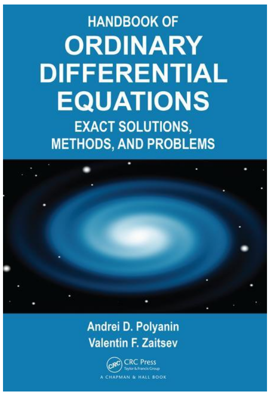

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

Andrei D. Polyanin, Valentin F. Zaitsev

Chapter 11: Group Methods for ODES

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Andrei D. Polyanin, Valentin F. Zaitsev
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Chapter 11

Group Methods for ODEs

11.1 Lie Group Method. Point Transformations

11.1.1 Local One-Parameter Lie Group of Transformations. Invariance Condition

► **Preliminary remarks.**

The Lie group method for ordinary differential equations presents a routine procedure that allows obtaining the following:

(i) transformations under which differential equations are invariant (such transformations bring the given equation to itself);

(ii) new variables (dependent and independent) in which differential equations become considerably simpler (so that the resulting equation can be completely integrated or has a lower order than the original equation).

Remark 11.1. The Lie group method for ordinary differential equations may be treated as a significant extension of the method outlined in Section 9.3.

► **Local one-parameter Lie group of transformations. Infinitesimal operator.**

Here, we examine transformations of the ordinary differential equation

$$y_x^{(n)} = F\bigl(x, y, \dots, y_x^{(n-1)}\bigr). \tag{11.1.1.1}$$

Consider the set of transformations

$$\mathrm{T}_\varepsilon = \begin{cases} \bar{x} = \varphi(x, y, \varepsilon), & \bar{x}|_{\varepsilon=0} = x, \\ \bar{y} = \psi(x, y, \varepsilon), & \bar{y}|_{\varepsilon=0} = y, \end{cases} \tag{11.1.1.2}$$

where φ, ψ are smooth functions of their arguments and ε is a real parameter. The set T_ε is called a *continuous one-parameter Lie group of point transformations* if, for any ε_1 and ε_2, the following relation holds:

$$\mathrm{T}_{\varepsilon_1} \circ \mathrm{T}_{\varepsilon_1} = \mathrm{T}_{\varepsilon_1+\varepsilon_2}, \tag{11.1.1.3}$$

i.e., consecutive application of two transformations of the form (11.1.1.1) with parameters ε_1 and ε_2 is equivalent to a single transformation of the same form with parameter $\varepsilon_1 + \varepsilon_2$.

In what follows, we consider local continuous one-parameter Lie groups of point transformations (briefly called point groups) corresponding to an infinitesimal transformation (11.1.1.2) for $\varepsilon \to 0$. Taylor's expansion of $\bar{x}$ and $\bar{y}$ in (11.1.1.2) with respect to the parameter ε about $\varepsilon = 0$ yields:

$$\bar{x} \simeq x + \xi(x,y)\varepsilon, \quad \bar{y} \simeq y + \eta(x,y)\varepsilon, \tag{11.1.1.4}$$

where

$$\xi(x,y) = \frac{\partial \varphi(x,y,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}, \quad \eta(x,y) = \frac{\partial \psi(x,y,\varepsilon)}{\partial \varepsilon}\Big|_{\varepsilon=0}.$$

At each point (x,y), the vector (ξ,η) is tangent to the curve described by the transformed points $(\bar{x},\bar{y})$.

S. LIE THEOREM. *Let the functions φ and ψ satisfy the group property* (11.1.1.3) *and allow the expansion* (11.1.1.4). *Then, these are solutions to the system of first-order ordinary differential equations (known as the Lie equations)*

$$\frac{d\varphi}{d\varepsilon} = \xi(\varphi,\psi), \qquad \frac{d\psi}{d\varepsilon} = \eta(\varphi,\psi) \tag{11.1.1.5}$$

subject to the initial conditions (11.1.1.2). *Conversely, for any smooth vector field (ξ,η), a solution to the Cauchy problem* (11.1.1.5), (11.1.1.2)*), which exists and is unique, satisfies the group property* (11.1.1.3).

The first-order linear differential operator

$$\mathrm{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} \tag{11.1.1.6}$$

corresponding to the infinitesimal transformation (11.1.1.4), is called the *infinitesimal operator* (or *infinitesimal generator*) of the group.

By definition, the *universal invariant* (briefly, invariant) of the group (11.1.1.2) and the operator (11.1.1.6) is a function $I_0(x,y)$, satisfying the condition

$$I_0(\bar{x},\bar{y}) = I_0(x,y).$$

Taylor's expansion with respect to the small parameter ε yields the following linear partial differential equation for I_0:

$$\mathrm{X}I_0 = \xi(x,y)\frac{\partial I_0}{\partial x} + \eta(x,y)\frac{\partial I_0}{\partial y} = 0. \tag{11.1.1.7}$$

► **Prolonged operator. Invariance condition and mth-order differential invariant.**

Equation (11.1.1.1) will be treated as a relation for $n+2$ variables x, y, y'_x, $\ldots$, $y_x^{(n)}$ with the differential constraints

$$y_x^{(k+1)} = \frac{dy^{(k)}}{dx}. \tag{11.1.1.8}$$

The space of these $n+2$ variables is called the space of nth prolongation; and in order to work with differential equations, one has to define the action of operator (11.1.1.6) on the "new" variables $y'_x, \ldots, y_x^{(n)}$, taking into account the differential constraints (11.1.1.8). For example, let us calculate the infinitesimal transformation of the first derivative. We have

$$\frac{d\bar{y}}{d\bar{x}} = \frac{D_x(y+\eta\varepsilon)}{D_x(x+\xi\varepsilon)} \simeq \frac{y'_x + (\eta_x + \eta_y y'_x)\varepsilon}{1 + (\xi_x + \xi_y y'_x)\varepsilon},$$
$$D_x = \frac{\partial}{\partial x} + y'_x \frac{\partial}{\partial y} + y''_{xx}\frac{\partial}{\partial y'_x} + \cdots,$$

where D_x is called the *operator of total derivative*. Expanding the right-hand side into a power series with respect to the parameter ε and preserving the first-order terms, we obtain

$$\bar{y}'_{\bar{x}} \simeq y'_x + \zeta_1(x, y, y'_x)\varepsilon,$$

where

$$\zeta_1 = \eta_x + (\eta_y - \xi_x)y'_x - \xi_y (y'_x)^2 = D_x(\eta) - y'_x D_x(\xi).$$

The action of the group on higher-order derivatives is determined by the recurrence formula:

$$\zeta_{k+1} = D_x(\zeta_k) - y_x^{(k+1)} D_x(\xi).$$

To a prolonged group there corresponds a *prolonged operator*:

$$\underset{n}{\mathrm{X}} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y} + \sum_{k=1}^{n} \zeta_k\bigl(x, y, y', \ldots, y_x^{(k)}\bigr)\frac{\partial}{\partial y_x^{(k)}}. \qquad (11.1.1.9)$$

The ordinary differential equation (11.1.1.1) admits the group (11.1.1.2) if

$$\underset{n}{\mathrm{X}}\bigl[y_x^{(n)} - F\bigl(x, y, y'_x, \ldots, y_x^{(n-1)}\bigr)\bigr]\big|_{y_x^{(n)}=F} = 0. \qquad (11.1.1.10)$$

Relation (11.1.1.10) is called the *invariance condition*.

Remark 11.2. The invariant I_0, which is a solution of equation (11.1.1.7), also satisfies the equation $\underset{n}{\mathrm{X}} I_0 = 0$.

By definition, an *mth-order differential invariant* of the operator X is a function $I_m = I_m\bigl(x, y, y'_x, \ldots, y_x^{(m)}\bigr)$, satisfying the linear partial differential equation $\underset{m}{\mathrm{X}} I_m = 0$ with the operator $\underset{m}{\mathrm{X}}$ defined by (11.1.1.9).

► Inverse problem.

In solving different modeling problems, it is required to construct a model equation that satisfies certain a priori conditions, for example, symmetry laws. Two different statements of the problem are possible here.

1°. Suppose there is a preset symmetry defined by the operator (11.1.1.6), with the coordinates $\xi(x,y)$ and $\eta(x,y)$ defined explicitly as specific functions. It is required to compute a universal invariant $I_0(x,y)$ and a first differential invariant $I_1(x, y, y'_x)$ of the operator (11.1.1.6). The class of nth-order equations that admit the operator (11.1.1.6) is given by the formula

$$\Phi\left(I_0, I_1, \frac{dI_1}{dI_0}, \ldots, \frac{d^{n-1}I_1}{d^{n-1}I_0}\right) = 0.$$

Thus, problem 1° is always solvable as long as two first invariants of the operator are known, with this being also valid for nonpoint operators.

