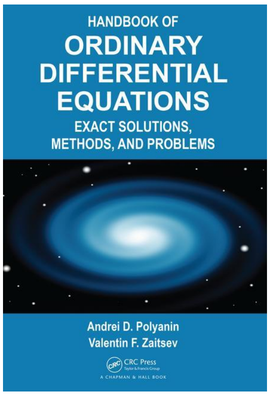

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

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Chapter 10: Methods for the Construction of Particular Solutions

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Chapter 10

Methods for the Construction of Particular Solutions

10.1 Two Problems on Searching for Particular Solutions to ODEs with Parameters

10.1.1 Preliminary Remarks. Traveling Wave Solutions

► **Preliminary remarks.**

In the theory of ordinary differential equations, it is customary to deal with methods† that allow one to find general solutions. However, methods for seeking particular solutions to nonlinear ODEs receive practically no attention. This hinders the development of related methods of the theory of partial differential equations for finding exact solutions to nonlinear PDEs that can be expressed in terms of elementary function, special functions or quadratures.

► **Traveling wave solutions for nonlinear PDEs and their relation to ODEs.**

The overwhelming majority of nonlinear equations of mathematical physics are of partial differential equations of the form

$$\Phi(w, w_z, w_t, w_{zz}, w_{zt}, w_{tt}, \dots) = 0, \tag{10.1.1.1}$$

which do not explicitly involve the independent variable; for simplicity, we consider equations with two independent variables, t and z, where t can be treated as time or a space coordinate.

In general, equation (10.1.1.1) admits solutions of the traveling wave type:

$$w = y(x), \quad x = a_1 z + a_2 t, \tag{10.1.1.2}$$

where a_1 and a_2 are arbitrary constants. Substituting (10.1.1.2) into (10.1.1.1) yields the ordinary differential equation

$$\Phi(y, a_1 y'_x, a_2 y'_x, a_1^2 y''_{xx}, a_1 a_2 y''_{xx}, a_2^2 y''_{xx}, \dots) = 0. \tag{10.1.1.3}$$

†Here and henceforth, we discuss exact methods for the integration of differential equations.

Thus, the ordinary differential equation (10.1.1.3) describes exact solutions to the special type of partial differential equations (10.1.1.1). Since the traveling wave solutions (10.1.1.2) are the most common type of exact solution to nonlinear equations of mathematical physics, it is of great importance to be able to find solutions to relevant ordinary differential equations.

Apart from the free parameters a_1 and a_2, equation (10.1.1.3) can often involve other parameters, which can also vary within certain ranges. In particular, for equations of the form (10.1.1.1), which can be represented in the divergence form (as a conservation law)

$$\begin{gathered}\frac{\partial}{\partial t}\Phi_1+\frac{\partial}{\partial z}\Phi_2=0,\\ \Phi_i=\Phi_i(w,w_z,w_t,w_{zz},w_{zt},w_{tt},\dots),\quad i=1,\,2,\end{gathered}\tag{10.1.1.4}$$

searching for traveling wave solutions (10.1.1.2) leads to the ordinary differential equation

$$\begin{gathered}a_2\Phi_1+a_1\Phi_2+a_3=0,\\ \Phi_i=\Phi_i(y,\,a_1y'_x,a_2y'_x,a_1^2y''_{xx},a_1a_2y''_{xx},a_2^2y''_{xx},\dots),\end{gathered}\tag{10.1.1.5}$$

involving three arbitrary constants: a_1, a_2, and a_3.

Importantly, methods of generalized and functional separation of variables reduce nonlinear PDEs to ODEs or systems of ODEs, which can include many free parameters that do not appear in the original equation. For relevant examples, see the literature cited at the end of the current section.

10.1.2 Two Problems for ODEs with Parameters. Conditional Capacity of Exact Solutions.

► Two problems for ODEs describing exact solutions to PDEs.

It follows from the above that there are a large number of equations in mathematical physics whose solutions can be expressed in terms of ordinary differential equations*

$$F(x,y,y'_x,\dots,y_x^{(n)};a_1,\dots,a_k)=0,\tag{10.1.2.1}$$

containing a set of free parameters a_i ($i=1,\dots,k$), which are not involved in the original partial differential equation. Below are two fundamentally different problems arising in dealing with equation (10.1.2.1).

PROBLEM 1. Find the values of the parameters a_i at which the general solution of equation (10.1.2.1) is possible (here and henceforth, we mean solutions that can be expressed in terms of elementary or special functions).

PROBLEM 2. Find the values of the parameters a_i at which the partial (exact) solutions of equation (10.1.2.1) are possible.

► Conditional capacity of exact solutions to nonlinear PDEs.

For a comparative analysis of the results of solving problems 1 and 2, the following definitions will be useful.

*The form of these solutions can differ from (10.1.1.2).

Definition. The conditional capacity of an exact solution of a nonlinear PDE is equal to the number of arbitrary constants involved in the solution but not the original equation. The conditional capacity of a solution will be denoted “cc.”

The practical sense of this definition is clear: the more arbitrary constants are involved in a solution, the more important, interesting, and valuable the solution is (the generality of a solution is determined by the number of arbitrary constants involved).

In problem 1, the general solution to the corresponding ordinary differential equation (10.1.2.1) can be obtained in closed form in only relatively few specific values of the parameters a_i or under certain limitations; in the latter case, there will be fewer free parameters, $a_1, \ldots, a_p$, and the other parameters, $a_{p+1}, \ldots, a_k$, will be dependent on them. The conditional capacities of such solutions is calculated as

$$\mathsf{cc}_1 = p + n, \qquad (10.1.2.2)$$

where n is the order of equation (10.1.2.1).

In problem 2, one often manages to obtain an exact solution to the ordinary differential equation (10.1.2.1) under fewer constraints on the parameters a_i, suggesting that more free parameters, $a_1, \ldots, a_q$, will remain than in problem 1 ($q \geq p$). In addition, the exact solution itself can depend on m constants of integration, with $m \leq n$. The conditional capacity of such solutions is evaluated as

$$\mathsf{cc}_2 = q + m. \qquad (10.1.2.3)$$

By comparing formulas (10.1.2.2) and (10.1.2.3), one can see that the conditional capacity of particular solutions to problem 2 can be lower than, equal to, or higher than that of general solutions to problem 1. This suggests that solutions to problems 1 and 2 are, in general, equally important with respect to the analysis of the original nonlinear PDEs.

Problem 1 is classical; it is solved using well-developed methods of integration of ordinary differential equations.

Problem 2 is nonclassical; solution methods for this problem have not yet been sufficiently well developed, which is primarily because problem 2 has not received much attention from the specialists in the area of ordinary differential equations. In the literature, there are relatively few methods for solving such problems, which, in addition, often have a very narrow area of application (these methods are most frequently used to treat autonomous equations with power-law nonlinearity).

► Two problems for ODEs with parameters.

The statements of problems 1 and 2 above can be arrived at from completely different considerations, without taking into account any relations between ordinary differential equations. For example, one can treat the ordinary differential equation (10.1.2.1) as dependent on physical-chemical constants a_i, which play an important role in applications and can vary within wide ranges. In this case, the role of the constants of integration, appearing in the general or particular solution to the equation, and the role of the physical-chemical constants a_i can be treated as equal; often, finding a particular solution to a wide class of equations can be much more useful than finding the general solution to a narrow class of equations.

Subsequent sections outline methods for constructing particular solutions to nonlinear ordinary differential equation with variable parameters without going into the physical or other meaning of these parameters.

⊙ *Literature for Section 10.1*: A. D. Polyanin and V. F. Zaitsev (2003, 2012), V. A. Galaktionov and S. R. Svirshchevskii (2006), A. D. Polyanin (2016).

10.2 Method of Undetermined Coefficients and Its Special Cases

10.2.1 General Description of the Method of Undetermined Coefficients

In general, the method of undetermined coefficients as applied to linear or nonlinear ordinary differential equations suggests particular solutions should be sought in a preset form dependent on a set of free (undetermined) parameters. On substituting the solution structure into the equation, one selects the values of the parameters so as to satisfy the equation exactly. Particular solutions are usually sought in the form of a finite sum

$$y = \sum_{k=0}^{n} a_k \varphi_k(x) \qquad (10.2.1.1)$$

where $\varphi_k(x)$ are given elementary functions and a_k are free (undetermined) parameters.

Most frequently, solutions are constructed using special cases of formulas (10.2.1.1):

$$y = \sum_{k=0}^{n} a_k \varphi^k(\lambda x) \quad \text{or} \quad y = \sum_{k=0}^{n} a_k \varphi^{m_k}(\lambda x). \qquad (10.2.1.2)$$

These are based on a single generating function $\varphi(z)$, which is present by the researcher. The constants n, a_k, m_k, and λ are to be determined; the second formula in (10.2.1.2) can include negative powers m_k. As $\varphi(z)$ in (10.2.1.2), one usually takes power-law, exponential, hyperbolic, or trigonometric functions (see Sections 10.2.2 and 10.2.3).

The determination of the constants n, a_k, m_k, and λ in (10.2.1.2) can often be simplified with modern computer algebra systems such as Maple or Mathematica, which allow one to perform a lot of cumbersome analytical calculations.

Remark 10.1. Seeking solutions using the first formula in (10.2.1.2) is equivalent to carrying out two consecutive actions: (i) performing the change of variable $\xi = \varphi(\lambda x)$ in the original ODE and (ii) searching for a solution to the transformed equation in the truncated series form $y = \sum_{k=0}^{n} a_k \xi^k$. This approach is technically simpler than the direct substitution of the first formula (10.2.1.2) into the original equation.

