21, 35]) focus on the solution of these equations, including generalized Mittag–Leffler function, Bessel's function, Struve function, *G*-function, *H*-function, and Aleph-function.

Haubold and Mathai [5] defined the fractional differential equation for the production and destruction of species. If the rate of change of reaction, $\mathcal{N} = \mathcal{N}(\mathrm{t})$, the rate of destruction, $\delta(\mathcal{N}_{\mathrm{t}})$, and the rate of growth, $p(\mathcal{N}_{\mathrm{t}})$, then:

$$\frac{d\mathcal{N}}{d\mathrm{t}} = -\delta(\mathcal{N}_{\mathrm{t}}) + p(\mathcal{N}_{\mathrm{t}}), \tag{8.1}$$

where, $\mathcal{N}_{\mathrm{t}}$ is given by $\mathcal{N}_{\mathrm{t}}(\mathrm{t}^*) = \mathcal{N}(\mathrm{t}-\mathrm{t}^*)$, $\mathrm{t}^* > 0$.

In addition, Haubold and Mathai [5] gave the limiting case of (8.1) when $\mathcal{N}(\mathrm{t})$ in the quantity of spatial fluctuations or homogeneities is ignored and given as

$$\frac{d\mathcal{N}_j}{d\mathrm{t}} = -c_j \mathcal{N}_j(\mathrm{t}), \tag{8.2}$$

where $\mathcal{N}_j(\mathrm{t}=0) = \mathcal{N}_0$ is the amount of density of species j at time $\mathrm{t} = 0$, $c_j > 0$. If we drop the index j and integrate the typical kinetic equation (8.2), we receive

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 = -c_0 D_{\mathrm{t}}^{-1} \mathcal{N}(\mathrm{t}), \tag{8.3}$$

where ${}_0D_{\mathrm{t}}^{-1}$ is the specialized case of Riemann–Liouville integral operator ${}_0D_{\mathrm{t}}^{-\nu}$ laid out as

$${}_0D_{\mathrm{t}}^{-\nu} f(\mathrm{t}) = \frac{1}{\Gamma(\nu)} \int_0^{\mathrm{t}} (\mathrm{t}-u)^{\nu-1} f(u)\,du, \quad (\mathrm{t} > 0, \Re(\nu) > 0). \tag{8.4}$$

Haubold and Mathai [5] gave a fractional direction to the classical kinetic equation by considering fractional derivative rather than total derivative in (8.2)

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 = -c^{\nu}\,{}_0D_{\mathrm{t}}^{-\nu} \mathcal{N}(\mathrm{t}), \tag{8.5}$$

then the solution for $\mathcal{N}(\mathrm{t})$ is a Mittag–Leffler function $E_\nu(.)$

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \sum_{r=0}^{\infty} \frac{(-1)^r (c\mathrm{t})^{\nu r}}{\Gamma(\nu r+1)} = \mathcal{N}_0 E_\nu(-c^\nu \mathrm{t}^\nu). \tag{8.6}$$

In addition, Saxena and Kalla [19] thought about subsequent fractional kinetic equations.

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 f(\mathrm{t}) = -c^\nu {}_0D_\mathrm{t}^{-\nu} \mathcal{N}(\mathrm{t}), \tag{8.7}$$

where $f(\mathrm{t}) \in L(0,\infty)$.

The Laplace transformation of the Riemann–Liouville fractional integration (8.4) is specified

$$L\left[{}_0D_\mathrm{t}^{-\nu} f(\mathrm{t}); \omega\right] = \omega^{-\nu} F(\omega), \left(\mathrm{t} > 0, \Re(\nu) > 0, \Re(\omega) > 0\right), \tag{8.8}$$

where $F(\omega)$ is the Laplace transform of the function $f(\mathrm{t})$.

The usual incomplete Gamma functions $\gamma(s,x)$ and $\Gamma(s,x)$ represented by

$$\gamma(s,x) := \int_0^x \mathrm{t}^{s-1} \ e^{-\mathrm{t}} \ d\mathrm{t} \qquad \left(\Re(s) > 0; x \geqq 0\right) \tag{8.9}$$

and

$$\Gamma(s,x) := \int_x^\infty \mathrm{t}^{s-1} \ e^{-\mathrm{t}} \ d\mathrm{t} \qquad \left(x \geqq 0; \Re(s) > 0 \quad \text{if} \quad x = 0\right), \tag{8.10}$$

satisfy the following rule of decomposition:

$$\gamma(s,x) + \Gamma(s,x) := \Gamma(s) \qquad \left(\Re(s) > 0\right). \tag{8.11}$$

Throughout this chapter, $\mathbb{N}$, Z^- and C stand for positive integer sets, negative integer sets, and complex numbers, resepctively.

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} \qquad \text{and} \qquad \mathbb{Z}_0^- := \mathbb{Z}^- \cup \{0\}.$$

In addition, the $x \geqq 0$ parameter dealt with in (8.9) and (8.10), as well as elsewhere in the present chapter, is independent of $R(z)$ of the complex number $z \in C$.