2°. Suppose there is a preset symmetry defined by the class of operators (11.1.1.6), with the coordinates $\xi(x,y)$ and $\eta(x,y)$ being arbitrary functions. A solution to the inverse problem is a class of equations of a given order that admit an arbitrary operator of the form (11.1.1.6). The universal method is the use of the similarity principle of one-parameter Lie groups of point transformations. Since any autonomous equation

$$y_x^{(n)} = \mathcal{F}\big(y, y'_x, \ldots, y_x^{(n-1)}\big)$$

admits translations along the x-axis (i.e., operator $\mathrm{X} = \partial_x$), the arbitrary invertible point transformation $x = f(t,u)$, $y = g(t,u)$ produces a general class on nth-order equations with two-functional arbitrariness that admit a certain point operator. See Section 11.1.2 for examples.

11.1.2 Group Analysis of Second-Order Equations

► Structure of an admissible operator for second-order equations.

For second-order nonlinear equations

$$y''_{xx} = F(x, y, y'_x), \tag{11.1.2.1}$$

the invariance condition (11.1.1.10) is written in the form

$$\begin{aligned}&\eta_{xx} + (2\eta_{xy} - \xi_{xx})y'_x + (\eta_{yy} - 2\xi_{xy})(y'_x)^2 - \xi_{yy}(y'_x)^3\\ &\qquad = (2\xi_x - \eta_y + 3\xi_y y'_x)F + \xi F_x + \eta F_y + [\eta_x + (\eta_y - \xi_x)y'_x - \xi_y(y'_x)^2]F_{y'_x},\end{aligned}$$

where $F = F(x, y, y'_x)$. This condition is in fact a second-order partial differential equation for two unknown functions $\xi(x,y)$ and $\eta(x,y)$. Since the unknown functions do not depend on the derivative y'_x, this equation can be represented (after F has been expanded in a power series with respect to y'_x, unless it is already a polynomial) in the form

$$\sum_{k=0}^{\infty} \Phi_k (y'_x)^k = 0, \tag{11.1.2.2}$$

with the Φ_k independent of y'_x. In order to ensure that condition (10) holds identically, one should set $\Phi_k = 0$, $k = 0, 1, \ldots$ Thus, the invariance condition for a second-order equation can be "split" and represented as a system of equations (whose number can generally be infinite).

► Illustrative examples.

Example 11.1. If $F = F(x,y)$, i.e., the right-hand side of equation (11.1.2.1) does not depend on y'_x, then the determining equation can be "split" and represented as the system:

$$\begin{aligned}&\xi_{yy} = 0,\\ &\eta_{yy} - 2\xi_{xy} = 0,\\ &2\eta_{xy} - \xi_{xx} - 3F(x,y)\xi_y = 0,\\ &\eta_{xx} + (\eta_y - 2\xi_x)F(x,y) - F_x(x,y)\xi - F_y(x,y)\eta = 0.\end{aligned}$$

From the first two equations we find that

$$\xi = a(x)y + b(x), \quad \eta = a'(x)y^2 + c(x)y + d(x),$$

where $a(x)$, $b(x)$, $c(x)$, and $d(x)$ are arbitrary functions. Substituting these expressions into the third and the fourth equations, we get

$$\begin{aligned} &3a''y + 2c' - b'' - 3F(x,y)a = 0, \\ &a'''y^2 + c''y + d'' + (c - 2b')F - (ay + b)F_x - (a'y^2 + cy + d)F_y = 0. \end{aligned} \tag{11.1.2.3}$$

In what follows, it is assumed that the function $F(x, y)$ is nonlinear with respect to the second argument. Then from the first equation in (11.1.2.3), we find that $a = 0$ and $c = \frac{1}{2}b' + \alpha$, where α is an arbitrary constant. The second equation in (11.1.2.3) becomes

$$\tfrac{1}{2}b'''y + d'' + \big(\alpha - \tfrac{3}{2}b'\big)F - bF_x - \big[\big(\tfrac{1}{2}b' + \alpha\big)y + d\big]F_y = 0. \tag{11.1.2.4}$$

Equation (11.1.2.4) enables us to solve two different problems.

1°. If the function $F(x, y)$ is given, then, splitting equation (11.1.2.4) with respect to powers of y (the unknown functions b and d are independent of y), we obtain a new system, from which b, d, and α can be found; i.e., we ultimately obtain an admissible operator.

2°. Assuming that the functions b, d and the constant α are known but arbitrary, one can regard relation (11.1.2.4) as an equation for the unknown function $F(x, y)$. Solving this equation, we obtain a class of equations admitting a point operator. Thus, problem 2° is stated as an inverse problem.

Example 11.2. Let $F(x, y) = Ax^n y^m$, i.e., we are dealing with the Emden–Fowler equation $y''_{xx} = Ax^n y^m$. Then equation (11.1.2.4) becomes

$$\tfrac{1}{2}b'''y + d'' + \big(\alpha - \tfrac{3}{2}b'\big)Ax^n y^m - bnAx^{n-1}y^m - \big[\big(\tfrac{1}{2}b' + \alpha\big)y + d\big]mAx^n y^{m-1} = 0.$$

This relation must be satisfied identically by any function $y = y(x)$, and therefore, the coefficients of different powers of y must be equal to zero. As a result, we obtain a new system whose structure essentially depends on the value of m.

1°. It was assumed above that $F(x, y)$ is nonlinear in its second argument, and therefore, $m \neq 0$ and $m \neq 1$. Let $m \neq 2$. Then the system has the form:

$$\begin{aligned} &d'' = 0, \\ &b''' = 0, \\ &d = 0, \\ &\big[\alpha(1 - m) - \tfrac{1}{2}(3 - m)b'\big]x - nb = 0. \end{aligned}$$

It follows that $d = 0$ and $b(x) = b_2x^2 + b_1x + b_0$, and the last equation of the system can be written in the form

$$(m + n + 3)b_2x^2 + \big[\tfrac{1}{2}(m + 2n + 3)b_1 + \alpha(m - 1)\big]x + nb_0 = 0. \tag{11.1.2.5}$$

To ensure relation (11.1.2.5), we equate all coefficients of this quadratic trinomial to zero to obtain

$$(m + n + 3)b_2 = 0, \quad \tfrac{1}{2}(m + 2n + 3)b_1 + \alpha(m - 1) = 0, \quad nb_0 = 0. \tag{11.1.2.6}$$

Analysis of system (11.1.2.6) yields solutions of the determining system corresponding to three different operators:

$$\begin{aligned} &\mathrm{X}_1 = (m - 1)x\partial_x - (n + 2)y\partial_y && \text{if } n \text{ and } m \text{ are arbitrary}, \\ &\mathrm{X}_2 = \partial_x && \text{if } n = 0, \\ &\mathrm{X}_3 = x^2\partial_x + xy\partial_y && \text{if } m + n + 3 = 0. \end{aligned}$$

2°. Let $m = 2$. Then equation (11.1.2.4) becomes

$$d'' + \big(\tfrac{1}{2}b''' - 2Adx^n\big)y - \big[\big(\tfrac{5}{2}b' + \alpha\big)x + nb\big]Ax^{n-1}y^2 = 0.$$

Equating the term d'' and the coefficient of y in parentheses to zero, we get

$$d(x) = d_1 x + d_0,$$
$$b(x) = \frac{4ad_1x^{n+4}}{(n+2)(n+3)(n+4)} + \frac{4ad_0x^{n+3}}{(n+1)(n+2)(n+3)} + b_2x^2 + b_1x + b_0,$$

where $n \neq -1, -2, -3, -4$. The expression in square brackets (the coefficient of y^2) can be split with respect to powers of x and we obtain an algebraic system which, to within nonzero coefficients, has the form:

$$(7n+20)d_1 = 0,$$
$$(7n+15)d_0 = 0,$$
$$(n+5)b_2 = 0,$$
$$(2n+5)b_1 + 2\alpha = 0,$$
$$nb_0 = 0.$$

The last three equations coincide with the corresponding equations of system (11.1.2.6), whose solutions are already known. The first two equations yield two cases of prolongation of the admissible group:

$$\mathrm{X}_1 = 343Ax^{8/7}\partial_x + 4\big(49Ax^{1/7}y - 3x\big)\partial_y \quad \text{if } n = -\tfrac{20}{7},$$
$$\mathrm{X}_2 = 343Ax^{6/7}\partial_x + 3\big(49Ax^{-1/7}y + 4\big)\partial_y \quad \text{if } n = -\tfrac{15}{7}.$$