One may succeed in searching for particular solutions in a more general form than (10.2.1.1):

$$y = \Phi(x; a_0, \dots, a_n), \qquad (10.2.1.3)$$

where $\Phi(x; a_0, \dots, a_n)$ is a given function and $a_0, \dots, a_n$ are free parameters.

A considerable limitation of such direct methods is that solutions are sought in explicit form, while the overwhelming majority of known general solutions to nonlinear equations are in implicit or parametric form (this follows from a statistical analysis of the results presented in the present handbook).

Most frequently, the method of undetermined coefficients is used to seek particular solutions to linear nonhomogeneous ODEs with constant coefficients. Table 4.1 lists recommended solution structures for such equations for special forms of the right-hand side (in particular, if the right-hand side of the equation is a polynomial, solutions are sought in the polynomial form).

Remark 10.2. The special cases of the method of undetermined coefficients discussed below in Sections 10.2.2 and 10.2.3 have become very common in searching for exact traveling-wave solutions to nonlinear partial differential equations (such solutions are described by ODEs following from the original PDEs).

10.2.2 Power-Law, Tanh-Coth, and Sine-Cosine Methods

► Methods based on power-law functions.

1°. *Power-law function method.* The main idea of the method is the assumption that a particular solution of the ODE can be expressed in terms of power-law functions, which corresponds to $\varphi_k(x) = x^{p_k}$ in (10.2.1.1), with the exponents p_k to be determined. In the special case $p_k = k$, such a solution will be a polynomial of degree n.

Example 10.1. Consider the generalized Emden–Fowler equation

$$y''_{xx} = Ax^n y^m (y'_x)^l. \tag{10.2.2.1}$$

Its particular solution will be sought in the form of a power-law function

$$y = ax^p. \tag{10.2.2.2}$$

Substituting (10.2.2.2) into (10.2.2.1) yields

$$ap(p-1)x^{p-2} = Aa^m(ap)^l x^{n+mp+l(p-1)}.$$

For this equation to be satisfied identically, one must set

$$p - 2 = n + mp + l(p-1), \qquad ap(p-1) = Aa^m(ap)^l.$$

On solving this system for a and p, we arrive at the constants determining solution (10.2.2.2):

$$p = \frac{n-l+2}{1-m-l}, \qquad a = \left(\frac{p-1}{Ap^{l-1}}\right)^{\frac{1}{m+l-1}}$$

with $n-l+2 \neq 0$, $m+l-1 \neq 0$, and $n+m+1 \neq 0$. Furthermore, for $l > 0$, there is a degenerate solution (10.2.2.2) with $p = 0$ and any a.

2°. *A modification.* The following fact may be useful in searching for particular solutions.

PROPOSITION. Suppose one deals with a nonlinear differential equation for $y = y(x)$, which has been reduced with a change of variable $y = f(x, w)$ to the equation

$$\begin{gathered}\Phi(w''_{xx}, \dots, w^{(n)}_x) + (b_1x^2 + b_2x + b_3)(w''_{xx})^2 + b_4ww''_{xx} + (b_5x^2 + b_6x + b_7)w''_{xx} \\ +b_8(w'_x)^2 + (b_9x + b_{10})w'_x + b_{11}w + b_{12}x^2 + b_{13}x + b_{14} = 0,\end{gathered} \tag{10.2.2.3}$$

where b_i are some constants. Then the original equation admits solutions of the form

$$y = f(x, Ax^2 + Bx + C), \tag{10.2.2.4}$$

where A, B, and C are determined from a system of three algebraic equations (omitted here).

Remark 10.3. The above remains valid also for equations of the form (10.2.2.3), where $b_k = b_k(w''_{xx}, \dots, w_x^{(n)})$ are arbitrary functions. The functions Φ and b_k can, in addition, depend on the combination $2ww''_{xx} - (w'_x)^2$.

Example 10.2. Consider the second-order nonautonomous equation

$$y''_{xx} = ay^m + (b_2x^2 + b_1x + b_0)y^n, \qquad m \neq n. \tag{10.2.2.5}$$

Let us see for which values of the parameters a, b_j, m, and n this equation admits solutions of the form (10.2.2.4).

Let us make the change of variable $y = w^p$, with the exponent p to be determined, and multiply the result by w^{2-p} to obtain

$$pww''_{xx} + p(p-1)(w'_x)^2 - aw^{(m-1)p+2} - (b_2x^2 + b_1x + b_0)w^{(n-1)p+2} = 0. \tag{10.2.2.6}$$

For equation (10.2.2.6) to fall in the class of equations (10.2.2.3), one must set

$$(m-1)p + 2 = 1, \qquad (n-1)p + 2 = 0.$$

This results in the relation between the exponents m and n and the desired expression of p:

$$n = 2m - 1, \qquad p = \frac{1}{1-m} \qquad (m \neq 1 \text{ is an arbitrary}). \tag{10.2.2.7}$$

(The remaining parameters, a and b_j, remain arbitrary for now.) Thus, for $n = 2m - 1$, the change of variable $y = w^{\frac{1}{1-m}}$ reduces equation (10.2.2.5) to

$$ww''_{xx} + s(w'_x)^2 + a(m-1)w + (m-1)(b_2x^2 + b_1x + b_0) = 0, \qquad s = \frac{m}{1-m}. \tag{10.2.2.8}$$

An exact solution to this equation has the form of a quadratic polynomial:

$$w = Ax^2 + Bx + C \qquad (y = w^{\frac{1}{1-m}}). \tag{10.2.2.9}$$

Substituting (10.2.2.9) into (10.2.2.8) and rearranging, we obtain

$$[2(2s+1)A^2 + a(m-1)A + b_2(m-1)]x^2 + [2(2s+1)AB + a(m-1)B + b_1(m-1)]x \\ + [2A + a(m-1)]C + sB^2 + b_0(m-1) = 0.$$

By equating the coefficients of the different powers of x to zero, we get the algebraic system of equations

$$\begin{aligned} 2(2s+1)A^2 + a(m-1)A + b_2(m-1) &= 0, \\ 2(2s+1)AB + a(m-1)B + b_1(m-1) &= 0, \\ [2A + a(m-1)]C + sB^2 + b_0(m-1) &= 0, \end{aligned} \tag{10.2.2.10}$$

with $s = \frac{m}{1-m}$. The first quadratic equation of system (10.2.2.10) serves to determine A (it has two distinct roots in a wide range of the parameters a, b_2, and m). By multiplying the first equation in (10.2.2.10) by B and the second by $-A$ and add together to obtain the simple relation

$$b_2B = b_1A, \tag{10.2.2.11}$$

which allows us to express B via A, provided that $b_2 \neq 0$, to get $B = (b_1/b_2)A$. Now C is easily determined from the last equation in (10.2.2.10).

For the autonomous equation (10.2.2.5) with $b_1 = b_2 = 0$, system (10.2.2.10) has the solution

$$A = \frac{a(m-1)^2}{2(m+1)}, \quad B \text{ is an arbitrary constant;} \quad C = \frac{m+1}{2a}\left[\frac{B^2}{(m-1)^2} - \frac{b_0}{m}\right]. \tag{10.2.2.12}$$

Let us focus on the special case of equation (10.2.2.5) with

$$a = b_1 = b_2 = 0, \quad m = -1, \quad n = -3.$$

It follows from system (10.2.2.10) that

$$A \text{ and } B \text{ are arbitrary constants,} \quad C = \frac{1}{4A}(B^2 + 4b_0).$$

Thus, we have found that the second-order equation $y''_{xx} = b_0 y^{-3}$ has the exact solution $y = \sqrt{Ax^2 + Bx + \frac{1}{4A}(B^2 + 4b_0)}$ involving two arbitrary constants. The general solution to this equation consists of two branches: $y = \pm\sqrt{Ax^2 + Bx + \frac{1}{4A}(B^2 + 4b_0)}$.

Example 10.3. The equation with an exponential nonlinearity

$$y''_{xx} + ce^{\lambda y} y'_x = ae^{\lambda y} + (b_2 x^2 + b_1 x + b_0)e^{2\lambda y} \tag{10.2.2.13}$$

can be reduced with the change of variable $y = -\frac{1}{\lambda}\ln w$ to a special case of equation (10.2.2.3):

$$ww''_{xx} - (w'_x)^2 + cw'_x + a\lambda w + \lambda(b_2 x^2 + b_1 x + b_0) = 0.$$

Hence, equation (10.2.2.13) admits an exact solution of the form $y = -\frac{1}{\lambda}\ln(Ax^2 + Bx + C)$. In particular, the autonomous equation $y''_{xx} = ae^{\lambda y} + b_0 e^{2\lambda y}$, which is the special case of equation (10.2.2.13) with $c = b_1 = b_2 = 0$, has a particular solution

$$y = -\frac{1}{\lambda}\ln\left(\frac{1}{2}a\lambda x^2 + Bx + \frac{B^2}{2a\lambda} - \frac{b_0}{2a}\right),$$

where B is an arbitrary constant.

► Tanh-coth and sinh-cosh methods.

1°. *Tanh-coth method.* The main idea of the tanh-coth method is the assumption that a particular solution can be expressed in terms of the hyperbolic tangent or hyperbolic cotangent functions, which corresponds to $\varphi(z) = \tanh z$ or $\varphi(z) = \coth z$ in (10.2.1.2).