Recently, Srivastava et al. [33] introduced incomplete pochhammer symbols and incomplete hypergeometric functions in terms of incomplete Gamma functions and explored importance of these functions in the field of communication theory, probability theory and groundwater pumping modelling.

Srivastava et al. [34] recently presented a pair of Mellin–Barnes contour integral representations of incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$, and incomplete $\bar{H}$-functions $\bar{\gamma}_{p,q}^{m,n}(z)$ and $\bar{\Gamma}_{p,q}^{m,n}(z)$, in view of the $\gamma(s,x)$ and $\Gamma(s,x)$ represented by (8.9) and (8.10), respectively,

$$\begin{aligned}\gamma_{p,q}^{m,n}(z) &= \gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\mathfrak{G}_1,x),(\mathfrak{g}_i,\mathfrak{G}_i)_{2,p}\\(\mathfrak{h}_j,\mathfrak{H}_j)_{1,q}\end{matrix}\right.\right]\\ &= \gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\mathfrak{G}_1,x),(\mathfrak{g}_2,\mathfrak{G}_2),\cdots,(\mathfrak{g}_p,\mathfrak{G}_p)\\(\mathfrak{h}_1,\mathfrak{H}_1),(\mathfrak{h}_2,\mathfrak{H}_2),\cdots,(\mathfrak{h}_q,\mathfrak{H}_q)\end{matrix}\right.\right]\\ &= \frac{1}{2\pi i}\int_{\mathcal{L}}\psi(s,x)\,z^{-s}\,ds,\end{aligned} \tag{8.12}$$

where

$$\psi(s,x)=\frac{\gamma(1-\mathfrak{g}_1-\mathfrak{G}_1 s,x)\prod_{j=1}^{m}\Gamma(\mathfrak{h}_j+\mathfrak{H}_j s)\prod_{j=2}^{n}\Gamma(1-\mathfrak{g}_j-\mathfrak{G}_j s)}{\prod_{j=m+1}^{q}\Gamma(1-\mathfrak{h}_j-\mathfrak{H}_j s)\prod_{j=n+1}^{p}\Gamma(\mathfrak{g}_j+\mathfrak{G}_j s)}, \tag{8.13}$$

and

$$\begin{aligned}\Gamma_{p,q}^{m,n}(z) &= \Gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\mathfrak{G}_1,x),(\mathfrak{g}_i,\mathfrak{G}_i)_{2,p}\\(\mathfrak{h}_j,\mathfrak{H}_j)_{1,q}\end{matrix}\right.\right]\\ &= \gamma_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\mathfrak{G}_1,x),(\mathfrak{g}_2,\mathfrak{G}_2),\cdots,(\mathfrak{g}_p,\mathfrak{G}_p)\\(\mathfrak{h}_1,\mathfrak{H}_1),(\mathfrak{h}_2,\mathfrak{H}_2),\cdots,(\mathfrak{h}_q,\mathfrak{H}_q)\end{matrix}\right.\right]\\ &= \frac{1}{2\pi i}\int_{\mathcal{L}}\Psi(s,x)\,z^{-s}\,ds,\end{aligned} \tag{8.14}$$

where

$$\Psi(s,x)=\frac{\Gamma(1-\mathfrak{g}_1-\mathfrak{G}_1 s,x)\prod_{j=1}^{m}\Gamma(\mathfrak{h}_j+\mathfrak{H}_j s)\prod_{j=2}^{n}\Gamma(1-\mathfrak{g}_j-\mathfrak{G}_j s)}{\prod_{j=m+1}^{q}\Gamma(1-\mathfrak{h}_j-\mathfrak{H}_j s)\prod_{j=n+1}^{p}\Gamma(\mathfrak{g}_j+\mathfrak{G}_j s)}, \tag{8.15}$$

The incomplete H-functions $\gamma_{p,q}^{m,n}(z)$ and $\Gamma_{p,q}^{m,n}(z)$ in (8.12) and (8.14) exist for all $x \geq 0$ within similar contours and conditions as stated in Mathai and Saxena [13]. The denotations (8.12) and (8.14) readily yield the succeeding division formula:

$$\gamma_{p,q}^{m,n}(z)+\Gamma_{p,q}^{m,n}(z)=H_{p,q}^{m,n}(z), \tag{8.16}$$

for the familiar H-function.

Jangid et al. [6] introduced a family of incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$, which leads to a natural generalization of a variety of I-functions:

$$\begin{aligned}{}^{\gamma}I_{p,q}^{m,n}(z) &= {}^{\gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(\mathfrak{g}_2,\alpha_2;\mathfrak{G}_2),\cdots,(\mathfrak{g}_p,\alpha_p;\mathfrak{G}_p)\\(\mathfrak{h}_1,\beta_1;\mathfrak{H}_1),(\mathfrak{h}_2,\beta_2;\mathfrak{H}_2),\cdots,(\mathfrak{h}_q,\beta_q;\mathfrak{H}_q)\end{matrix}\right.\right]\\ &=\frac{1}{2\pi i}\int_{\pounds}\phi(s,x)\,z^s\,ds,\end{aligned} \tag{8.17}$$