Example 11.3. Let us look at the inverse problem of Item 2° in Example 11.1. The solution of equation (11.1.2.4) is

$$F = b^{-3/2}E\left\{\Phi(u) + \int \left[\frac{1}{2}bb'''(u+V) + b^{1/2}d''E^{-1}\right]dx\right\}, \tag{11.1.2.7}$$

where $b(x)$ and $d(x)$ are arbitrary functions, Φ is an arbitrary function of its argument,

$$u = b^{-1/2}E^{-1}y - V, \quad V = \int b^{-3/2}dE^{-1}\,dx, \quad E = \exp\left(\alpha\int\frac{dx}{b(x)}\right),$$

and α is an arbitrary constant. The integral in formula (11.1.2.7) can be expressed in terms of V and E as

$$F = b^{-3/2}E\left[\Phi(u) + \alpha^2V\right] + \frac{2bb'' - (b')^2}{4b^2}y + \frac{2bd' - b'd + 2\alpha d}{2b^2}.$$

A similar method is always used to solve the inverse problem for the equation of arbitrary (nth) order

$$y_x^{(n)} = \mathcal{F}\big(x, y, y'_x, \ldots, y_x^{(n-2)}\big),$$

provided that the right-hand side $\mathcal{F}$ does not contain the derivative $y_x^{(n-1)}$.

Example 11.4. Consider the problem from the previous example for the general second-order equation

$$y''_{xx} = F(x, y, y'_x). \tag{11.1.2.8}$$

Obviously, the most general class of equations admitting the translation group along the $\widetilde{x}$-axis is a subclass of autonomous equations from the class (11.1.2.8); specifically,

$$\widetilde{y}''_{\widetilde{x}\widetilde{x}} = F(\widetilde{y}, \widetilde{y}'_{\widetilde{x}}). \tag{11.1.2.9}$$

The translation operator $\mathrm{X} = \partial_{\widetilde{x}}$ can be converted into any operator of the form $\mathrm{X} = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y$ by the point transformation

$$\widetilde{x} = \varphi(x,y), \quad \widetilde{y} = \psi(x,y), \quad \varphi_x\psi_y - \varphi_y\psi_x \neq 0. \tag{11.1.2.10}$$

It is clear that the substitution of (11.1.2.10) into equation (11.1.2.9) results in a subclass of *all* equations (11.1.2.8) admitting a point operator:

$$(\varphi_x\psi_y-\psi_x\varphi_y)y''_{xx}+(\varphi_y\psi_{yy}-\psi_y\varphi_{yy})(y'_x)^3+(\varphi_x\psi_{yy}-\psi_x\varphi_{yy}+ \\ +2\varphi_y\psi_{xy}-2\psi_y\varphi_{xy})(y'_x)^2+(\varphi_y\psi_{xx}-\psi_y\varphi_{xx}+2\varphi_x\psi_{xy}-2\psi_x\varphi_{xy})y'_x+ \\ +\varphi_x\psi_{xx}-\psi_x\varphi_{xx}=(\varphi_x+\varphi_y y'_x)^3 F\left(\psi,\frac{\psi_x+\psi_y y'_x}{\varphi_x+\varphi_y y'_x}\right).$$

11.1.3 Utilization of Local Groups for Reducing the Order of Equations and Their Integration

Suppose that an ordinary differential equation (11.1.1.1) admits an infinitesimal operator X of the form (11.1.1.6). Then the order of the equation can be reduced by one. Below we describe two methods for reducing the order of ODEs.

► **First method for reducing the order of equations.**

The transformation

$$t=f(x,y),\qquad u=g(x,y),\tag{11.1.3.1}$$

with f and g ($g\not\equiv 0$) being arbitrary particular solutions of the first-order linear partial differential equations

$$\begin{aligned}\xi(x,y)\frac{\partial f}{\partial x}+\eta(x,y)\frac{\partial f}{\partial y}&=k,\\ \xi(x,y)\frac{\partial g}{\partial x}+\eta(x,y)\frac{\partial g}{\partial y}&=0,\end{aligned}\tag{11.1.3.2}$$

reduces equation (11.1.1.1) to an autonomous equation (the constant $k\neq 0$ can be chosen arbitrarily). The function $g=g(x,y)$ is a universal invariant of the operator X.

Suppose that the general solution of the characteristic equation

$$\frac{dx}{\xi(x,y)}=\frac{dy}{\eta(x,y)}$$

has the form

$$U(x,y)=C,$$

where C is an arbitrary constant. Then the general solutions of equations (11.1.3.2) are given by (see A. D. Polyanin, V. F. Zaitsev, and A. Moussiaux, 2002):

$$\begin{aligned}f&=k\int\frac{dx}{\bar\xi(x,U)}+\Psi_1(U),\\ g&=\Psi_2(U),\qquad U=U(x,y),\end{aligned}$$

where $\Psi_1(U)$ and $\Psi_2(U)$ are arbitrary functions, $\bar\xi\big(x,U(x,y)\big)\equiv\xi(x,y)$, and U in the integral is regarded as a parameter.

Example 11.5. The Emden–Fowler equation $y''_{xx}=Ax^{-15/7}y^2$ admits the operator (cf. the operator X_2 in Item 2° of Example 11.2):

$$X=\xi(x,y)\frac{\partial}{\partial x}+\eta(x,y)\frac{\partial}{\partial y},\quad\text{where}\quad \xi(x,y)=343Ax^{6/7},\ \ \eta(x,y)=147Ax^{-1/7}y+12.$$

Equations (11.1.3.2) for $k = 49A$ admit the particular solutions

$$f = x^{1/7}, \qquad g = x^{-3/7}y + \frac{6}{49A}x^{-2/7}.$$

Solving (11.1.3.1) for x and y, we obtain the transformation

$$x = t^7, \qquad y = t^3 u - \frac{6}{49A}t,$$

which reduces the original equation to the autonomous equation

$$u''_{tt} = 49Au^2,$$

which can easily be integrated by quadrature.

► **Second method for reducing the order of equations.**

Suppose that we know two invariants of the admissible operator X:

$$I_0 = I_0(x, y) \qquad \text{(universal invariant)}, \tag{11.1.3.3}$$

$$I_1 = I_1(x, y, y'_x) \qquad \text{(first differential invariant)}. \tag{11.1.3.4}$$

Then the second differential invariant can be found by differentiation,

$$I_2(x, y, y'_x, y''_{xx}) = \frac{dI_1}{dI_0}, \tag{11.1.3.5}$$

where $dI_m = (D_x I_m)\, dx$. Using (11.1.3.4)–(11.1.3.5), let us eliminate the derivatives y'_x and y''_{xx} from the original equation and take into account relation (11.1.3.3). Thus we obtain the first-order equation:

$$\frac{dI_1}{dI_0} = G(I_0, I_1).$$

Example 11.6. The Emden–Fowler equation $y''_{xx} = Ax^{-6}y^3$ admits an operator whose first prolongation has the form:

$$\underset{1}{\mathrm{X}} = x^2\partial_x + xy\partial_y + (y - xy')\partial_{y'}.$$

This operator admits the invariants:

$$I_0 = y/x, \qquad I_1 = xy'_x - y, \tag{11.1.3.6}$$

which form an integral basis of the first-order linear partial differential equation

$$x^2\frac{\partial I}{\partial x} + xy\frac{\partial I}{\partial y} + (y - xy')\frac{\partial I}{\partial y'} = 0.$$

Using (11.1.3.5) and (11.1.3.6), we find the second invariant:

$$I_2 = \frac{dI_1}{dI_0} = \frac{x^3 y''_{xx}}{xy'_x - y}. \tag{11.1.3.7}$$

Let us express the unknown function and its derivatives from (11.1.3.6)–(11.1.3.7) to obtain

$$y = ux, \quad y'_x = \frac{ux + w}{x}, \quad y''_{xx} = \frac{ww'_u}{x^3}, \qquad \text{where} \quad u = I_0, \;\; w = I_1.$$

Substituting these expressions into the original equation, we see that the variable x is canceled and the equation takes the form

$$ww'_u = Au^3,$$

i.e., it becomes a first-order separable equation.

11.1.4 Seeking Particular Solutions

Particular solutions can be sought using Marius Sophus Lie's method, since the admitted group converts a solution of the equation into another solution. Therefore, a particular solution that is not invariant under this group generates a one-parameter family of particular solutions. Under the same condition, a particular solution to a first-order equation generates the general solution as a function of the group parameter a, which can be treated as an arbitrary constant.