Example 10.4. Consider the second-order nonlinear differential equation

$$y''_{xx} + by - cy^3 = 0. \tag{10.2.2.14}$$

We seek particular solutions of the equation in the form

$$y = \sum_{k=0}^{n} a_k z^k, \quad z = \tanh(\lambda x), \tag{10.2.2.15}$$

with a_k, λ, and n to be determined. Differentiating (10.2.2.15) twice and taking into account that $z'_x = \lambda/\cosh^2(\lambda x) = \lambda(1 - z^2)$, we obtain

$$\begin{aligned} y'_x &= y'_z z'_x = \left(\sum_{k=0}^{n} a_k k z^{k-1}\right)\lambda(1-z^2) = \lambda\sum_{k=0}^{n} a_k k z^{k-1} - \lambda\sum_{k=0}^{n} a_k k z^{k+1}, \\ y''_{xx} &= (y'_x)_z z'_x = \lambda\left(\sum_{k=0}^{n} a_k k(k-1)z^{k-2} - \lambda\sum_{k=0}^{n} a_k k(k+1)z^k\right)\lambda(1-z^2) \\ &= \lambda^2\sum_{k=0}^{n} a_k k(k-1)z^{k-2} - 2\lambda^2\sum_{k=0}^{n} a_k k^2 z^k + \lambda^2\sum_{k=0}^{n} a_k k(k+1)z^{k+2}. \end{aligned} \tag{10.2.2.16}$$

From (10.2.2.15) and (10.2.2.16) it follows that the terms in equation (10.2.2.14) are represented as:

y''_{xx} is a linear combination of different powers of z up to z^{n+2} inclusive,
y is a linear combination of different powers of z up to z^n inclusive,
y^3 is a linear combination of different powers of z up to z^{3n} inclusive.

For ODE (10.2.2.14) to be satisfied identically, the terms with highest power of z must be matched up. Hence, the equality $n+2=3n$ must hold, resulting in $n=1$.

Substituting formulas (10.2.2.15) and (10.2.2.16) with $n=1$ into (10.2.2.14) and rearranging, we arrive at a cubic equation for z:

$$a_1(2\lambda^2 - a_1^2 c)z^3 - 3a_0 a_1^2 c z^2 + a_1(b - 2\lambda^2 - 3a_0^2 c)z + a_0(b - a_0^2 c) = 0.$$

Equating the coefficients of the different powers of z to zero results in the overdetermined system of algebraic equations

$$a_1(2\lambda^2 - a_1^2 c) = 0, \quad a_0 a_1 c = 0, \quad a_1(b - 2\lambda^2 - 3a_0^2 c) = 0, \quad a_0(b - a_0^2 c) = 0.$$

This system can be satisfied, for example, with $a_0 = 0$, $a_1 = \pm\sqrt{b/c}$, and $\lambda = \pm\sqrt{b/2}$. As a result, we get the following particular solutions to equation (10.2.2.14):

$$y = \pm\sqrt{b/c}\,\tanh(\sqrt{b/2}\,x). \tag{10.2.2.17}$$

Remark 10.4. Since equation (10.2.2.14) is invariant to the translation transformation $x \Longrightarrow x + \text{const}$, it also admits the solutions $y = \pm\sqrt{b/c}\,\tanh(\sqrt{b/2}\,x + s)$, where s is an arbitrary constant.

Remark 10.5. In a similar fashion, we can also obtain the following particular solutions to equation (10.2.2.14): $y = \pm\sqrt{b/c}\,\coth(\sqrt{b/2}\,x)$ and $y = \pm\sqrt{b/c}\,\coth(\sqrt{b/2}\,x + s)$.

2°. *Sinh-cosh method.* The sinh-cosh method is based on the assumption that a particular solution can be expressed in terms of the hyperbolic sine or hyperbolic cosine functions, and corresponds to $\varphi(z) = \sinh z$ or $\varphi(z) = \cosh z$ in (10.2.1.2).

Example 10.5. Consider the fourth-order nonlinear differential equation

$$y''''_{xxxx} = b_1[yy''_{xx} - (y'_x)^2] + b_2 y + b_3. \tag{10.2.2.18}$$

We seek particular solutions to the equation in the form

$$y = a_0 + a_1 \sinh(\lambda x). \tag{10.2.2.19}$$

Substituting (10.2.2.19) into (10.2.2.18) and rearranging taking into account the identity $\cosh^2 z - \sinh^2 z = 1$, we obtain

$$a_1(\lambda^4 - a_0 b_1 \lambda^2 - b_2)\sinh(\lambda x) + a_1^2 b_1 \lambda^2 - a_0 b_2 - b_3 = 0.$$

For this equation to be satisfied identically for any x, one must set

$$\lambda^4 - a_0 b_1 \lambda^2 - b_2 = 0, \qquad a_1^2 b_1 \lambda^2 - a_0 b_2 - b_3 = 0.$$

Solving these equations for a_0 and a_1 yields

$$a_0 = \frac{\lambda^4 - b_2}{b_1 \lambda^2}, \qquad a_1 = \pm\frac{1}{b_1\lambda^2}\sqrt{b_1\lambda^4 + b_1 b_3 \lambda^2 - b_2^2}. \tag{10.2.2.20}$$

Formulas (10.2.2.19) and (10.2.2.20) define particular solutions of equation (10.2.2.18) involving one free parameter, λ, with the restrictions that $\lambda \neq 0$ and the radicand must be positive.

Remark 10.6. Likewise, one can obtain more general, two-parameter particular solutions to equation (10.2.2.18):

$$y = a_0 + a_1\sinh(\lambda x) + a_2\cosh(\lambda x),$$
$$a_0 = \frac{\lambda^4 - b_2}{b_1\lambda^2}, \qquad a_1 = \pm\frac{1}{b_1\lambda^2}\sqrt{a_2^2 b_1^2 \lambda^4 + b_1\lambda^4 + b_1 b_3\lambda^2 - b_2^2},$$

where a_2 and λ are arbitrary constants.

► **Sine-cosine and tan-cot methods.**

1°. *Sine-cosine method.* The sine-cosine method is based on the assumption that a particular solution can be expressed in terms of the sine or cosine function, which corresponds to $\varphi(z) = \sin z$ or $\varphi(z) = \cos z$ in (10.2.1.2).

Example 10.6. Consider once again equation (10.2.2.18). We seek particular solutions of the form

$$y = a_0 + a_1 \sin(\lambda x). \tag{10.2.2.21}$$

Substituting (10.2.2.21) in (10.2.2.18) and rearranging while taking into account the identity $\cos^2 z + \sin^2 z = 1$, we obtain

$$a_1(\lambda^4 + a_0 b_1 \lambda^2 - b_2)\sin(\lambda x) + a_1^2 b_1 \lambda^2 - a_0 b_2 - b_3 = 0.$$

For this equation to be satisfied identically for any x, one must set

$$\lambda^4 + a_0 b_1 \lambda^2 - b_2 = 0, \qquad a_1^2 b_1 \lambda^2 - a_0 b_2 - b_3 = 0.$$

Solving these equations for a_0 and a_1 gives

$$a_0 = \frac{b_2 - \lambda^4}{b_1 \lambda^2}, \qquad a_1 = \pm \frac{1}{b_1 \lambda^2} \sqrt{b_2^2 + b_1 b_3 \lambda^2 - b_1 \lambda^4}. \tag{10.2.2.22}$$

Formulas (10.2.2.19) and (10.2.2.22) define particular solutions to equation (10.2.2.18) involving one free parameter, λ, with the restriction that $\lambda \neq 0$ and the radicand must be positive.

Remark 10.7. Since equation (10.2.2.18) is invariant to translation, $x \Longrightarrow x + \text{const}$, it also admits solutions of the form $y = a_0 + a_1 \sin(\lambda x + c)$, where a_0 and a_1 are given by (10.2.2.22), while c and λ are arbitrary constants.

2°. *Tan-cot method.* The tan-cot method is based on the assumption that a particular solution can be expressed in terms of the tangent or cotangent functions, which corresponds to $\varphi(z) = \tan z$ or $\varphi(z) = \cot z$ in (10.2.1.2).

Example 10.7. Consider equation (10.2.2.14) with $b < 0$ and $c > 0$. We seek particular solutions of the form $y = \sum_{k=0}^{n} a_k z^k$ with $z = \tan(\lambda x)$, where a_k, λ, and n are undetermined constants. Arguing in the same way as in Example 10.4, we find that $n = 1$ and $a_0 = 0$. As a result, we obtain the particular solutions

$$y = \pm\sqrt{-b/c}\,\tan\bigl(\sqrt{-b/2}\,x\bigr).$$

Remark 10.8. Since equation (10.2.2.14) is invariant to translation, $x \Longrightarrow x + \text{const}$, it also admits the solutions $y = \pm\sqrt{-b/c}\,\tan\bigl(\sqrt{-b/2}\,x + s\bigr)$, where s is an arbitrary constant.

10.2.3 Exp-Function, Q-Expansion and Related Methods

► **Exp-function method. The simplest version.**

In the simplest case, the exp-function method is based on the assumption that a particular solution can be expressed in terms of the exponential function, which corresponds to $\varphi(z) = \exp z$ in (10.2.1.2).

Importantly, the autonomous differential equation of arbitrary order with a quadratic nonlinearity

$$\sum_{k=1}^{n} S_k[w] w_x^{(k)} + b_0 = 0, \qquad S_k[w] = \sum_{s=0}^{k} b_{ks} w_x^{(s)}, \quad w_x^{(0)} = w \tag{10.2.3.1}$$

admits particular solutions of the form

$$w = A + Be^{\lambda x}, \tag{10.2.3.2}$$

provided that there is a single relation between the coefficients b_{ks} and b_0. The constants A and λ are to be determined, while B is arbitrary. Therefore, one should first try to reduce the equation by a change of variable $y = f(w)$ to an equation with a quadratic nonlinearity (10.2.3.1) and then look for its particular solutions of the form (10.2.3.1).