and

$$\begin{aligned}{}^{\Gamma}I_{p,q}^{m,n}(z) &= {}^{\Gamma}I_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(\mathfrak{g}_2,\alpha_2;\mathfrak{G}_2),\cdots,(\mathfrak{g}_p,\alpha_p;\mathfrak{G}_p)\\(\mathfrak{h}_1,\beta_1;\mathfrak{H}_1),(\mathfrak{h}_2,\beta_2;\mathfrak{H}_2),\cdots,(\mathfrak{h}_q,\beta_q;\mathfrak{H}_q)\end{matrix}\right.\right]\\ &=\frac{1}{2\pi i}\int_{\pounds}\Phi(s,x)\,z^s\,ds,\end{aligned} \tag{8.18}$$

$\forall\ z \neq 0$, provided

$$\phi(s,x)=\frac{\{\gamma(1-\mathfrak{g}_1+\alpha_1 s,x)\}^{\mathfrak{G}_1}\prod_{j=1}^{m}\{\Gamma(\mathfrak{h}_j-\beta_j s)\}^{\mathfrak{H}_j}\prod_{j=2}^{n}\{\Gamma(1-\mathfrak{g}_j+\alpha_j s)\}^{\mathfrak{G}_j}}{\prod_{j=n+1}^{p}\{\Gamma(\mathfrak{g}_j-\alpha_j s)\}^{\mathfrak{G}_j}\prod_{j=m+1}^{q}\{\Gamma(1-\mathfrak{h}_j+\beta_j s)\}^{\mathfrak{H}_j}}, \tag{8.19}$$

and

$$\Phi(s,x)=\frac{\left\{\Gamma\left(1-\mathfrak{g}_1+\alpha_1 s,x\right)\right\}^{\mathfrak{G}_1}\prod_{j=1}^{m}\left\{\Gamma\left(\mathfrak{h}_j-\beta_j s\right)\right\}^{\mathfrak{H}_j}\prod_{j=2}^{n}\left\{\Gamma\left(1-\mathfrak{g}_j+\alpha_j s\right)\right\}^{\mathfrak{G}_j}}{\prod_{j=n+1}^{p}\left\{\Gamma\left(\mathfrak{g}_j-\alpha_j s\right)\right\}^{\mathfrak{G}_j}\prod_{j=m+1}^{q}\left\{\Gamma\left(1-\mathfrak{h}_j+\beta_j s\right)\right\}^{\mathfrak{H}_j}}, \quad (8.20)$$

The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$ in (8.17) and (8.18) exist for all $x \geq 0$ within similar contours and conditions as stated in Rathie [18].

Further, if we set $\mathfrak{H}_1 = \cdots = \mathfrak{H}_m = 1$ and $\mathfrak{G}_{n+1} = \cdots = \mathfrak{G}_p = 1$, we define the following new incomplete $\bar{I}$-functions:

$${}^{\gamma}\bar{I}_{p,q}^{m,n}(z) = {}^{\gamma}\bar{I}_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),\cdots,(\mathfrak{g}_n,\alpha_n;\mathfrak{G}_n),\\(\mathfrak{h}_1,\beta_1;1),\cdots,(\mathfrak{h}_m,\beta_m;1),\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(\mathfrak{g}_{n+1},\alpha_{n+1};1),\cdots,(\mathfrak{g}_p,\alpha_p;1)\\(\mathfrak{h}_{m+1},\beta_{m+1};\mathfrak{H}_{m+1}),\cdots,(\mathfrak{h}_q,\beta_q;\mathfrak{H}_q)\end{matrix}\right], \quad (8.21)$$

and

$${}^{\Gamma}\bar{I}_{p,q}^{m,n}(z) = {}^{\Gamma}\bar{I}_{p,q}^{m,n}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),\cdots,(\mathfrak{g}_n,\alpha_n;\mathfrak{G}_n),\\(\mathfrak{h}_1,\beta_1;1),\cdots,(\mathfrak{h}_m,\beta_m;1),\end{matrix}\right.\right.$$
$$\left.\begin{matrix}(\mathfrak{g}_{n+1},\alpha_{n+1};1),\cdots,(\mathfrak{g}_p,\alpha_p;1)\\(\mathfrak{h}_{m+1},\beta_{m+1};\mathfrak{H}_{m+1}),\cdots,(\mathfrak{h}_q,\beta_q;\mathfrak{H}_q)\end{matrix}\right]. \quad (8.22)$$

The incomplete I-functions ${}^{\gamma}I_{p,q}^{m,n}(z)$ and ${}^{\Gamma}I_{p,q}^{m,n}(z)$ defined in (8.17) and (8.18) exist for $x \geq 0$, within the set of circumstances outlined by Rathie [18], with

$$\Delta > 0, \left|\arg(z)\right| < \Delta\pi / 2,$$

where

$$\Delta = \sum_{j=1}^{m}\mathfrak{H}_j\beta_j - \sum_{j=m+1}^{q}\mathfrak{H}_j\beta_j + \sum_{j=1}^{n}\mathfrak{G}_j\alpha_j - \sum_{j=n+1}^{p}\mathfrak{G}_j\alpha_j.$$

For more details on developments in incomplete functions and their applications, see recent research [6, 7, 9, 10, 15, 17].