Example 11.7. Consider the equation

$$y''_{xx} = Ax^{-15/7}y^2. \tag{11.1.4.1}$$

Its general solution is written in parametric form as

$$x = \alpha C_1^7\tau^7, \quad y = \beta C_1\tau(\tau^2\wp \mp 1), \qquad A = \pm\frac{6}{49}\alpha^{1/7}\beta^{-1}, \tag{11.1.4.2}$$

where $\wp$ is the Weierstrass function (see also 14.3.1.20). Equation (11.1.4.1) can be integrated by the classical group method, since it admits a two-dimensional point Lie algebra with operators

$$\mathrm{X}_1 = 7x\partial_x + y\partial_y, \quad \mathrm{X}_2 = 343x^{6/7}\partial_x + 3\left(49Ax^{-1/7}y + 4\right)\partial_y.$$

The finite-group of transformations for the operator X_2 is given by

$$\widetilde{x} = \left(49Aa + x^{1/7}\right)^7, \quad \widetilde{y} = \frac{49^2A^2\left(49Ay + 6x^{1/7}\right)}{x^{3/7}}\left(a + \frac{x^{1/7}}{49A}\right)^3 - 6\left(a + \frac{x^{1/7}}{49A}\right). \tag{11.1.4.3}$$

The Emden–Fowler equation, with (11.1.4.1) being its special case, has a particular solution in the form of a power-law function:

$$y_0 = -\frac{6}{49A}x^{1/7}. \tag{11.1.4.4}$$

Solution (11.1.4.4) is invariant under the group determined by the operator X_1. However, it turns out that the solution is also invariant under the transformations (11.1.4.3). Consequently, solution (11.1.4.4) is unsuitable for multiplication.

Example 11.8. Note that equation (11.1.4.1) has the trivial (zero) solution $y = 0$. It is also invariant under the operator X_1, but not under the operator X_2. This allows us to construct a one-parameter family of particular solutions to equation (11.1.4.1) by applying transformation (11.1.4.3) to y:

$$y = \frac{6}{49A}\left(\frac{x^{3/7}}{\left(49AC + x^{1/7}\right)^2} - x^{1/7}\right). \tag{11.1.4.5}$$

The group parameter a has been replaced with the arbitrary constant C. It is noteworthy that the family of algebraic functions (11.1.4.5) appears in the general solution (11.1.4.2), which is not obvious.

Thus, there is a simple algorithm for seeking families of particular solutions to broad classes of ordinary differential equations admitting a certain operator. As initial solutions, it is the easiest to use simple solutions such as constants, which are easy to verify.

If a differential equation has the zero solution ($y = 0$) and admits the operator

$$\mathrm{X} = \xi(x, y)\partial_x + \eta(x, y)\partial_y,$$

the condition $\eta \neq 0$ must hold for the solution to be suitable for multiplication to obtain a one-parameter family of particular solutions.

Remark 11.3. Unfortunately, the above condition is not sufficient for the construction of a nontrivial family of solutions. Indeed, any equation that does not involve the variable y explicitly admits the operator $\mathrm{X} = \partial_y$. However, the zero solution (if any) generates the family $y = C$, which is too obvious and does not make much sense.

Example 11.9. Consider the equation

$$(y - x)y''_{xx} - (y'_x)^2 + (xy - x^2 - 2)y'_x - xy = 0.$$

It has the trivial solution and admits the operator

$$\mathrm{X} = (y - x)^{-1}\partial_y.$$

The finite group of transformations for the operator X has the form

$$\widetilde{x} = x, \quad \widetilde{y} = x - \sqrt{(y - x)^2 + a}.$$

On applying this transformation to y, we arrive at the nontrivial family of particular solutions

$$y = x - \sqrt{x^2 + C}.$$

⊙ *Literature for Section 11.1*: G. W. Bluman and J. D. Cole (1974), L. V. Ovsiannikov (1982), J. M. Hill (1982), P. J. Olver (1986), G. W. Bluman and S. Kumei (1989), H. Stephani (1989), N. H. Ibragimov (1994), A. D. Polyanin and V. F. Zaitsev (2003), V. F. Zaitsev and L. V. Linchuk (2009, 2014).

11.2 Contact and Bäcklund Transformations. Formal Operators. Factorization Principle

11.2.1 Contact Transformations

► **Continuous one-parameter Lie group of tangential transformations.**

The set of transformations

$$\mathrm{T}_\varepsilon = \begin{cases} \bar{x} = \varphi(x, y, y'_x, \varepsilon), & \bar{x}|_{\varepsilon=0} = x, \\ \bar{y} = \psi(x, y, y'_x, \varepsilon), & \bar{y}|_{\varepsilon=0} = y, \\ \bar{y}'_{\bar{x}} = \chi(x, y, y'_x, \varepsilon), & \bar{y}'_{\bar{x}}|_{\varepsilon=0} = y'_x \end{cases} \tag{11.2.1.1}$$

(here, φ, ψ, χ are smooth functions of their arguments and ε is a real parameter) is called a *continuous one-parameter Lie group of tangential transformations* (or simply, a *tangential* or *contact group*) if $\mathrm{T}_{\varepsilon_1} \circ \mathrm{T}_{\varepsilon_2} = \mathrm{T}_{\varepsilon_1+\varepsilon_2}$, i.e., if successive application of transformations (11.2.1.1) with parameters ε_1 and ε_2 is equivalent to the same transformation with parameter $\varepsilon_1 + \varepsilon_2$. The transformed derivative $\bar{y}'_{\bar{x}}$ depends only on the first derivative y'_x and does not depend on the second derivative. Thus, the functions φ and ψ in (11.2.1.1) cannot be arbitrary but are related by (see Section 1.9.1):

$$\frac{\partial\psi}{\partial y'_x}\Big(\frac{\partial\varphi}{\partial x} + y'_x\frac{\partial\varphi}{\partial y}\Big) - \frac{\partial\varphi}{\partial y'_x}\Big(\frac{\partial\psi}{\partial x} + y'_x\frac{\partial\psi}{\partial y}\Big) = 0,$$

where the function χ is defined by

$$\chi = \frac{\partial\psi}{\partial y'_x}\Big/\frac{\partial\varphi}{\partial y'_x}.$$

► **Infinitesimal operator. Invariance condition.**

Proceeding as in Section 11.1.1, we consider the Taylor expansions of $\bar{x}$, $\bar{y}$, and $\bar{y}'_{\bar{x}}$ in (11.2.1.1) with respect to the parameter ε about $\varepsilon = 0$, preserving only the first-order terms. We have

$$\bar{x} \simeq x + \xi(x, y, y'_x)\varepsilon, \quad \bar{y} \simeq y + \eta(x, y, y'_x)\varepsilon, \quad \bar{y}'_{\bar{x}} \simeq y'_x + \zeta(x, y, y'_x)\varepsilon,$$

where

$$\xi(x, y, y'_x) = \frac{\partial\varphi(x, y, y'_x, \varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}, \quad \eta(x, y, y'_x) = \frac{\partial\psi(x, y, y'_x, \varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0},$$
$$\zeta(x, y, y'_x) = \frac{\partial\chi(x, y, y'_x, \varepsilon)}{\partial\varepsilon}\bigg|_{\varepsilon=0}.$$

On the other hand,

$$\bar{y}'_{\bar{x}} \equiv \frac{d\bar{y}}{d\bar{x}} = \frac{D_x(y + \eta\varepsilon)}{D_x(x + \xi\varepsilon)} \simeq \frac{y'_x + (\eta_x + \eta_y y'_x + \eta_{y'_x} y''_{xx})\varepsilon}{1 + (\xi_x + \xi_y y'_x + \xi_{y'_x} y''_{xx})\varepsilon}. \tag{11.2.1.2}$$

Expanding (11.2.1.2) with respect to ε and requiring that ζ be independent of y''_{xx}, we find that the three functions ξ, η, and ζ are expressed in terms of a single function $W(x, y, y'_x)$ as follows:

$$\xi = -\frac{\partial W}{\partial y'_x}, \quad \eta = W - y'_x \frac{\partial W}{\partial y'_x}, \quad \zeta = \frac{\partial W}{\partial x} + y'_x \frac{\partial W}{\partial y}. \tag{11.2.1.3}$$

To an infinitesimal tangential transformation (11.2.1.1) there corresponds the infinitesimal operator:

$$\mathrm{X} = \xi(x, y, y'_x)\frac{\partial}{\partial x} + \eta(x, y, y'_x)\frac{\partial}{\partial y} + \zeta(x, y, y'_x)\frac{\partial}{\partial y'_x} \tag{11.2.1.4}$$

whose coordinates satisfy relations (11.2.1.3).