Example 10.8. Consider the equation

$$y''_{xx} + (a_1 + a_2 y^{m-1})y'_x = by + cy^m. \tag{10.2.3.3}$$

First, we make the change of variable $y = w^p$, with the exponent p to be determined. Then, on multiplying the result by w^{2-p}, we get

$$pww''_{xx} + p(p-1)(w'_x)^2 + p(a_1 w + a_2 w^{(m-1)p+1})w'_x = bw^2 + cw^{(m-1)p+2}.$$

In order to obtain an equation with a quadratic nonlinearity, one must set $p = \frac{1}{1-m}$. Thus, the change of variable $y = w^{\frac{1}{1-m}}$ reduces equation (10.2.3.3) to the form

$$ww''_{xx} + s(w'_x)^2 + a_1 ww'_x + a_2 w'_x + b(m-1)w^2 + c(m-1)w = 0, \qquad s = \frac{m}{1-m}. \tag{10.2.3.4}$$

Substituting (10.2.3.2) into (10.2.3.4) and rearranging, we obtain

$$B^2[(s+1)\lambda^2 + a_1\lambda + b(m-1)]E^2 + B\{A[\lambda^2 + a_1\lambda + 2b(m-1)] + a_2\lambda + c(m-1)\}E$$
$$+ (m-1)A(Ab+c) = 0, \qquad E = e^{\lambda x}, \qquad s = \frac{m}{1-m}.$$

Equating the coefficients of the various powers of E to zero results in the algebraic system of equations

$$\begin{aligned} (s+1)\lambda^2 + a_1\lambda + b(m-1) &= 0, \\ A[\lambda^2 + a_1\lambda + 2b(m-1)] + a_2\lambda + c(m-1) &= 0, \\ A(Ab+c) &= 0. \end{aligned} \tag{10.2.3.5}$$

The trivial cases of $B = 0$ (constant solution) and $m = 1$ (linear equation) have been discarded.

The first quadratic equation in system (10.2.3.5) serves to determine λ (in a wide range of the parameters a_1, b, and m, it has two distinct roots). From the last equation in (10.2.3.5) one can see that there are two possibilities, $A = 0$ and $A \neq 0$, which need to be treated separately.

1. In the degenerate case of $A = 0$ and $a_2 \neq 0$, we get the solution

$$w = Be^{\lambda x}, \qquad \lambda = \frac{c(1-m)}{a_2},$$

which exists under the condition that

$$a_2 c^2 + a_1 a_2 c - a_2^2 b = 0.$$

2. In the nondegenerate case $A \neq 0$, the first and third equations in (10.2.3.5) give the parameters of two particular solutions (10.2.3.2):

$$A = -\frac{c}{b}, \qquad \lambda_{1,2} = \frac{1}{2}(m-1)\Big(a_1 \pm \sqrt{a_1^2 + 4b}\Big).$$

The second equation of system (10.2.3.5), with $A = -c/b$ and the value of λ_1 (or $A = -c/b$ and λ_2) inserted, determines the relationship between the coefficients of the original equation (10.2.3.3) required for the existence of such a solution (the relationship is omitted).

► **Q-expansion and logistic function methods.**

1°. There is a more complex method based on the usage of exponential functions suggesting that particular solutions to autonomous equations with polynomial nonlinearity should be sought in the form

$$y = \sum_{k=0}^{n} a_k Q^k(\lambda x), \qquad Q(z) = \frac{1}{1 + Ce^z}, \tag{10.2.3.6}$$

where C is an arbitrary constant and the remaining constants, a_k, λ, and n, are to be determined.

Expression (10.2.3.6) is substituted into the ODE of interest and then, after multiplying by Q^{-n} and matching the coefficients of like powers of e^{kz}, one arrives at a system of algebraic equations for the unknowns a_k, λ, and n as well as the coefficients involved in the equation of interest.

The representation of solutions in form (10.2.3.6) with $C = \lambda = 1$ constitutes the *Q-expansion method* and corresponds to $\varphi(z) = (1 + e^z)^{-1}$ in the first formula in (10.2.1.2). In the special case $C = 1$ and $\lambda = -1$, the function $Q(z) = (1 + e^{-x})^{-1}$ appearing in the solution is called a *logistic function* (or the *sigmoid function*).

The function $Q(z)$ in (10.2.3.6) can be represented equivalently in terms of hyperbolic functions as follows:

$$Q(z) = \frac{1}{1 + e^{z+z_0}} = \frac{1}{2}\left[1 - \tanh\left(\frac{z + z_0}{2}\right)\right], \qquad z_0 = \ln C, \qquad \text{if } C > 0; \tag{10.2.3.7}$$

$$Q(z) = \frac{1}{1 - e^{z+z_0}} = \frac{1}{2}\left[1 - \coth\left(\frac{z + z_0}{2}\right)\right], \qquad z_0 = \ln |C|, \qquad \text{if } C < 0. \tag{10.2.3.8}$$

The comparison of formulas (10.2.3.6), (10.2.3.7), and (10.2.3.6), (10.2.3.8) with formula (10.2.1.2) at $\varphi(z) = \tanh z$ and $\varphi(z) = \coth z$ shows that the current modification of the exp-function method allows one to cover all the solutions that can be obtained using the tanh-coth methods. Furthermore, the representation of solutions in the form (10.2.3.6) is more compact and is simpler as it does not require the knowledge of hyperbolic functions or relations between them.

2°. The current method admits an alternative and more economical usage based on the fact that the function $Q = Q(z)$ is the general solution of the Bernoulli equation

$$Q'_z = Q^2 - Q. \tag{10.2.3.9}$$

Differentiating (10.2.3.9) with respect to z and eliminating Q'_z with the help of (10.2.3.9), we find successively

$$\begin{aligned} Q''_{zz} &= 2QQ'_z - Q'_z = 2Q^3 - 3Q^2 + Q, \\ Q'''_{zzz} &= (6Q^2 - 6Q + 1)Q'_z = 6Q^4 - 12Q^3 + 7Q^2 - Q, \\ Q''''_{zzzz} &= (24Q^3 - 36Q^2 + 14Q - 1)Q'_z = 24Q^5 - 60Q^4 + 50Q^3 - 15Q^2 + Q. \end{aligned} \tag{10.2.3.10}$$

In a similar fashion, we we can obtain the representation of the derivative $Q^{(k)}_z$ as a polynomial $P_{k+1}(Q)$.

Using formulas (10.2.3.9)–(10.2.3.10), we obtain

$$
\begin{aligned}
y'_x &= \lambda y'_z = \lambda\biggl(\sum_{k=0}^{n} a_k k Q^{k-1}\biggr) Q'_z = \lambda \sum_{k=0}^{n} a_k k Q^{k+1} - \lambda \sum_{k=0}^{n} a_k k Q^k,\\
y''_{xx} &= \lambda^2\biggl(\sum_{k=0}^{n} a_k k(k+1) Q^k - \sum_{k=0}^{n} a_k k^2 Q^{k-1}\biggr) Q'_z \qquad (10.2.3.11)\\
&= \lambda^2 \sum_{k=0}^{n} a_k k(k+1) Q^{k+2} - \lambda^2 \sum_{k=0}^{n} a_k k(2k+1) Q^{k+1} + \lambda^2 \sum_{k=0}^{n} a_k k^2 Q^k.
\end{aligned}
$$

In a similar fashion, we can express the derivative $y_x^{(k)}$ in terms of a polynomial $\widetilde{P}_{n+k}(Q)$.

The degree n of the polynomial (10.2.3.6) is obtained as follows. We replace the terms of the ODE under consideration by the rule

$$
y_x^{(k)} \Longrightarrow Q^{n+k}, \quad y^m \Longrightarrow Q^{nm} \quad (k, m = 0,\, 1,\, \dots) \qquad (10.2.3.12)
$$

and then match up the two (or more) terms with the largest powers of Q. As a result, we obtain a simple equation for n. We can use this technique if n is a positive integer. In the case of noninteger n, we have to use a transformation of the solution $y = y(x)$. For example, if we obtain $n = \frac{1}{m}$, where m is an integer, we can transform the solution as $y = u^m$, where $u = u(x)$ is the new function.

On determining n, we substitute (10.2.3.6) into the differential equation of interest and replace the derivatives $Q_z^{(k)}$ with the expressions (10.2.3.9)–(10.2.3.10) to obtain an algebraic equation for Q. Equating the coefficients of this equation to zero results in an algebraic system for the coefficients a_k and λ.

Example 10.9. Consider the nonlinear second-order equation with a quadratic nonlinearity*

$$
y''_{xx} + by'_x + c(y - y^3) = 0. \qquad (10.2.3.13)
$$

We look for solutions to equation (10.2.3.13) as the sum (10.2.3.6). On replacing the terms of the equation by the rule (10.2.3.12), we equate the exponents of the highest-order terms in Q (which have the correspondence $y''_{xx} \Longrightarrow Q^{n+2}$ and $y^2 \Longrightarrow Q^{3n}$) to obtain $n = 1$.