Srivastava [32] introduced the general class of polynomials with dimension n, ($n = 0,1,2,\cdots$) as follows:

$$S_n^m\left(x\right)=\sum_{s=0}^{[n/m]}\frac{\left(-n\right)_{ms}}{s!}A_{n,s}x^s. \tag{8.23}$$

$A_{n,\,s} \in$ R (or C) are unrestricted positive constants, while m is a positive integer. Pochhammer symbol and largest integer function are denoted by the correspondences $(-n)_m$ and "[.]", respectively. Upon adequately specializing the coefficient $A_{n,\,s}$, Srivastava's polynomials yield a set of relevant polynomials as special instances.

In this chapter, we propose an extended model of the fractional kinetic equation and possible solutions. Together with the well-known Laplace transform technique, we provide another approach for deriving solutions to kinetic equations that incorporates a class of polynomials and the incomplete I-functions and incomplete $\bar{I}$-functions.

8.2 Solution of Generalized Fractional Kinetic Equations

Theorem 1

Assume that ζ, η, $\mathfrak{g}$, $\mathfrak{h} > 0$ *and* $\mu > 0$, *so the solution of*

$$\mathcal{N}\left(\mathrm{t}\right)-\mathcal{N}_0\mathrm{t}^{\mu-1}S_n^m\left[\mathfrak{g}\mathrm{t}^{\zeta}\right]^{\Gamma}I_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right]=-c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}\left(\mathrm{t}\right), \tag{8.24}$$

is provided as

$$\mathcal{N}\left(\mathrm{t}\right)=\mathcal{N}_0\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\mathrm{t}^{\nu}\right)^i\sum_{k=0}^{[n/m]}\frac{\left(-n\right)_{mk}A_{n,k}}{k!}\left(\mathfrak{g}\mathrm{t}^{\zeta}\right)^k \times {}^{\Gamma}I_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left|\begin{matrix}\left(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x\right),\left(1-\mu-\zeta k,\eta;1\right),\left(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j\right)_{2,u}\\ \left(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j\right)_{1,v},\left(1-\mu-\zeta k-\nu i,\eta;1\right)\end{matrix}\right.\right]. \tag{8.25}$$

Proof. The Laplace transform approach is used to establish the result. Take the Laplace transform of (8.24), and using (8.18), (8.23) and (8.8), after a little simplification we obtain

$$\left[1+c^{\nu}\omega^{-\nu}\right]\mathcal{N}\left(\omega\right)=\mathcal{N}_0\sum_{k=0}^{[n/m]}\frac{\left(-n\right)_{mk}A_{n,k}\mathfrak{g}^k}{k!}\frac{1}{2\pi i}\int_{\mathcal{L}}\Phi\left(s,x\right)\mathfrak{h}^s\,\frac{\Gamma\left(\mu+\zeta k+\eta s\right)}{\omega^{\mu+\zeta k+\eta s}}\,ds, \tag{8.26}$$

where $\mathcal{N}(\omega) = \mathrm{L}\{\mathcal{N}(\mathrm{t});\omega\}$ and $\Phi(s, x)$ is defined in (8.20). Since $(1+x)^{-1} = \sum_{r=0}^{\infty}(-1)^r x^r$, therefore (8.26) implies that

$$\mathcal{N}(\omega) = \mathcal{N}_0 \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k} \mathfrak{g}^k}{k!} \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s,x) \mathfrak{h}^s \Gamma(\mu + \zeta k + \eta s)\, ds \times \sum_{i=0}^{\infty} \left(-c^\nu\right)^i \omega^{-(\mu+\zeta k+\eta s+\nu i)}. \tag{8.27}$$

Taking the inverse Laplace transform of (8.27) gives us

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k} \mathfrak{g}^k}{k!} \times \frac{1}{2\pi i} \int_{\mathcal{L}} \Phi(s,x) \mathfrak{h}^s\ \Gamma(\mu + \zeta k + \eta s)\, ds \times \sum_{i=0}^{\infty} \left(-c^\nu\right)^i \frac{\mathrm{t}^{(\mu+\zeta k+\eta s+\nu i-1)}}{\Gamma(\mu + \zeta k + \eta s + \nu i)}. \tag{8.28}$$

Finally, using (8.18), we achieve the desired outcome (8.25).