The action of the group on higher derivatives is determined by the recurrence formula:

$$\zeta_{k+1} = D_x(\zeta_k) - y_x^{(k+1)} D_x(\xi),$$

where $\zeta_1 = \zeta$. The invariance condition and the algorithm of finding tangential operators (11.2.1.4) admitted by ordinary differential equations are similar to those for point operators. The only difference is that the coordinates of the tangential operator depend on the first derivative; therefore, the determining equation can be split and reduced to a system only in the case of equations whose order is greater or equal to three.

Remark 11.4. There are no tangential transformations of finite order $k > 1$ other than prolonged point transformations and contact transformations [these transformations are described by formulas similar to (11.2.1.1) and, in addition to y'_x, $\bar{y}'_{\bar{x}}$, contain higher derivatives of up to order k inclusive].

11.2.2 Bäcklund Transformations. Formal Operators and Nonlocal Variables

► **Lie–Bäcklund groups. Operator in canonical form.**

1°. If the coordinates of the infinitesimal operator are allowed to depend on the derivatives of arbitrary (up to infinity) orders, we obtain Lie–Bäcklund groups (of tangential transformations of infinite order). However, on the manifold determined by an ordinary differential equation, all higher derivatives are expressed through finitely many lower derivatives, as dictated by the structure of the equation itself and the differential relations obtained from the equation. The substitution of the right-hand side of equation (11.2.1.1) into an infinite series with derivatives usually results in very cumbersome formulas hardly suitable for practical calculations. For this reason, the Lie–Bäcklund groups are widely used only for the investigation of partial differential equations, whereas in the case of ordinary differential equations, a more effective approach is that based on the canonical form of an operator and the notion of a formal operator.

2°. The canonical form $\widetilde{\mathrm{X}}$ is defined by the relation

$$\widetilde{\mathrm{X}} = \mathrm{X} - \xi(x,y)D_x = [\eta(x,y) - \xi(x,y)y'_x]\frac{\partial}{\partial y},$$

where $\mathrm{X} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}$ is the infinitesimal operator of the group [see formula (11.1.1.6) in Section 11.1.1], and D_x is the operator of total derivative. The operators X and $\widetilde{\mathrm{X}}$ are equivalent in the sense that if one of them is admissible for the equation, then the other is also admissible (the operator of total derivative is admissible for any ordinary differential equation). The function $I_0(x,y) \equiv x$ is an invariant of any operator in canonical form.

The action of the group on higher order derivatives for an operator in canonical form is determined by the simple recurrence formula $\widetilde{\zeta}_{k+1} = D_x(\widetilde{\zeta}_k)$. The order of an equation that admits an operator in canonical form can be reduced on the basis of the algorithm described in Section 11.1.3 (see Paragraph “Second method”).

► Formal operators and nonlocal variables.

By definition, a formal operator is an infinitesimal operator of the form

$$X = \Phi\,\partial_y, \tag{11.2.2.1}$$

where the function Φ depends on x, y, y'_x, …, $y_x^{(k)}$ (with k smaller than the order of the equation under investigation) and auxiliary variables whose definition involves the symbol of indefinite integral, for instance,

$$\int \zeta(x,y,y'_x)\,dx$$

(the integration is with respect to the variable x which is involved both explicitly and implicitly, through the dependence of y on x). Such auxiliary variables are called nonlocal, in contrast to the coordinates of the prolonged space defined pointwise. The nonlocal variables depend on derivatives of arbitrarily high order, for instance,

$$\int y\,dx = \sum_{m=0}^{\infty}(-1)^m\frac{x^{m+1}}{(m+1)!}y_x^{(m)}.$$

This formula is obtained by successive integration by parts of its left-hand side. Thus, a nonlocal variable can be represented as an infinite formal series; and this enables us to express the coordinates of the Lie–Bäcklund operator in concise form.

A formal operator is a far-reaching generalization of an operator in canonical form. The function $I_0(x,y) \equiv x$ is an invariant of the formal operator (11.2.2.1) for any Φ.

When solving the direct problem, one usually prescribes the nonlocal operator in the general form

$$\mathrm{X} = \Big[\eta_1 \exp\Big(\int \zeta\,dx\Big) + \eta_2\Big]\partial_y \quad \text{or} \quad \mathrm{X} = \Big(\eta_1 \int \zeta\,dx + \eta_2\Big)\partial_y, \tag{11.2.2.2}$$

and then, in order to find an admissible operator, one uses a search algorithm similar to that described in Section 11.1.2. The coordinates of the prolonged operator are found by the formulas $\zeta_k = D_x(\zeta_{k-1})$, where $\zeta_0 = \Phi$. In contrast to the method of finding a point operator, in the present case, there are three unknown functions (η_1, η_2, ζ); and the splitting procedure to obtain a system can be realized with respect to all "independent" variables, in particular, the nonlocal variables.

Suppose that the differential equation

$$y_x^{(n)} = F\big(x, y, y'_x, \dots, y_x^{(n-1)}\big) \tag{11.2.2.3}$$

can be written in new variables $x = I_0$, $z = I_1(x, y, y'_x)$, z'_x, z''_{xx}, $\dots$, $z_x^{(n-1)}$, where I_0 and I_1 are invariants of an admissible operator of the form (11.2.2.1). Then the coordinate Φ of this operator satisfies the equation

$$\frac{\partial I_1}{\partial y}\Phi + \frac{\partial I_1}{\partial y'_x} D_x[\Phi] = 0,$$

which is an analogue of a linear ordinary differential equation for a function of several variables, since it involves the total derivative of the unknown function (exact differential equation). Its solution has the form:

$$\Phi = \exp\Big(-\int \frac{\partial I_1/\partial y}{\partial I_1/\partial y'_x}\,dx\Big), \tag{11.2.2.4}$$

where the integral is taken with respect to x involved explicitly and implicitly (through the dependence of y, y'_x, …on x), which means that this representation of an operator through a nonlocal variable is most universal. The function (11.2.2.4) generates a nonlocal exponential operator of the form (11.2.2.1) [the class of nonlocal exponential operators is specified by the first expression in (11.2.2.2) with $\eta_2 \equiv 0$].

THEOREM 1. *Any first-order equation*

$$y'_x = F(x, y) \tag{11.2.2.5}$$

admits a ***unique*** *formal operator (up to identical transformations on the manifold (11.2.2.5)) with coordinate explicitly independent of derivatives*

$$X = \exp\left(\int \frac{\partial F}{\partial y}\,dx\right)\partial_y. \tag{11.2.2.6}$$

Indeed, the invariance condition for equation (11.2.2.5) is

$$\underset{1}{X}\big[y'_x - F(x,y)\big]\Big|_{[F]} = 0.$$

It follows that

$$D_x[\Phi] - \frac{\partial F}{\partial y}\Phi = 0,$$

and hence,

$$\Phi = \exp\left(\int \frac{\partial F}{\partial y}\,dx\right).$$

Remark 11.5. It follows from Theorem 1 that equation (11.2.2.5) is integrable by quadrature if

$$\frac{\partial F}{\partial y} = D_x\Phi(x,y)\Big|_{y'_x=F(x,y)}, \tag{11.2.2.7}$$

where $\Phi(x,y)$ is some function. Indeed, if condition (11.2.2.7) holds, (11.2.2.6) is a point operator.

Example 11.10. The equation

$$y''_{xx} = 0$$

admits two Lie–Bäcklund operators:

$$\mathrm{X}_1 = \xi(x,y,y'_x)D_x, \quad \mathrm{X}_2 = \sum_{m=0}^{\infty} D_x^m\Big[yg(y-xy'_x,y'_x) + h(y-xy'_x,y'_x)\Big]\frac{\partial}{\partial y_x^{(m)}},$$

where ξ, g, h are arbitrary functions of their variables. The first operator is trivial (the operator of total derivative is admissible for any differential equation), while the second operator determines the maximal group of contact transformations admitted by the equation under consideration.

► Construction methods for Lie–Bäcklund operators admitted by ODEs.

A Lie–Bäcklund operator admitted by an ordinary differential equation can by found by three methods:

(i) in the form of an infinite formal series;

(ii) by passing to an equivalent system of ordinary first-order differential equations:

$$y'_1 = y_2, \quad y'_2 = y_3, \quad \ldots, \quad y'_n = F(x,y_1,y_2,\ldots,y_n),$$

and finding an admissible point group;

(iii) by its representation as a formal operator whose coordinates depend on nonlocal variables (the general form of the operator is chosen by the investigator).