Using formulas (10.2.3.6) and (10.2.3.11) with $n = 1$, we get

$$
\begin{aligned}
y &= a_0 + a_1 Q,\\
y'_x &= \lambda a_1 (Q^2 - Q),\\
y''_{xx} &= \lambda^2 a_1 (2Q^3 - 3Q^2 + Q).
\end{aligned}
$$

Inserting these expressions into (10.2.3.13) yields a polynomial of degree 3 in Q. Equating its coefficients of the different powers of Q to zero and dividing by $a_1 \neq 0$ and $c \neq 0$, we arrive at the algebraic system equations

$$
\begin{aligned}
2\lambda^2 - ca_1^2 &= 0, &(10.2.3.14)\\
3\lambda^2 - b\lambda + 3a_0 a_1 c &= 0, &(10.2.3.15)\\
\lambda^2 - b\lambda + c - 3a_0^2 c &= 0, &(10.2.3.16)\\
a_0(1 - a_0^2) &= 0. &(10.2.3.17)
\end{aligned}
$$

*This equation arises when one looks for exact solutions to the nonlinear Burgers–Huxley PDE $u_t = u_{\xi\xi} + c(u - u^3)$ in the form of a traveling wave, $u = y(x)$ with $x = \xi - bt$, where b is an arbitrary constant.

Subtracting equation (10.2.3.16) from (10.2.3.15) and eliminating λ^2 with the help of (10.2.3.14), we obtain the relation between a_0 and a_1:

$$3a_0^2 + 3a_0 a_1 + a_1^2 - 1 = 0. \qquad (10.2.3.18)$$

The cubic equation (10.2.3.17) has three roots: $a_0 = 0$, $a_0 = 1$, and $a_0 = -1$. To each a_0 there correspond two roots of the quadratic equation (10.2.3.18) for a_1, and to each a_1 there correspond two roots of equation (10.2.3.14). Substituting a_0, a_1, and λ into equation (10.2.3.15) (or (10.2.3.16)), we arrive at the relationship between the coefficients b and c that ensures the existence of a solution. To sum up, there are the following possibilities:

$$\begin{aligned}
&\text{(i)} && a_0 = 0, && a_1 = 1, && \lambda = \pm\sqrt{c/2}, && b = \pm 3\sqrt{c/2};\\
&\text{(ii)} && a_0 = 0, && a_1 = -1, && \lambda = \pm\sqrt{c/2}, && b = \pm 3\sqrt{c/2};\\
&\text{(iii)} && a_0 = 1, && a_1 = -1, && \lambda = \pm\sqrt{c/2}, && b = \mp 3\sqrt{c/2};\\
&\text{(iv)} && a_0 = 1, && a_1 = -2, && \lambda = \pm\sqrt{2c}, && b = 0;\\
&\text{(v)} && a_0 = -1, && a_1 = 1, && \lambda = \pm\sqrt{c/2}, && b = \mp 3\sqrt{c/2};\\
&\text{(vi)} && a_0 = -1, && a_1 = 2, && \lambda = \pm\sqrt{2c}, && b = 0.
\end{aligned}$$

It follows from equation (10.2.3.13) that if y is a solution, then $-y$ is also a solution. The above formulas for the coefficients determine three pairs of solutions of the form $y = a_0 + \dfrac{a_1}{1 + Ce^{\lambda x}}$, which differ in sign.

3°. Another convenient technique to seek particular solutions of the form (10.2.3.6) is based on using the change of variable $\xi = \dfrac{1}{1 + Ce^{\lambda x}}$ in the equation of interest followed by representing solutions in the form of finite power series in ξ: $y = \sum_{k=0}^{n} a_k \xi^k$ (see Remark 10.1).

► Solutions in the form of the ratio of exponential polynomials.

Particular solutions to ODEs can also be sought in the form of the ratio of exponential polynomials

$$y(x) = \frac{\sum_{k=-r}^{s} a_k e^{kz}}{\sum_{j=-p}^{q} b_j e^{jz}} = \frac{a_{-r}e^{-rz} + \cdots + a_s e^{sz}}{b_{-p}e^{-pz} + \cdots + b_q e^{qz}}, \qquad z = \lambda x, \qquad (10.2.3.19)$$

where r, s, p and q are unknown positive integers to be determined and a_k, b_j, and λ are unknown constants. Symbolic computations with computer algebra systems (such as Maple or Mathematica) can often be very helpful in searching for such solutions.

For example, in the special case $p = q = r = s = 1$, (10.2.3.19) becomes

$$y(x) = \frac{a_{-1}e^{-z} + a_0 + a_1 e^z}{b_{-1}e^{-z} + b_0 + b_1 e^z}, \qquad z = \lambda x.$$

By substituting this expression into the ODE and by matching the coefficients of like powers of e^{kz}, we generate the system of algebraic equations for the unknowns a_{-1}, a_0, a_1, b_{-1}, b_0, b_1, and λ as well as the coefficients involved in the equation.

⊙ *Literature for Section 10.2*: W. Malfliet (1992), W. Malfliet and W. Hereman (1996), E. Fan (2000), A. M. Wazwaz (2004, 2007a, 2007b, 2008), D.-S. Wang, Y.-J. Ren, and H.-Q. Zhang (2005), J.-H. He and X.-H. Wu (2006), A. Ebaid (2007), J.-H. He and M. A. Abdou (2007), S. Zhang (2007), A. Bekir (2008), L. Wazzan (2009), N. A. Kudryashov (2010b, 2012, 2013, 2015), E. J. Parkes (2010), N. A. Kudryashov and D. I. Sinelshchikov (2012), A. D. Polyanin and V. F. Zaitsev (2012), N. A. Kudryashov and A. S. Zakharchenko (2014).

10.3 Method of Differential Constraints

10.3.1 Preliminary Remarks. First-Order Differential Constraints and Their Applications

The main idea of the method is that exact solutions to a complex (nonintegrable) equation are sought by jointly analyzing this equation and an auxiliary simpler (integrable) equation, called a *differential constraint*.*

The order of a differential constraint is the order of the highest derivative involved. Usually, the order of the differential constraint is less than that of the equation; first-order differential constraints are simplest and most common. The equation and differential constraint must involve a set of free parameters (or even arbitrary functions) whose values are chosen by ensuring that the equation and the constraint are consistent. After the consistency analysis, all solutions obtained by integrating the differential constraint will be simultaneously solutions to the original equation. The method makes it possible to find particular solutions to the original equation for some values of the determining parameters.

For simplicity, we first consider autonomous ordinary differential equations of the form

$$F(y, y'_x, \dots, y_x^{(n)}; \mathbf{a}) = 0, \qquad (10.3.1.1)$$

which do not involve the independent variable x explicitly and depend on a vector of free parameters $\mathbf{a} = \{a_1, \dots, a_k\}$. For equations (10.3.1.1), one should take first-order differential constraints in the autonomous form

$$G(y, y'_x; \mathbf{b}) = 0, \qquad (10.3.1.2)$$

dependent on a vector of free parameters $\mathbf{b} = \{b_1, \dots, b_s\}$.

By differentiating relation (10.3.1.2) successively several times, one can express higher-order derivatives in terms of y and y'_x: $y_x^{(k)} = \varphi_k(y, y'_x; \mathbf{b})$. Substituting these expressions into the original equation (10.3.1.1), one arrives at a first-order equation

$$\mathcal{F}(y, y'_x; \mathbf{a}, \mathbf{b}) = 0. \qquad (10.3.1.3)$$

By eliminating the derivative y'_x from (10.3.1.2) and (10.3.1.3), one obtains an algebraic/transcendental equation

$$P(y; \mathbf{a}, \mathbf{b}) = 0. \qquad (10.3.1.4)$$

Further, one looks for the values of $\mathbf{a}$ and $\mathbf{b}$ at which equation (10.3.1.4) is satisfied identically for any y (this may result in some restrictions on the components of the vector $\mathbf{a}$). After this, one expresses the vector $\mathbf{b}$ in terms of $\mathbf{a}$, so that $\mathbf{b} = \mathbf{b}(\mathbf{a})$, and substitutes it back into the differential constraint (10.3.1.2) to obtain a first-order ordinary differential equation

$$g(y, y'_x; \mathbf{a}) = 0 \qquad (g = G|_{\mathbf{b}=\mathbf{b}(\mathbf{a})}). \qquad (10.3.1.5)$$

*The ideas of this method as applied to searching for exact solutions to nonlinear PDEs were first put forward by Yanenko (1964). The studies by Galaktionov (1994), Olver and Vorob'ev (1996), Andreev, Kaptsov, Pukhnachov, and Rodionov (1998), Kaptsov and Verevkin (2003), Polyanin and Zaitsev (2004, 2012), Polyanin, Zaitsev, and Zhurov (2005) give a number of nontrivial examples of how to use this method to construct exact solutions (other than traveling wave solutions) to different nonlinear PDEs of mathematical physics.

This equation is consistent with the original equation (10.3.1.1); in other words, the original equation is a consequence of equation (10.3.1.5) and, therefore, inherits all of its solutions. Finally, by solving for the derivative, equation (10.3.1.5) is reduced to a separable equation, which is integrated to obtain a general solution. The general solution of equation (10.3.1.5) is also an exact solution of the original equation (10.3.1.1).

Remark 10.9. If a first-order differential constraint is defined in explicit form, $y'_x = h(y; \mathbf{b})$, the successive differentiation enables one to express the higher-order derivatives in terms of y, so that

$$y''_{xx} = (y'_x)'_y y'_x = hh'_y, \quad y'''_{xxx} = (y''_{xx})'_y y'_x = h(hh'_y)'_y, \quad \dots$$

Using these expressions and the differential constraint to eliminate the derivatives from (10.3.1.1), one immediately arrives at an algebraic/transcendental equation of the form (10.3.1.4).

Remark 10.10. Instead of y'_x, one can eliminate the dependent variable y from (10.3.1.2) and (10.3.1.3) to obtain an algebraic/transcendental equation for the derivative: $Q(y'_x; \mathbf{a}, \mathbf{b}) = 0$.