Theorem 2

Assume that ζ, η, $\mathfrak{g}$, $\mathfrak{h} > 0$ and $\mu > 0$, then the solution of

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 \mathrm{t}^{\mu-1} S_n^m\left[\mathfrak{g}\mathrm{t}^\zeta\right]^\gamma I_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^\eta\right] = -c^\nu {}_0D_t^{-\nu} \mathcal{N}(\mathrm{t}), \tag{8.29}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \mathrm{t}^{\mu-1} \sum_{i=0}^{\infty} \left(-c^\nu \mathrm{t}^\nu\right)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,\,k}}{k!} \left(\mathfrak{g}\mathrm{t}^\zeta\right)^k \times {}^\gamma I_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^\eta \left| \begin{matrix} (\mathfrak{g}_1, \alpha_1; \mathfrak{G}_1 : x), (1-\mu-\zeta k, \eta; 1), (\mathfrak{g}_j, \alpha_j; \mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j, \beta_j; \mathfrak{H}_j)_{1,v}, (1-\mu-\zeta k - \nu i, \eta; 1) \end{matrix} \right.\right]. \tag{8.30}$$

Proof. The proof is the immediate consequence of definitions (8.17), (8.23) and parallel to Theorem 1, and hence we skip the proof.

With the following relationships, incomplete I-functions are connected to incomplete $\overline{I}$-functions:

$${}^\Gamma \overline{I}_{u,\,v}^{r,\,s}(z) = {}^\Gamma I_{u,\,v}^{r,\,s}\left[z \left| \begin{matrix} (\mathfrak{g}_1, \alpha_1; \mathfrak{G}_1, x), (\mathfrak{g}_j, \alpha_j; \mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j, \beta_j; 1)_{1,r}, (\mathfrak{h}_j, \beta_j; \mathfrak{H}_j)_{r+1,v} \end{matrix} \right.\right] \tag{8.31}$$

and

$$ {}^{\gamma}\bar{I}_{u,\,v}^{r,\,s}(z) = {}^{\gamma}I_{u,\,v}^{r,\,s}\left[z \left| \begin{matrix} (\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1,x),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j,\beta_j;1)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v} \end{matrix} \right. \right]. \tag{8.32} $$

If we give specific values to the parameters, such as $\mathfrak{H}_j = 1$ $(j = 1,\cdots,r)$ in (8.24), (8.25), (8.29) and (8.30) and utilizing the connection (8.31) and (8.32), we get the following corollaries:

Corollary 1

Assume that $\zeta,\ \eta,\ \mathfrak{g},\ \mathfrak{h} > 0$ *and* $\mu > 0$, *so the solution of*

$$ \mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1}S_n^m\left[\mathfrak{g}\mathrm{t}^{\zeta}\right]^{\Gamma}\bar{I}_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.33} $$

is provided as

$$ \begin{aligned} \mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\mathrm{t}^{\nu}\right)^i \sum_{k=0}^{[n/m]}\frac{(-n)_{mk}A_{n,k}}{k!}\left(\mathfrak{g}\mathrm{t}^{\zeta}\right)^k \\ \times {}^{\Gamma}\bar{I}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta} \left| \begin{matrix} (\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(1-\mu-\zeta k,\eta;1),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j,\beta_j;1)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v},(1-\mu-\zeta k-\nu i,\eta;1) \end{matrix} \right. \right]. \end{aligned} \tag{8.34} $$

Corollary 2

Assume that $\zeta,\ \eta,\ \mathfrak{g},\ \mathfrak{h} > 0$ *and* $\mu > 0$, *then the solution of*

$$ \mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1}S_n^m\left[\mathfrak{g}\mathrm{t}^{\zeta}\right]^{\gamma}\bar{I}_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.35} $$

is provided as

$$ \begin{aligned} \mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\mathrm{t}^{\nu}\right)^i \sum_{k=0}^{[n/m]}\frac{(-n)_{mk}A_{n,k}}{k!}\left(\mathfrak{g}\mathrm{t}^{\zeta}\right)^k \\ \times {}^{\gamma}\bar{I}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta} \left| \begin{matrix} (\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(1-\mu-\zeta k,\eta;1),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j,\beta_j;1)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v},(1-\mu-\zeta k-\nu i,\eta;1) \end{matrix} \right. \right]. \end{aligned} \tag{8.36} $$

The incomplete *I*-functions are related to the incomplete $\bar{H}$-functions, as in the relation below (see [34]):

$$\begin{aligned}\bar{\Gamma}_{u,v}^{r,s}(z) &= {}^{\Gamma}I_{u,v}^{r,s}\left[z \left| \begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,s},(\mathfrak{g}_j,\alpha_j;1)_{s+1,u}\\(\mathfrak{h}_j,\beta_j;1)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v}\end{matrix}\right.\right]\\&=\bar{\Gamma}_{u,v}^{r,s}\left[z \left| \begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,s},(\mathfrak{g}_j,\alpha_j)_{s+1,u}\\(\mathfrak{h}_j,\beta_j)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v}\end{matrix}\right.\right],\end{aligned} \tag{8.37}$$

and

$$\begin{aligned}\bar{\gamma}_{u,v}^{r,s}(z) &= {}^{\gamma}I_{u,v}^{r,s}\left[z \left| \begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,s},(\mathfrak{g}_j,\alpha_j;1)_{s+1,u}\\(\mathfrak{h}_j,\beta_j;1)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v}\end{matrix}\right.\right]\\&=\bar{\gamma}_{u,v}^{r,s}\left[z \left| \begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,s},(\mathfrak{g}_j,\alpha_j)_{s+1,u}\\(\mathfrak{h}_j,\beta_j)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v}\end{matrix}\right.\right].\end{aligned} \tag{8.38}$$