In all cases, the search algorithm amounts to solving the determining system which is constructed by a procedure similar to that of Section 11.1. From the standpoint of simplicity and the possibility of integrating equations, the third method seems to be the most effective if one takes into account that an equation admitting an operator can be written in terms of new variables—invariants of the admissible operator—as a new ordinary differential equation whose order is by one less than that of the original equation.

11.2.3 Factorization Principles

► Factorization principle: a special case.

The use of formal operators enables us to formulate universal principles for reducing the order of an equation, independently of the specific structure of the operator (it can be a point operator, a tangential or nonlocal operator, or a Lie–Bäcklund operator).

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THEOREM 1. *An arbitrary nth-order differential equation (11.2.2.3) can be factorized to a system of special form*

$$z_x^{(n-1)} = G\big(x, z, z'_x, \dots, z_x^{(n-2)}\big), \qquad z = H(x, y, y'_x), \tag{11.2.3.1}$$

if and only if equation (11.2.2.3) admits the nonlocal exponential operator:

$$\mathrm{X} = \exp\Big(-\int \frac{H_y}{H_{y'_x}}\,dx\Big)\frac{\partial}{\partial y}. \tag{11.2.3.2}$$

The function $H(x, y, y'_x)$ is the first differential invariant of the operator (11.2.3.2). Therefore, having found an admissible operator (11.2.3.2) of the form

$$\mathrm{X} = \Phi\,\frac{\partial}{\partial y}, \qquad \Phi \equiv \exp\Big[\int Q(x, y, y'_x)\,dx\Big], \tag{11.2.3.3}$$

we can calculate H by solving the first-order linear partial differential equation

$$H_y + QH_{y'_x} = 0.$$

The function $Q(x, y, y'_x)$ is found as a solution of the determining system obtained by “splitting” the invariance condition for operator (11.2.3.3):

$$\underset{n}{\mathrm{X}}\big[y_x^{(n)} - F\big(x, y, y'_x, \dots, y_x^{(n-1)}\big)\big]\big|_{y_x^{(n)}=F} = 0,$$

where

$$\underset{n}{\mathrm{X}} = \sum_{k=0}^{n} \Phi_k \frac{\partial}{\partial y_x^{(k)}}, \qquad \Phi_k = D_x\Phi_{k-1}, \qquad \Phi_0 = \Phi,$$

$$D_x = \frac{\partial}{\partial x} + y'_x\frac{\partial}{\partial y} + y''_{xx}\frac{\partial}{\partial y'_x} + \cdots.$$

Theorem 1 generalizes the classical Lie algorithm, which is restricted to the case of unconditional solvability of the second equation of system (11.2.3.1). On the other hand, the introduction of the factor system (11.2.3.1) allows for two more cases, since the first equation is independent of y. These cases are the following:

1°. The first equation of system (11.2.3.1) allows for the reduction of the order or is solvable.

2°. The first equation of system (11.2.3.1) has some special properties, for instance, admits a fundamental system of solutions.

Example 11.11. The equation

$$y''_{xx} = f(x)y + g'_x(x)y^{-1} - [g(x)]^2y^{-3} \tag{11.2.3.4}$$

for arbitrary functions $f(x)$ and $g(x)$ is the only equation of the form (its uniqueness is to within a Kummer–Liouville equivalence transformation; see Section 2.2.1)

$$y''_{xx} = F(x, y)$$

admitting the nonlocal exponential operator:

$$\mathrm{X} = \exp\Big(\int \zeta\,dx\Big)\eta\frac{\partial}{\partial y} = \exp\Big[\int\Big(\zeta + \frac{\eta_x + \eta_y y'_x}{\eta}\Big)\,dx\Big]\frac{\partial}{\partial y}, \qquad \eta = \eta(x, y), \quad \zeta = \zeta(x, y).$$

The second prolongation of the operator X has the form:

$$\underset{2}{\mathrm{X}} = \exp\Bigl(\int \zeta\,dx\Bigr)\bigl\{\eta\partial_y + (\eta_x + \eta_y y'_x + \eta\zeta)\partial_{y'_x} + \bigl[\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + (2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y)y'_x + \eta_{yy}(y'_x)^2 + \eta_y y''_{xx}\bigr]\partial_{y''_{xx}}\bigr\}.$$

Applying this operator to the equation $y''_{xx} = F(x,y)$ and replacing all instances of y''_{xx} by $F = F(x,y)$, we obtain the invariance condition in the form:

$$\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + \eta_y F - \eta F_y + (2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y)y'_x + \eta_{yy}(y'_x)^2 = 0.$$

Splitting this relation with respect to powers of the "independent" variable y'_x, we obtain the following system of three equations for the functions η, ζ, and F:

$$\begin{aligned}&\eta_{yy} = 0,\\ &2\eta_{xy} + 2\zeta\eta_y + \eta\zeta_y = 0,\\ &\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \zeta^2\eta + \eta_y F - \eta F_y = 0.\end{aligned}$$

From the first two equations it follows that

$$\begin{aligned}\eta &= a(x)y + b(x),\\ \zeta &= -\frac{aa'y^2 + 2a'by + c(x)}{(ay+b)^2},\end{aligned}$$

where $a = a(x)$, $b = b(x)$, and $c = c(x)$ are arbitrary functions. The third equation can be treated as a first-order linear differential equation for the unknown function $F = F(x,y)$:

$$\frac{dF}{dy} - \frac{\eta_y}{\eta}F = \frac{1}{\eta}(\eta_{xx} + 2\zeta\eta_x + \eta\zeta_x + \eta\zeta^2).$$

Substituting the above expressions of η and ζ into this relation and integrating the result, we obtain

$$F(x,y) = (ay+b)f(x) + \frac{[aa'' - 2(a')^2]b - (ab'' - 2a'b')a}{a^3} - \frac{[aa'' - 3(a')^2]b^2 + 2aa'bb' - (ac' - 2a'c)a}{2a^3(ay+b)} - \frac{(a'b^2 - ac)^2}{4a^3(ay+b)^3},$$

where $f(x)$ is an arbitrary function.

The differential invariant z of the operator X satisfies the linear partial differential equation

$$\eta\frac{\partial z}{\partial y} + (\eta_x + \eta_y y'_x + \eta\zeta)\frac{\partial z}{\partial y'_x} = 0$$

(obtained after the division by $\exp(\int \zeta\,dx)$). Substituting the above η and ζ into this equation, we pass to the characteristic equation

$$\frac{dw}{dy} = \frac{aw}{ay+b} + \frac{2aa'y^2 + (3a'b + ab')y + bb' + c}{(ay+b)},$$

where $w = y'_x$. Integrating this equation, we find the differential invariant:

$$z = \frac{y'_x}{ay+b} - \frac{a'b - ab'}{a^2(ay+b)} + \frac{a'b^2 - ac}{2a^2(ay+b)^2}.$$

Having calculated the derivative z'_x, one can find y''_{xx} and, taking into account the known structure of the function $F(x,y)$, one obtains the factorization of the original equation:

$$\begin{aligned}&z'_x + az^2 + (a'/a)z = f,\\ &(ay+b)y'_x = (ay+b)^2 z + a^{-2}(a'b - ab')(ay+b) - \tfrac{1}{2}a^{-2}(a'b^2 - ac).\end{aligned}$$

An equivalence transformation of the form $ay + b \to y$, combined with the corresponding transformation of the independent variable and changed notation, yields:

$$
\begin{aligned}
&z'_x + z^2 = f(x),\\
&y'_x = zy + g(x)y^{-1}.
\end{aligned}
\tag{11.2.3.5}
$$

The first equation of system (11.2.3.5) is a Riccati equation. Its general solution can be represented in terms of a fundamental system of solutions of the "truncated" linear equation:

$$
y''_{xx} = f(x)y, \tag{11.2.3.6}
$$

which coincides with (11.2.3.4) for $g \equiv 0$. The second equation of system (11.2.3.5) is a Bernoulli equation. It can be integrated by quadrature for an arbitrary function $z = z(x, C)$. Therefore, the general solution of equation (11.2.3.4) can be expressed in terms of a fundamental system of solutions of the linear equation (11.2.3.6). Note that in the general case, equation (11.2.3.4) admits no point groups.