The structure of the nonlinearity of the differential constraint (10.3.1.2) can often be taken to be similar to that of the original equation (10.3.1.1) so as to have different determining parameters. This will be illustrated below by specific examples of second-, third-, fourth-, and higher-order equations.

Example 10.10. Consider the second-order ordinary differential equation with a power-law nonlinearity

$$y''_{xx} - cy'_x = ay + by^n, \tag{10.3.1.6}$$

which arises in the theory of chemical reactors, combustion theory, and mathematical biology.*

Let us supplement equation (10.3.1.6) with the first-order differential constraint

$$y'_x = \alpha y + \beta y^m, \tag{10.3.1.7}$$

which is a separable equation and is easy to integrate. The form of the right-hand side of (10.3.1.7) has been chosen to be similar to that of the original equation (10.3.1.6).

The equation and differential constraint involve seven parameters: a, b, c, n, m, α, and β. The further analysis aims at determining the parameters α, β, and m of the differential constraint so as to express them in terms of a, b, c, and n. Simultaneously, restrictions on the equation parameters will be found.

Differentiating (10.3.1.7) and replacing the first derivative with the right-hand side of (10.3.1.7), we get

$$\begin{aligned} y''_{xx} &= (\alpha + m\beta y^{m-1})y'_x = (\alpha + m\beta y^{m-1})(\alpha y + \beta y^m) \\ &= \alpha^2 y + \alpha\beta(m+1)y^m + m\beta^2 y^{2m-1}. \end{aligned} \tag{10.3.1.8}$$

Eliminating the first and second derivatives from (10.3.1.6) using (10.3.1.7) and (10.3.1.8) and rearranging, we obtain

$$(\alpha^2 - \alpha c - a)y + \beta[\alpha(m+1) - c]y^m + m\beta^2 y^{2m-1} - by^n = 0.$$

For this equation to hold for all y, one must set

$$\begin{aligned} &\alpha^2 - \alpha c - a = 0, \\ &\alpha(m+1) - c = 0, \\ &2m - 1 = n, \\ &m\beta^2 - b = 0. \end{aligned} \tag{10.3.1.9}$$

*Equations (10.3.1.6) and (10.3.1.11) describe traveling-wave solutions of the Kolmogorov–Petrovskii–Piskunov PDE, $u_t = u_{\xi\xi} - f(u)$, for some forms of the kinetic function $f(u)$. In this case, we have $u = y(x)$ with $x = \xi + ct$.

If conditions (10.3.1.9) hold, then solutions to equation (10.3.1.7) are also solutions to the more complex equation (10.3.1.6). The determining system of four equations (10.3.1.9) contains seven parameters a, b, c, n, m, α, and β. The three parameters b, c, and n of the original equation can be regarded as arbitrary and the other parameters are expressed as follows:

$$a = -\frac{2c^2(n+1)}{(n+1)^2}, \quad m = \frac{n+1}{2}, \quad \alpha = \frac{2c}{n+3}, \quad \beta = \pm\sqrt{\frac{2b}{n+1}}. \tag{10.3.1.10}$$

It is apparent that for equations (10.3.1.6) and (10.3.1.7) to be consistent, the original equation parameter a must be connected with two other parameters, c and n. In this case, two families of parameters (10.3.1.10) of equation (10.3.1.7) can be identified that determine two different one-parameter solutions to equations (10.3.1.6) and (10.3.1.7); recall that equation (10.3.1.7) is separable and is easy to integrate.

Example 10.11. The second-order equation with an exponential nonlinearity

$$y''_{xx} - cy'_x = a + be^{\lambda y} \tag{10.3.1.11}$$

can be investigated in a similar manner. The equation will be considered in conjunction with the first-order differential constraint

$$y'_x = \alpha + \beta e^{\mu y}. \tag{10.3.1.12}$$

The analysis shows that three parameters of the original equation, b, c, and λ, can be regarded as arbitrary and the other parameters are expressed as

$$a = -\frac{2c^2}{\lambda}, \quad \alpha = \frac{2c}{\lambda}, \quad \beta = \pm\sqrt{\frac{2b}{\lambda}}, \quad \mu = \frac{\lambda}{2}. \tag{10.3.1.13}$$

It is apparent that for equations (10.3.1.11) and (10.3.1.12) to be consistent, the parameter a must be related in a certain way to two other parameters of the equation, c and λ. In this case, two families of parameters (10.3.1.13) of the differential constraint (10.3.1.12) can be identified, which determine two different one-parameter solutions to equations (10.3.1.11) and (10.3.1.12). Equation (10.3.1.12) is separable and is easy to integrate.

Example 10.12. Consider the nonlinear third-order equation

$$y'''_{xxx} = ay^4 + by^2 + c \tag{10.3.1.14}$$

in conjunction with the first-order differential constraint

$$y'_x = \alpha y^2 + \beta. \tag{10.3.1.15}$$

Using (10.3.1.15), we find the derivatives

$$\begin{aligned} y''_{xx} &= 2\alpha y y'_x = 2\alpha y(\alpha y^2 + \beta) = 2\alpha^3 y^3 + 2\alpha\beta y, \\ y'''_{xxx} &= (6\alpha^2 y^2 + 2\alpha\beta)y'_x = (6\alpha^2 y^2 + 2\alpha\beta)(\alpha y^2 + \beta) = 6\alpha^3 y^4 + 8\alpha^2\beta y^2 + 2\alpha\beta^2. \end{aligned}$$

For the last equation to coincide with (10.3.1.14), the relations

$$a = 6\alpha^3, \quad b = 8\alpha^2\beta, \quad c = 2\alpha\beta^2$$

must hold. On solving the first two equations for α and β and substituting the resulting expressions into the last equation, we obtain

$$\alpha = \left(\frac{a}{6}\right)^{1/3}, \quad \beta = \left(\frac{a}{6}\right)^{-2/3}\frac{b}{8}, \quad c = \frac{3b^2}{16a}. \tag{10.3.1.16}$$

It follows that with this c, the third-order equation (10.3.1.14) has a particular solution resulting from solving the first-order separable equation (10.3.1.15) whose parameters are connected with those of the original equation by the first two relations in (10.3.1.16).

Example 10.13. Consider the nonlinear fourth-order equation

$$y''''_{xxxx} = ay^n + by^{2n+3} \tag{10.3.1.17}$$

in conjunction with the first-order differential constraint

$$(y'_x)^2 = \alpha y^m + \beta. \tag{10.3.1.18}$$

Differentiating (10.3.1.18), we get the derivatives

$$\begin{aligned} y''_{xx} &= \tfrac{1}{2}\alpha m y^{m-1} \qquad \text{(after canceling by } y'_x\text{)},\\ y'''_{xxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y'_x,\\ y''''_{xxxx} &= \tfrac{1}{2}\alpha m(m-1)y^{m-2}y''_{xx} + \tfrac{1}{2}\alpha m(m-1)(m-2)y^{m-3}(y'_x)^2\\ &= \tfrac{1}{2}\alpha\beta m(m-1)(m-2)y^{m-3} + \tfrac{1}{4}\alpha^2 m(m-1)(3m-4)y^{2m-3}. \end{aligned} \tag{10.3.1.19}$$

Comparing the right-hand side of (10.3.1.17) and that of the last equation in (10.3.1.19) enables us to draw the following conclusions about the consistency of (10.3.1.17) and (10.3.1.19).

1°. For any values of the parameters of the original equation (10.3.1.17) except for $n \neq -1, -2, -3, -\frac{5}{3}$ and $b \neq 0$, one can calculate the parameters of the differential constraint (10.3.1.18) by the formulas

$$m = n+3, \quad \alpha = \pm 2\sqrt{\frac{b}{(n+2)(n+3)(3n+5)}}, \quad \beta = \frac{2a}{\alpha(n+1)(n+2)(n+3)}.$$

2°. For $b = 0$ and $n = -\frac{5}{3}$, we have

$$m = \frac{4}{3}, \quad \beta = -\frac{27a}{4\alpha}, \quad \alpha \neq 0 \text{ is an arbitrary constant.}$$

In this case, the solution to equation (10.3.1.18) will depend on two arbitrary constants (α plays the role of an additional constant of integration).

Remark 10.11. For $b=0$ and $n=-\frac{5}{3}$, one can find the general solution of equation (10.3.1.17) (see Eq. 1 in Section 16.2.1).

Example 10.14. The fourth-order equation with an exponential nonlinearity

$$y''''_{xxxx} = ae^{\lambda y} + b^{2\lambda y} \tag{10.3.1.20}$$

can be analyzed using the differential constraint

$$(y'_x)^2 = \alpha e^{\lambda y} + \beta. \tag{10.3.1.21}$$

Analysis shows that for any values of the parameters of the original equation (10.3.1.20) satisfying the condition $b\lambda > 0$, two families of parameters of the differential constraint (10.3.1.21) can be found using the formulas

$$\alpha = \pm\frac{a}{\lambda^2}\left(\frac{3\lambda}{b}\right)^{1/2}, \quad \beta = \pm\frac{2}{\lambda}\left(\frac{b}{3\lambda}\right)^{1/2}.$$

Here, one takes either the upper or lower signs simultaneously.