If we give specific values to the parameters, such as $\mathfrak{H}_j(j = 1,\cdots,r) = 1$ and $\mathfrak{G}_j(j = s+1,\cdots,u) = 1$ in (8.24), (8.25), (8.29) and (8.30), and utilizing the relation (8.37) and (8.38), we get the preceding corollaries:

Corollary 3

Suppose that $\zeta,\ \eta,\ \mathfrak{g},\ \mathfrak{h} > 0$ *and* $\mu > 0$, *therefore the solution of*

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0\mathrm{t}^{\mu-1}S_n^m\left[\mathfrak{g}\mathrm{t}^{\zeta}\right]\bar{\Gamma}_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.39}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathrm{t}) = \mathcal{N}_0\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\mathrm{t}^{\nu}\right)^i\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}A_{n,k}}{k!}\left(\mathfrak{g}\mathrm{t}^{\zeta}\right)^k\\ \times\bar{\Gamma}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left| \begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(1-\mu-\zeta k,\eta;1),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,s},(\mathfrak{g}_j,\alpha_j)_{s+1,u}\\(\mathfrak{h}_j,\beta_j)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v},(1-\mu-\zeta k-\nu i,\eta;1)\end{matrix}\right.\right].\end{aligned} \tag{8.40}$$

Corollary 4

Assume that ζ, η, $\mathfrak{g}$, $\mathfrak{h} > 0$ *and* $\mu > 0$, *then the solution of*

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 \mathrm{t}^{\mu-1} S_n^m\left[\mathfrak{g}\mathrm{t}^{\zeta}\right] \bar{\gamma}_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}\,{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.41}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \mathrm{t}^{\mu-1} \sum_{i=0}^{\infty}\left(-c^{\nu}\mathrm{t}^{\nu}\right)^i \sum_{k=0}^{[n/m]} \frac{(-n)_{mk} A_{n,k}}{k!}\left(\mathfrak{g}\mathrm{t}^{\zeta}\right)^k \times \bar{\gamma}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta} \left| \begin{matrix} (\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(1-\mu-\zeta k,\eta;1),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,s},(\mathfrak{g}_j,\alpha_j)_{s+1,u} \\ (\mathfrak{h}_j,\beta_j)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v},(1-\mu-\zeta k-\nu i,\eta;1) \end{matrix}\right.\right]. \tag{8.42}$$

The incomplete $\bar{H}$-functions are related to the incomplete H-functions $\Gamma_{u,v}^{r,s}$ and $\gamma_{u,v}^{r,s}$ (see [34]). In consequence of (8.37) and (8.38), the incomplete I-functions are related to the incomplete H-functions as below:

$$\Gamma_{u,v}^{r,s}(z) = {}^{\Gamma}I_{u,v}^{r,s}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;1:x),(\mathfrak{h}_j,\alpha_j;1)_{2,u}\\(\mathfrak{h}_j,\beta_j;1)_{1,v}\end{matrix}\right.\right] = \Gamma_{u,v}^{r,s}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1:x),(\mathfrak{g}_j,\alpha_j)_{2,u}\\(\mathfrak{h}_j,\beta_j)_{1,v}\end{matrix}\right.\right], \tag{8.43}$$

and

$$\gamma_{u,v}^{r,s}(z) = {}^{\gamma}I_{u,v}^{r,s}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;1:x),(\mathfrak{g}_j,\alpha_j;1)_{2,u}\\(\mathfrak{h}_j,\beta_j;1)_{1,v}\end{matrix}\right.\right] = \gamma_{u,v}^{r,s}\left[z\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1:x),(\mathfrak{g}_j,\alpha_j)_{2,u}\\(\mathfrak{h}_j,\beta_j)_{1,v}\end{matrix}\right.\right]. \tag{8.44}$$

If we put $\mathfrak{H}_j(j = 1,\cdots,v) = 1$ and $\mathfrak{G}_j(j=1,\cdots,u)=1$ in (8.24), (8.25), (8.29) and (8.30), and making use of connection (8.43) and (8.44), then we get the known results due to [7]:

Corollary 5

Assume that $\zeta,\ \eta,\ \mathfrak{g},\ \mathfrak{h} > 0$ *and* $\mu > 0$*, then the solution of*

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1} S_n^m\left[\mathfrak{g}\mathrm{t}^{\zeta}\right]\Gamma_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}\,{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.45}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\,\mathrm{t}^{\nu}\right)^{i}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}\,A_{n,k}}{k!}\left(\mathfrak{g}\mathrm{t}^{\zeta}\right)^{k}\\ \times\,\Gamma_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1,x),(1-\mu-\zeta k,\eta),(\mathfrak{g}_j,\alpha_j)_{2,u}\\ (\mathfrak{h}_j,\beta_j)_{1,v},(1-\mu-\zeta k-\nu i,\eta)\end{matrix}\right.\right].\end{aligned} \tag{8.46}$$