Theorem 1 can be made more general. Let the second-order ordinary differential equation

$$
y''_{xx} = F(x, y, y'_x), \tag{11.2.3.7}
$$

admit the exponential nonlocal operator

$$
\mathrm{X} = \bigl[\xi(x, y, y'_x)\partial_x + \eta(x, y, y'_x)\partial_y\bigr]\Omega, \quad \Omega = \exp\left(\int \zeta(x, y, y'_x)\,dx\right). \tag{11.2.3.8}
$$

To describe *all* equations of the form (11.2.3.7) admitting factorization, it suffices to consider the operator (11.2.3.8). Equation (11.2.3.7) is then factorized to the system

$$
\begin{aligned}
&\dot{u}_t = G(t, u),\\
&u(t) = H_1(x, y, y'_x),\\
&t = H_0(x, y),
\end{aligned}
\tag{11.2.3.9}
$$

where H_0 and H_1 are invariants of the operator (11.2.3.8). The last two equations in (11.2.3.9) are essentially one equation determining the function $u(t)$ in parametric form.

► Factorization principle: the general case.

If an operator admitted by equation (11.2.1.1) has no differential invariants of the first-order, then it is possible to apply the general factorization principle.

THEOREM 2. *An arbitrary nth-order differential equation (11.2.1.1) can be factorized to the system of special structure*

$$
\begin{aligned}
&z_x^{(n-k)} = G\bigl(x, z, z'_x, \dots, z_x^{(n-k-1)}\bigr),\\
&z = H\bigl(x, y, y'_x, \dots, y_x^{(k)}\bigr), \qquad \frac{\partial z}{\partial y_x^{(k)}} \neq 0,
\end{aligned}
\tag{11.2.3.10}
$$

provided that equation (11.2.2.3) admits a formal operator of the form (11.2.2.1) for which $H\bigl(x, y, y'_x, \dots, y_x^{(k)}\bigr)$ is a lower-order differential invariant on the manifold defined by (11.2.2.3). The coordinate Φ of this operator satisfies the linear equation with total derivatives:

$$
\Phi\frac{\partial z}{\partial y} + D_x[\Phi]\frac{\partial z}{\partial y'} + \dots + D_x^{(k)}[\Phi]\frac{\partial z}{\partial y^{(k)}} = 0. \tag{11.2.3.11}
$$

Equation (11.2.3.11) plays a crucial role in both the direct and inverse problems. It can be regarded as an equation for the determination of the coordinate of the canonical operator (if one knows the invariant z). It can also be regarded as an equation for the determination of an invariant (if one knows the coordinate Φ). In the latter case, this is a first-order partial differential equation.

Example 11.12. The third-order nonlinear equation

$$yy'''_{xxx} + (y''_{xx})^2 - y'_x y''_{xx} - f(x)y^2 = 0 \tag{11.2.3.12}$$

admits two operators

$$\mathrm{X}_1 = y\,\partial_y, \quad \mathrm{X}_2 = \Big(y\int y^{-2}\,dx\Big)\partial_y, \tag{11.2.3.13}$$

which can be found with the help of the direct algorithm, if the structure of the operator is specified by the second expression in (11.2.2.2). The first operator, X_1, is the usual point operator of scaling (the original equation is homogeneous) and provides the usual reduction of order of equation (11.2.3.12) by one. The second operator, X_2, is nonlocal.

Let us construct differential invariants of the operator X_2. To this end, we should solve the equations:

$$\begin{aligned} &\Phi\frac{\partial I_1}{\partial y} + D_x[\Phi]\frac{\partial I_1}{\partial y'_x} = 0, \quad \Phi = y\int y^{-2}\,dx,\\ &\Phi\frac{\partial I_2}{\partial y} + D_x[\Phi]\frac{\partial I_2}{\partial y'_x} + D_x^2[\Phi]\frac{\partial I_2}{\partial y''_{xx}} = 0. \end{aligned} \tag{11.2.3.14}$$

After differentiation with respect to Φ, the first equation in (11.2.3.14) becomes

$$\Big(y\int y^{-2}\,dx\Big)\frac{\partial I_1}{\partial y} + \Big[y^{-1} + \Big(y'_x\int y^{-2}\,dx\Big)\Big]\frac{\partial I_1}{\partial y'_x} = 0.$$

Let us show that this equation admits no solutions depending only on x, y, y'_x, and $\partial I_1/\partial y'_x \neq 0$, i.e., there are no first-order differential invariants. The nonlocal expression $\int y^{-2}\,dx$ depends on derivatives of arbitrarily high orders and can be regarded as an independent quantity. Therefore, the first equation (11.2.3.14) can be split and we obtain the system:

$$y\frac{\partial I_1}{\partial y} + y'_x\frac{\partial I_1}{\partial y'_x} = 0, \qquad y^{-1}\frac{\partial I_1}{\partial y'_x} = 0.$$

It follows that $\partial I_1/\partial y'_x = 0$.

Let us find a second-order differential invariant. After differentiation with respect to Φ, the second equation in (11.2.3.14) becomes

$$\Big(y\int y^{-2}\,dx\Big)\frac{\partial I_2}{\partial y} + \Big[y^{-1} + \Big(y'_x\int y^{-2}\,dx\Big)\Big]\frac{\partial I_2}{\partial y'_x} + \Big(y''_{xx}\int y^{-2}\,dx\Big)\frac{\partial I_2}{\partial y''_{xx}} = 0.$$

Splitting this equation with respect to the nonlocal variable $\int y^{-2}\,dx$, we find that $\partial I_2/\partial y'_x = 0$. In the remaining equation, the nonlocal variable is canceled,

$$y\frac{\partial I_2}{\partial y} + y''_{xx}\frac{\partial I_2}{\partial y''_{xx}} = 0.$$

It follows that $I_2 = z = y''_{xx}/y$, and equation (11.2.3.12) is factorized to the system

$$\begin{aligned} &z'_x + z^2 = f(x),\\ &y''_{xx} - yz = 0. \end{aligned}$$

► **Applications to third-order ODEs.**

Consider the nonlocal nonexponential operator

$$\mathrm{X} = \big(\xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y\big)\int \zeta(x,y,y'_x,y''_{xx})\,dx \tag{11.2.3.15}$$

(see the previous example). It can be used to factorize the third-order ODE

$$y'''_{xxx} = F(x,y,y'_x,y''_{xx}). \tag{11.2.3.16}$$

It is clear that the universal invariant of the operator (11.2.3.15) coincides with the invariant of the point operator

$$\mathrm{X}_0 = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y. \tag{11.2.3.17}$$

The following theorem holds.

THEOREM 1. *The third-order ODE (11.2.3.16) admitting the nonlocal operator (11.2.3.15) always admits the point operator (11.2.3.17).*

Consequently, one needs to look for the nonlocal operator (11.2.3.15) for equation (11.2.3.16) only if the equation possesses a point symmetry. It may seem that this fact reduces the value of operators of the form (11.2.3.15). In fact, there are a large number of equations (in particular, third-order equations) for which no operators are known other than a single point operator. Therefore, the presence of at least one operator, even though a nonlocal one, admitted by the equation can significantly facilitate the integration and investigation of the original equation.

1°. *Preliminary remarks*. Consider the problem of seeking a class of third-order equations admitting a nonlocal nonexponential operator of the form

$$\mathrm{X} = \eta(x,y,y'_x)\left(\int \zeta(x,y,y'_x)\,dx\right)\partial_y. \tag{11.2.3.18}$$

By virtue of Theorem 1, we can restrict ourselves to the class of autonomous equations, thus setting $\eta \equiv y'$ and looking for an operator in the form

$$\mathrm{X} = y'_x\left(\int \zeta(x,y,y'_x)\,dx\right)\partial_y. \tag{11.2.3.19}$$

Then, we can find all such classes of equations by applying an arbitrary point transformation.