Example 10.15. The nonlinear nth-order equation

$$y^{(n)}_x = ae^{\lambda y}$$

admits the first-order differential constraint

$$y'_x = \alpha e^{\mu y}.$$

The successive differentiation of the differential constraint gives $y^{(n)}_x = \alpha^n\mu^{n-1}(n-1)!\,e^{n\mu y}$. Comparing this expression with the equation yields $\lambda = n\mu$ and $a = \alpha^n\mu^{n-1}(n-1)!$, or

$$\mu = \frac{\lambda}{n}, \quad \alpha = \left[\frac{an^{n-1}}{\lambda^{n-1}(n-1)!}\right]^{1/n}.$$

10.3.2 Differential Constraints of Arbitrary Order. General Consistency Method for Two Equations

In general, a differential constraint is an ordinary differential equation of arbitrary order. Therefore, it is necessary to be able to analyze overdetermined systems of two ordinary differential equations for consistency. Outlined below is the general algorithm for the analysis of such systems.

1°. First, let us consider two ordinary differential equations of the same order

$$F_1(x, y, y'_x, \dots, y_x^{(n)}) = 0, \tag{10.3.2.1}$$

$$F_2(x, y, y'_x, \dots, y_x^{(n)}) = 0; \tag{10.3.2.2}$$

here and henceforth, it is assumed that the equations depend on free parameters, which are omitted for brevity. We eliminate the highest derivative (by solving one of the equations for $y_x^{(n)}$ and substituting the resulting expression into the other equation) to obtain the $(n-1)$st-order equation

$$G_1(x, y, y'_x, \dots, y_x^{(n-1)}) = 0. \tag{10.3.2.3}$$

Differentiating (10.3.2.3) with respect to x and eliminating the derivative $y_x^{(n)}$ from the resulting equation using either of the equations (10.3.2.1) and (10.3.2.2), one arrives at another $(n-1)$st-order equation

$$G_2(x, y, y'_x, \dots, y_x^{(n-1)}) = 0. \tag{10.3.2.4}$$

Thus, the analysis of two nth-order equations (10.3.2.1) and (10.3.2.2) is reduced to the analysis of two $(n-1)$st-order equations (10.3.2.3) and (10.3.2.4). By reducing the order of equations in a similar manner further, one ultimately arrives at a single algebraic/transcendental equation (since two first-order differential equations are reducible to a single algebraic equation). The analysis of the resulting algebraic equation presents no fundamental difficulties and is performed in the same way as previously in Section 10.3.1 for the case of a first-order differential constraint.

2°. Suppose there are two ordinary differential equations having different orders:

$$F_1(x, y, y'_x, \dots, y_x^{(n)}) = 0, \tag{10.3.2.5}$$

$$F_2(x, y, y'_x, \dots, y_x^{(m)}) = 0, \tag{10.3.2.6}$$

with $m < n$. Then, by differentiating (10.3.2.6) $n - m$ times, one reduces system (10.3.2.5)–(10.3.2.6) to a system of the form (10.3.2.1)–(10.3.2.2), in which both equations have the same order n.

Example 10.16. Consider the fourth-order equation with a quadratic nonlinearity

$$y''''_{xxxx} = a(y''_{xx})^2 - by^2 + c \tag{10.3.2.7}$$

in conjunction with the second-order differential constraint

$$y''_{xx} = \alpha y + \beta. \tag{10.3.2.8}$$

Differentiating (10.3.2.8) twice gives $y''''_{xxxx} = \alpha^2 y + \alpha\beta$. Using this expression and the differential constraint (10.3.2.8) to eliminate the derivatives from (10.3.2.7), one arrives at a quadratic equation for y, which is satisfied identically if the conditions

$$a\alpha^2 - b = 0, \quad \alpha - 2a\beta = 0, \quad c = \alpha\beta - a\beta^2$$

hold. Two parameters of the original equation, a and b, can be regarded as arbitrary and the third parameter, c, with coefficients of differential constraint (10.3.2.8) are expressed in terms of them as follows:

$$c = \frac{b}{4a^2}, \quad \alpha = \pm\sqrt{\frac{b}{a}}, \quad \beta = \pm\frac{1}{2a}\sqrt{\frac{b}{a}}.$$

Example 10.17. The equation of order mn with a quadratic nonlinearity

$$y_x^{(mn)} = a[y_x^{(n)}]^2 + byy_x^{(n)} + cy_x^{(n)} + dy^2 + ky + p \qquad (m \text{ is positive integer}),$$

which generalizes equation (10.3.2.7), can be investigated using the nth-order differential constraint

$$y_x^{(n)} = \alpha y + \beta.$$

3°. The general autonomous second-order differential constraint

$$y''_{xx} = f(y)$$

is equivalent to the autonomous first-order differential constraint

$$(y'_x)^2 = F(y),$$

where $F(y) = 2\int f(y)\,dy + C$ and C is an arbitrary constant. This is proved by differentiating the latter relation and comparing with the original differential constraint.

With this in mind, the second-order differential constraint (10.3.2.8) in Example 10.16 could be replaced by the first-order constraint $(y'_x)^2 = \alpha y^2 + 2\beta y + \gamma$, where γ is an extra free parameter. However, the differential constraint (10.3.2.8) is linear and is easy to integrate.

4°. In principle, any differential constraint of arbitrary order (10.3.2.6) can be replaced by a suitable first-order differential constraint. Indeed, the above algorithm for successive order reduction of system (10.3.2.5)–(10.3.2.6) leads, in the nondegenerate case, to a system of first-order equations, one of which can be treated as a first-order differential constraint.

10.3.3 Using Point Transformations in Combination with the Method of Differential Constraints

► **General description of the solution-seeking procedure.**

1°. In some cases, it is first useful to reduce the ODE of interest, with a point transformation, to another equation (simpler or more convenient for investigation), which can then be analyzed using a suitable differential constraint. With this approach, solutions to the autonomous equation (10.3.1.1) are sought in the form

$$y = G(w; \mathbf{b}), \tag{10.3.3.1}$$

where G is a given function and $w = w(x)$ is a function satisfying the first-order differential equation (the differential constraint)

$$H(w, w'_x; \mathbf{c}) = 0. \tag{10.3.3.2}$$

The functions G and H in (10.3.3.1) and (10.3.3.2) depend on the vectors of free parameters **b** and **c**.

The introduction of the new variable w defined by relation (10.3.3.1) reduces equation (10.3.1.1) to a new ODE with one differential constraint (10.3.3.2); this creates the standard situation discussed in Section 10.3.1).

► **Examples of constructing particular solutions.**

Example 10.18. Let us look at the second-order ordinary differential equation with a power-law nonlinearity of arbitrary degree

$$y''_{xx} - cy'_x = ay + by^n + dy^{2n-1}, \tag{10.3.3.3}$$

which generalizes equation (10.3.1.6) to the case of $d \neq 0$.

We choose the linking dependence (10.3.3.1) in the power-law form

$$y = w^p, \tag{10.3.3.4}$$

with the exponent p to be determined. Substituting (10.3.3.4) into (10.3.3.3) and multiplying by w^{2-p}, we obtain

$$pww''_{xx} + p(p-1)(w'_x)^2 - cpww'_x = aw^2 + bw^{k+1} + dw^{2k}, \quad k = p(n-1)+1. \tag{10.3.3.5}$$

Let us discuss a few possibilities for choosing p that allow one to find exact solutions.

Case 1. Suppose that

$$p = \frac{1}{1-n} \quad (k=0). \tag{10.3.3.6}$$

The change of variable (10.3.3.4), (10.3.3.6) converts the original equation (10.3.3.3) into the equation with a quadratic nonlinearity

$$ww''_{xx} + \frac{n}{1-n}(w'_x)^2 - cww_x = a(1-n)w^2 + b(1-n)w + d(1-n), \tag{10.3.3.7}$$

which is more convenient for analysis.

1.1. Let us supplement equation (10.3.3.7) with the linear differential constraint

$$w'_x = \alpha w + \beta. \tag{10.3.3.8}$$

We use this relation to eliminate the derivatives in (10.3.3.7) to obtain a quadratic equation for w, which is satisfied identically if the conditions (determining system of algebraic equations)

$$\frac{1}{1-n}\alpha^2 - c\alpha = a(1-n), \quad \frac{1+n}{1-n}\alpha\beta - c\beta = b(1-n), \quad n\beta^2 = d(1-n)^2$$

hold. The first and last equations give two pairs of solutions each,

$$\alpha_{1,2} = \frac{1}{2}(1-n)\big(c \pm \sqrt{c^2+4a}\big), \quad \beta_{1,2} = \pm\frac{1}{1-n}\sqrt{\frac{d}{n}}, \tag{10.3.3.9}$$

which are then substituted into the second equation. As a result, for each pair α_i, β_j we obtain one constraint (not written out here) that connects the parameters a, b, c, d, and n.

It is apparent from (10.3.3.9) that for $a = 0$, equation (10.3.3.7) admits a simple, degenerate first-order differential constraint

$$w'_x = \beta = \text{const}.$$

1.2. For $c = 0$, equation (10.3.3.7) can be supplemented with the differential constraint

$$(w'_x)^2 = \alpha w^2 + \beta w + \gamma. \tag{10.3.3.10}$$

A simple analysis shows that the constraint coefficients in (10.3.3.10) are expressed in terms of the equation coefficients in (10.3.3.3) as follows:

$$\alpha = \frac{a(1-n)^2}{2-n}, \quad \beta = b(1-n)^2, \quad \gamma = \frac{d(1-n)^2}{n}.$$

Remark 10.12. For $a = c = 0$, equation (10.3.3.7) admits a solution of the form $w = Ax^2 + Bx + C$.