Corollary 6

Assume that $\zeta,\ \eta,\ \mathfrak{g},\ \mathfrak{h} > 0$ *and* $\mu > 0$*, then the solution of*

$$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1} S_n^m\left[\mathfrak{g}\mathrm{t}^{\zeta}\right]\gamma_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}\,{}_0D_{t}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.47}$$

is provided as

$$\begin{aligned}\mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\,\mathrm{t}^{\nu}\right)^{i}\sum_{k=0}^{[n/m]}\frac{(-n)_{mk}\,A_{n,k}}{k!}\left(\mathfrak{g}\mathrm{t}^{\zeta}\right)^{k}\\ \times\,\gamma_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1,x),(1-\mu-\zeta k,\eta),(\mathfrak{g}_j,\alpha_j)_{2,u}\\ (\mathfrak{h}_j,\beta_j)_{1,v},(1-\mu-\zeta k-\nu i,\eta)\end{matrix}\right.\right].\end{aligned} \tag{8.48}$$

8.3 Applications

In this section, we look at some implications and applications of the preceding findings. Particular examples of derived results can be generated by appropriately specializing the coefficient $A_{n,\,s}$ to obtain a wide number of known polynomial spectrums.

Example 1

Show that the solution of:

(i) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1\Gamma}I_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}\,{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.49}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\,\mathrm{t}^{\nu}\right)^{i} \times {}^{\Gamma}I_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;\,\mathfrak{G}_1:x),(1-\mu,\eta;\,1),(\mathfrak{g}_j,\alpha_j;\,\mathfrak{G}_j)_{2,u}\\(\mathfrak{h}_j,\beta_j;\,\mathfrak{H}_j)_{1,v},(1-\mu-\nu i,\eta;1)\end{matrix}\right.\right]. \tag{8.50}$$

(ii) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1\Gamma}\bar{I}_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}\,{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.51}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\,\mathrm{t}^{\nu}\right)^{i} \times {}^{\Gamma}\bar{I}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(1-\mu,\eta;1),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,u}\\(\mathfrak{h}_j,\beta_j;1)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v},(1-\mu-\nu i,\eta;1)\end{matrix}\right.\right]. \tag{8.52}$$

(iii) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1}\bar{\Gamma}_{u,v}^{r,s}\left[\mathfrak{h}\,\mathrm{t}^{\eta}\right] = -c^{\nu}\,{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.53}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\,\mathrm{t}^{\nu}\right)^{i} \times \bar{\Gamma}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left|\begin{matrix}(\mathfrak{g}_1,\alpha_1;\mathfrak{G}_1:x),(1-\mu,\eta;1),(\mathfrak{g}_j,\alpha_j;\mathfrak{G}_j)_{2,s},\\(\mathfrak{h}_j,\beta_j)_{1,r},(\mathfrak{h}_j,\beta_j;\mathfrak{H}_j)_{r+1,v},(1-\mu-\nu i,\,\eta;1)\end{matrix}\right.\right]. \tag{8.54}$$

(iv) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0\,\mathrm{t}^{\mu-1}\Gamma_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}\,{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.55}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0\,\mathrm{t}^{\mu-1}\sum_{i=0}^{\infty}\left(-c^{\nu}\,\mathrm{t}^{\nu}\right)^{i}\Gamma_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta}\left|\begin{matrix}(\mathfrak{g}_1,\mathfrak{G}_1,x),(1-\mu,\eta),(\mathfrak{g}_j,\mathfrak{G}_j)_{2,u}\\(\mathfrak{h}_j,\mathfrak{H}_j)_{1,v},(1-\mu-\nu i,\eta)\end{matrix}\right.\right]. \tag{8.56}$$

Solution. *If we set* $m = 1$, $\mathfrak{g} = 1$, $\zeta = 0$ *and* $A_{n,s} = \frac{s!}{(-n)_{ms}}$ *for* $s = 0$ *and* $A_{n,s} = 0$ *for* $s \neq 0$ *(i.e.,* $S_n^m\left[\mathfrak{g}t^{\zeta}\right] = 1$*) in* (8.24), (8.33), (8.39) *and* (8.45). *Thus, assertions of the example follow from Theorem 1, Corollary 1, Corollary 3, and Corollary 5.*

Remark 1

It is noteworthy that $x = 0$, $\mathfrak{G}_j = 1(j = 1, \cdots, u)$ *and* $\mathfrak{H}_j = 1$ $(j = 1, \cdots, v)$, *and the kinetic equation and its solution given by* (8.49) *and* (8.50) *respectively yield the corresponding results given earlier by Choi and Kumar [2].*

Example 2

Show that the solution of:

(i) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 \mathrm{t}^{\mu+\frac{n}{2}-1} \mathrm{H}_n\left(\frac{1}{2\sqrt{\mathrm{t}}}\right) {}^{\Gamma}I_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.57}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \mathrm{t}^{\mu-1} \sum_{i=0}^{\infty} \left(-c^{\nu}\mathrm{t}^{\nu}\right)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k \mathrm{t}^k}{k!(n-2k)!} \times {}^{\Gamma}I_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta} \middle| \begin{matrix} (\mathfrak{g}_1, \alpha_1; \mathfrak{G}_1 : x), (1-\mu-k, \eta; 1), (\mathfrak{g}_j, \alpha_j; \mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j, \beta_j; \mathfrak{H}_j)_{1,v}, (1-\mu-k-\nu i, \eta; 1) \end{matrix}\right]. \tag{8.58}$$

(ii) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 \mathrm{t}^{\mu+\frac{n}{2}-1} \mathrm{H}_n\left(\frac{1}{2\sqrt{\mathrm{t}}}\right) \overline{I}_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.59}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \mathrm{t}^{\mu-1} \sum_{i=0}^{\infty} \left(-c^{\nu}\mathrm{t}^{\nu}\right)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k \mathrm{t}^k}{k!(n-2k)!} \times {}^{\Gamma}\overline{I}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta} \middle| \begin{matrix} (\mathfrak{g}_1, \alpha_1; \mathfrak{G}_1 : x), (1-\mu-k, \eta; 1), (\mathfrak{g}_j, \alpha_j; \mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j, \beta_j; 1)_{1,r}, (\mathfrak{h}_j, \beta_j; \mathfrak{H}_j)_{r+1,v}, (1-\mu-k-\nu i, \eta; 1) \end{matrix}\right]. \tag{8.60}$$

(iii) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 \mathrm{t}^{\mu+\frac{n}{2}-1} \mathrm{H}_n\left(\frac{1}{2\sqrt{t}}\right) \overline{\Gamma}_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.61}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \mathrm{t}^{\mu-1} \sum_{i=0}^{\infty} \left(-c^{\nu}\mathrm{t}^{\nu}\right)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k \mathrm{t}^k}{k!(n-2k)!} \times \bar{\Gamma}_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta} \middle| \begin{matrix} (\mathfrak{g}_1, \alpha\mathfrak{g}_1; A_1 : x), (1-\mu-k, \eta; 1), (\mathfrak{g}_j, \alpha_j; \mathfrak{G}_j)_{2,s}, (\mathfrak{g}_j, \alpha_j)_{s+1,u} \\ (\mathfrak{h}_j, \beta_j)_{1,r}, (\mathfrak{h}_j, \beta_j; \mathfrak{H}_j)_{r+1,v}, (1-\mu-k-\nu i, \eta; 1) \end{matrix}\right]. \tag{8.62}$$

(iv) $$\mathcal{N}(\mathrm{t}) - \mathcal{N}_0 \mathrm{t}^{\mu+\frac{n}{2}-1} \mathrm{H}_n\left(\frac{1}{2\sqrt{\mathrm{t}}}\right) \Gamma_{u,v}^{r,s}\left[\mathfrak{h}\mathrm{t}^{\eta}\right] = -c^{\nu}{}_0D_{\mathrm{t}}^{-\nu}\mathcal{N}(\mathrm{t}), \tag{8.63}$$

is provided as

$$\mathcal{N}(\mathrm{t}) = \mathcal{N}_0 \mathrm{t}^{\mu-1} \sum_{i=0}^{\infty} \left(-c^{\nu}\mathrm{t}^{\nu}\right)^i \sum_{k=0}^{[n/2]} \frac{(-1)^k \mathrm{t}^k}{k!(n-2k)!} \times \Gamma_{u+1,v+1}^{r,s+1}\left[\mathfrak{h}\mathrm{t}^{\eta} \middle| \begin{matrix} (\mathfrak{g}_1, \mathfrak{G}_1, x), (1-\mu-k, \eta), (\mathfrak{g}_j, \mathfrak{G}_j)_{2,u} \\ (\mathfrak{h}_j, \mathfrak{H}_j)_{1,v}, (1-\mu-k-\nu i, \eta) \end{matrix}\right]. \tag{8.64}$$

Solution. *If we set* $m = 2$, $\mathfrak{g} = 1$, $\zeta = 1$ *and* $A_{n,s} = (-1)^s$ *(i.e.,* $S_n^2[\mathrm{t}] = \mathrm{t}^{n/2}\mathrm{H}_n\left(1/2\sqrt{\mathrm{t}}\right)$*, where* $\mathrm{H}_n(\mathrm{t})$ *is a Hermite polynomial) in* (8.24), (8.33), (8.39), *and* (8.45). *Thus, assertions of the example follow from Theorem 1, Corollary 1, Corollary 3, and Corollary 5.*

Remark 2

From results (8.29), (8.35), (8.41), *and* (8.47), *we can derive a number of results.*

8.4 Conclusions

In this chapter, we investigated a new fractional generalization of the standard kinetic equation, which includes a family of polynomials and the incomplete I-functions, incomplete $\bar{I}$-functions, incomplete $\bar{H}$-functions, and incomplete H-functions, and presented their solutions using the well-known Laplace transform method. All of the derived results are of the natural type, yielding a wide range of fractional kinetic equations and solutions.

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