Remark 10.13. For $n = -1/3$, a particular solution of equation (10.3.3.7) can be obtained with the help of the differential constraint

$$w'_x = \alpha w + \beta\sqrt{w} + \gamma.$$

Case 2. Suppose that

$$p = \frac{1}{n-1} \quad (k = 2). \tag{10.3.3.11}$$

The change of variable (10.3.3.4), (10.3.3.11) converts the original equation (10.3.3.3) into the equation with a nonlinearity of fourth degree

$$ww''_{xx} + \frac{2-n}{n-1}(w'_x)^2 - cww_x = a(n-1)w^2 + b(n-1)w^3 + d(n-1)w^4, \tag{10.3.3.12}$$

We look for particular solutions to equation (10.3.3.7) using the quadratic differential constraint

$$w'_x = \alpha w^2 + \beta w + \gamma. \tag{10.3.3.13}$$

We use this relation to eliminate the derivatives from (10.3.3.12) to obtain an equation of fourth degree for w. Equating its coefficients to zero results in the algebraic system

$$\begin{aligned}
&n\alpha^2 = d(n-1)^2,\\
&\frac{(n+1)\alpha\beta}{n-1} - c\alpha = b(n-1),\\
&\frac{\beta^2 + 2\alpha\gamma}{n-1} - c\beta = a(n-1),\\
&\gamma\left(\frac{3-n}{n-1}\beta - c\right) = 0,\\
&(2-n)\gamma^2 = 0.
\end{aligned} \tag{10.3.3.14}$$

The cases $\gamma = 0$ and $n = 2$ need to be considered; these correspond to solutions of the last equation.

2.1. For $\gamma = 0$, we determine the original coefficients and a particular solution using the first, second, and fourth equations of (10.3.3.14) as well as the differential constraint (10.3.3.13):

$$\alpha = \pm(n-1)\sqrt{\frac{d}{n}}, \quad \beta = \frac{n-1}{n+1}\left(c \pm b\sqrt{\frac{n}{d}}\right), \quad \gamma = 0, \quad w = -\frac{\beta}{\alpha + Ce^{-\beta x}}, \tag{10.3.3.15}$$

where C is an arbitrary constant. The third equation of (10.3.3.14) defines a necessary relation between the coefficients of the equation of interest:

$$a = \frac{1}{(n+1)^2}\left(c \pm b\sqrt{\frac{n}{d}}\right)\left(-nc \pm b\sqrt{\frac{n}{d}}\right).$$

Either the upper or lower signs must be taken in the above formulas.

2.2. For $n = 2$, the coefficients of the differential constraint are determined from the first, third, and fourth equations of (10.3.3.14):

$$\alpha = \pm\sqrt{d/2}, \quad \beta = c, \quad \gamma = \pm\frac{a}{\sqrt{2d}}. \tag{10.3.3.16}$$

The second equation defined the relation between the equation coefficients: $b = \pm c\sqrt{2d}$ (either the upper or lower signs must be taken in all formulas). The desired solution is determined by integrating the separable equation (10.3.3.13) taking into account (10.3.3.16).

Case 3. On setting $k = 3$ in (10.3.3.5), we can look for a solution in the form $w = a_0 + a_1Q + a_2Q^2$ with $Q'_x = b_2Q^2 + b_1Q + b_0$.

Example 10.19. Consider the second-order ordinary differential equation with an exponential nonlinearity

$$y''_{xx} - cy'_x = a + be^{\lambda y} + de^{2\lambda y}. \tag{10.3.3.17}$$

The linking dependence (10.3.3.1) will be taken in the logarithmic form

$$y = \frac{k}{\lambda} \ln w, \tag{10.3.3.18}$$

with the coefficient k to be determined. Substituting (10.3.3.18) into (10.3.3.17) and multiplying by λw^2, we obtain

$$kww''_{xx} - k(w'_x)^2 - ckww'_x = a\lambda w^2 + b\lambda w^{k+2} + d\lambda w^{2k+2}. \tag{10.3.3.19}$$

Let us look at a few possibilities of choosing the coefficient k that allow us to find exact solutions.

Case $k=-1$. The substitution (10.3.3.18) with $k=-1$ converts the original equation (10.3.3.17) into the equation with a quadratic nonlinearity

$$ww''_{xx} - (w'_x)^2 - cww_x = -a\lambda w^2 - b\lambda w - d, \tag{10.3.3.20}$$

which only differs from equation (10.3.3.7) in coefficients. The differential constraint (10.3.3.8) allows one to find a particular solution to equation (10.3.3.20) (details are omitted).

Remark 10.14. For $a = c = 0$, equation (10.3.3.20) admits a solution of the form $w = Ax^2 + Bx + C$.

Case $k = 1$. The substitution (10.3.3.18) with $k = 1$ converts the original equation (10.3.3.17) into the equation with a quartic nonlinearity

$$ww''_{xx} - (w'_x)^2 - cww'_x = a\lambda w^2 + b\lambda w^3 + d\lambda w^4, \tag{10.3.3.21}$$

which only differs from equation (10.3.3.12) in coefficients. The differential constraint (10.3.3.13) allows one to find a particular solution to equation (10.3.3.21) (details are omitted).

► Modification of the solution-seeking procedure.

If the differential constraint (10.3.3.2) can be solved for the derivative,

$$w'_x = h(w; \mathbf{b}), \tag{10.3.3.22}$$

then a different order of actions may be more convenient.

By differentiating relation (10.3.3.22) repeatedly and expressing the derivatives via w, one obtains relations of the form $w_x^{(k)} = \varphi_k(w; \mathbf{b})$. Then, on substituting (10.3.3.1) into the equation of interest, one eliminates the derivatives with the help of (10.3.3.22).

Most frequently, one uses differential constraints of the form (10.3.3.22). These represent separable Riccati or Bernoulli equations.

Example 10.20. Example 10.9 considered previously demonstrates the work of this method for the original equation taken in the form (10.2.3.13), relation (10.3.3.1) taken in the form of a finite sum (10.2.3.6) with $w \equiv Q$, and the differential constraint (10.3.3.22) taken in the form of the Bernoulli equation (10.2.3.9).

10.3.4 Using Several Differential Constraints. G'/G-Expansion Method and Simplest Equation Method

► **Using several differential constraints.**

In some situations, the equation under study is supplemented with several differential constraints containing additional unknown constants. To be specific, let us return to the nth-order autonomous equation (10.3.1.1) and supplement it with two first-order differential constraints

$$y = G(w, w'_x; \mathbf{b}), \tag{10.3.4.1}$$

$$H(w, w'_x; \mathbf{c}) = 0, \tag{10.3.4.2}$$

where **b** and **c** are vectors of free parameters. On substituting (10.3.4.1) into (10.3.1.1), one obtains an $(n+1)$st-order equation for $w = w(x)$:

$$F_1(w, w'_x, \dots, w_x^{(n+1)}; \mathbf{b}, \mathbf{c}) = 0. \tag{10.3.4.3}$$

This equation in conjunction with the differential constraint (10.3.4.2) is analyzed with the method outlined in Sections 10.3.1 and 10.2.3. There is an insignificant distinction that the order of equation (10.3.4.3) is higher than that of the original equation (10.3.1.1).

Remark 10.15. The differential constraints (10.3.4.1) and (10.3.4.2) can involve higher derivatives of w with respect to x (see below).

► ***G'/G*-expansion method.**

With the G'/G-expansion method, one looks for particular solutions to autonomous equations using two differential constraints (10.3.4.1) and (10.3.4.2) of the special form

$$y = \sum_{k=0}^{n} b_k \left(\frac{w'_x}{w}\right)^k, \tag{10.3.4.4}$$

$$w''_{xx} - c_1 w'_x - c_0 w = 0. \tag{10.3.4.5}$$

Remark 10.16. In the original papers by Wang, Li, and Zhang (2008) and subsequent publications, the notation $w = G$ was used.

It was shown by Kudryashov (2010) that searching for particular solutions of an ODE with the G'/G-expansion method based on the differential constraints (10.3.4.4)–(10.3.4.5) with $c_1^2 + 4c_0 > 0$ leads to the same results as the tanh method (see Section 10.2.2). For $c_1^2 + 4c_0 < 0$, the G'/G-expansion method is equivalent to the tan method.

For specific examples of how to use the G'/G-expansion method, see the list of literature cited at end of this section.

► **Simplest equation method.**

The simplest equation method, devised by Kudryashov (2005) for seeking particular solutions, is equivalent to using two differential constraints. In his papers, Kudryashov chose the differential constraint (10.3.4.2) in one of the following three forms:

$$w_x + w^2 - c_1 w - c_2 = 0, \qquad (10.3.4.6)$$

$$(w'_x)^2 - 4w^3 - c_1 w^2 - c_2 w - c_3 = 0, \qquad (10.3.4.7)$$

$$(w'_x)^2 - w^4 - c_1 w^3 - c_2 w^2 - c_3 w - c_4 = 0. \qquad (10.3.4.8)$$

The differential constraint (10.3.4.1) was chosen from the class of functions

$$y = \sum_{k=0}^{K} b_{1k} w^k + w'_x \sum_{l=0}^{L} b_{2l} w^l + \sum_{m=1}^{M} b_{3m} \left(\frac{w'_x}{w}\right)^m. \qquad (10.3.4.9)$$

For the differential constraint (10.3.4.6), it was assumed that $K = M$ and $b_{2l} = 0$ ($l = 1, \dots, L$) in (10.3.4.9). This resulted in a number of new exact solutions to nonlinear second-, third-, and fourth-order equations.

Remark 10.17. All equations (10.3.4.6)–(10.3.4.8) are reduced to separable equations, whose solutions are expressed in terms of elementary functions or/and integrals of elementary functions. A solution to (10.3.4.7) can be expressed through the Weierstrass function $\wp = \wp(z, g_2, g_3)$. A solution to equation (10.3.4.8) is expressed through the Jacobi elliptic function.

For specific examples of how to use the simplest equation method, see the articles cited below.

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