Let us find the prolongation of the operator (11.2.3.19). Let I denote the nonlocal variable:

$$I = \int \zeta(x,y,y'_x)\,dx.$$

Since $\widetilde{\eta} = y'_x I$, we get

$$\begin{aligned}
\widetilde{\eta}_1 &= D_x\widetilde{\eta} = y''_{xx}I + y'_x\zeta,\\
\widetilde{\eta}_2 &= D_x^2\widetilde{\eta} = y'''_{xxx}I + 2y''_{xx}\zeta + y'_x\big(\zeta_x + \zeta_y y'_x + \zeta_{y'_x}y''_{xx}\big),\\
\widetilde{\eta}_3 &= D_x^3\widetilde{\eta} = y_x^{(4)}I + 3y'''_{xxx}\zeta + 3y''_{xx}\big(\zeta_x + \zeta_y y'_x + \zeta_{y'_x}y'_x\big) + y'_x\big[\zeta_{xx} + 2\zeta_{xy}y'_x\\
&\quad + \zeta_{yy}(y'_x)^2 + 2\zeta_{xy'_x}y''_{xx} + 2\zeta_{yy'_x}y'_x y''_{xx} + \zeta_{y'_x y'_x}\big(y''_{xx}\big)^2 + \zeta_y y''_{xx} + \zeta_{y'_x}y'''_{xxx}\big].
\end{aligned}$$

2°. *Equations of the form* $y'''_{xxx} = F(y)$. The invariance condition is written in the form

$$\left(\widetilde{\eta}_3 - \widetilde{\eta}\frac{\partial F}{\partial y}\right)\bigg|_{y'''_{xxx}=F(y)} = 0.$$

Replacing y'''_{xxx} and $y^{(4)}_x$ with the help of the original equation and its differential consequence $y^{(4)}_x = y'_x F'(y)$ and splitting the remaining expression into powers of y''_{xx} and I, we arrive at a system that only has the trivial solution ($F \equiv 0$ or F is any but $\zeta \equiv 0$). Therefore, the following statement holds.

THEOREM 2. *There is no equation of the form* $y'''_{xxx} = F(y)$, *other than the trivial equation, admitting a nonlocal operator of the form (11.2.3.18).*

3°. *Equations of the form* $y'''_{xxx} = F(y, y'_x)$. Now consider the autonomous third-order equation without the second derivative

$$y'''_{xxx} = F(y, y'_x), \tag{11.2.3.20}$$

admitting the nonlocal nonexponential operator (11.2.3.19). The invariance condition is written as

$$\left(\widetilde{\eta}_3 - \widetilde{\eta}\frac{\partial F}{\partial y} - \widetilde{\eta}_1\frac{\partial F}{\partial y'_x}\right)\bigg|_{y'''_{xxx}=F(y,y'_x)} = 0.$$

The determining system has the form

$$\begin{aligned}
&3\zeta_{y'_x} + y'_x\zeta_{y'_x y'_x} = 0,\\
&3\zeta_x + 2y'_x(\zeta_{xy'_x} + 2\zeta_y + \zeta_{yy'_x}y'_x) = 0,\\
&(3\zeta + y'_x\zeta_{y'_x})F - y'_x\zeta F_{y'_x} + y'_x\zeta_{xx} + 2\zeta_{xy}(y'_x)^2 + \zeta_{yy}(y'_x)^3 = 0.
\end{aligned}$$

THEOREM 3. *Equation (11.2.3.20) admits the nonlocal nonexponential operator (11.2.3.19) if and only if the right-hand side has the form*

$$F(y, y'_x) = y'_x\big[C(y'_x)^2 + G(y)\big]H(y) - \frac{1}{2C}G''(y)y'_x, \tag{11.2.3.21}$$

with

$$\zeta(x, y, y'_x) = C + \frac{G(y)}{(y'_x)^2}, \tag{11.2.3.22}$$

where $G(y)$ *and* $H(y)$ *are arbitrary functions and* $C \neq 0$ *is an arbitrary constant.*

Remark 11.6. The value $C = 0$ is possible only if $G''(y) \equiv 0$. However, in this case, the original equation is trivial and easy to integrate.

The operator (11.2.3.19) has no first differential invariant (more precisely, it has no invariant dependent on the first derivative alone). To compute the second differential invariant of the operator

$$\mathrm{X} = y'_x\left[\int\left(C + \frac{G(y)}{(y'_x)^2}\right)dx\right]\partial_y, \tag{11.2.3.23}$$

we have to solve the equation

$$\widetilde{\eta}\frac{\partial \Phi}{\partial y} + \widetilde{\eta}_1\frac{\partial \Phi}{\partial y'_x} + \widetilde{\eta}_2\frac{\partial \Phi}{\partial y''_{xx}} = 0.$$

Inserting the coordinates of the operator (11.2.3.23) and splitting the equation in the nonlocal variable I, we obtain the system of two equations

$$y'_x\left[C+\frac{G(y)}{(y'_x)^2}\right]\frac{\partial\Phi}{\partial y'_x}+\big(2Cy''_{xx}+G'(y)\big)\frac{\partial\Phi}{\partial y''_{xx}}=0,$$
$$y'_x\frac{\partial\Phi}{\partial y}+y''_{xx}\frac{\partial\Phi}{\partial y'_x}+\left[y'_x\Big(C(y'_x)^2+G(y)\Big)H(y)-\frac{1}{2C}G''(y)y'_x\right]\frac{\partial\Phi}{\partial y''_{xx}}=0. \tag{11.2.3.24}$$

Note that in the second equation, the derivative y'''_{xxx} is replaced with the right-hand side of equation (11.2.3.21); that is, the invariant is sought *on the manifold* of solutions of the original equation. The solution of the first equation in (11.2.3.24) is

$$\Omega\left(y,\frac{2Cy''_{xx}+G'(y)}{C(y'_x)^2+G(y)}\right). \tag{11.2.3.25}$$

Inserting (11.2.3.25) into the second equation of the system results in a first-order linear partial differential equation for Ω:

$$\frac{\partial\Omega}{\partial y}+\left[H(y)-2C\omega^2\right]\frac{\partial\Omega}{\partial\omega}=0, \tag{11.2.3.26}$$

where ω is the second argument of the function Ω. The equation in characteristics for (11.2.3.26) is a canonical Riccati equation, which is always reduced to a second-order linear equation. In a large number of cases, the solution to (11.2.3.26) can be expressed in closed form (in terms of elementary or special functions). The actual representation significantly depends on the function $H(y)$.

Example 11.13. If $H(y)=y^k$ or $H(y)=e^y$, the second differential invariant is expressed in terms of Bessel functions. In addition, in the case of the power-law function, we get a special Riccati equation and if the fraction $\frac{k+3}{k+2}$ is a half-integer, the second differential invariant is an elementary function. For example, if $k=0$, we get

$$\Omega=\sqrt{2C}y-\operatorname{arth}\left(\frac{2Cy''_{xx}+G'(y)}{\sqrt{2C}\,(C(y'_x)^2+G(y))}\right).$$

By direct verification, one can see that $\Omega'_x=0$ by virtue of the original equation $\Omega'_x=0$, suggesting the factorization

$$\Omega'_x\Big|_{y'''_{xxx}=F}=0,$$
$$\Omega=\sqrt{2C}y-\operatorname{arth}\left(\frac{2Cy''_{xx}+G'(y)}{\sqrt{2C}\,(C(y'_x)^2+G(y))}\right).$$

Thus, the function Ω is an *autonomous first integral* of the original equation, while the symmetry is analogous to variational symmetry (see Section 11.3).

Example 11.14. The equation

$$y'''_{xxx}=f(x)y+y^{-1}(y''_{xx})^2+y^{-4}(y^3y'_x+2A)y''_{xx}+A^2y^{-7}$$

can be factorized to the system

$$z'_x=z^2+f(x),$$
$$z=\frac{y''_{xx}}{y}+Ay^{-4}.$$

The first equation is a Riccati equation and the second one is the Ermakov equation; therefore, the solution to the original equation is here uniquely determined by two fundamental systems of solutions of two linear second-order equations. Note that this system can be factorized further, since the Ermakov equation always admits an exponential nonlocal operator.

Apparently, it makes sense to perform a *test for a nonexponential operator* for any third- or higher-order equation possessing a point symmetry. This test allows one to find an additional nonexponential nonlocal operator admitted by the equation in order to facilitate its subsequent integration or seek an unobvious first integral.

⊙ *Literature for Section 11.2*: R. L. Anderson and N. H. Ibragimov (1979), O. N. Pavlovskii and G. N. Yakovenko (1982), N. H. Ibragimov (1985), P. J. Olver (1986), V. F. Zaitsev (2001), A. D. Polyanin and V. F. Zaitsev (2003), V. F. Zaitsev and L. V. Linchuk (2014).

11.3 First Integrals (Conservation Laws)

11.3.1 Algorithm of Finding First Integrals of ODEs

A function $P = P\big(x, y, y'_x, \dots, y_x^{(n-1)}\big)$ is called a *first integral* (*conservation law*) of the ordinary differential equation

$$y_x^{(n)} = F\big(x, y, \dots, y_x^{(n-1)}\big) \tag{11.3.1.1}$$

if the total derivative of the function P along the trajectories of equation (11.3.1.1) is zero or, equivalently, if

$$D_x[P] \equiv M\big(x, y, y'_x, \dots, y_x^{(n-1)}\big)\big[y_x^{(n)} - F\big(x, y, \dots, y_x^{(n-1)}\big)\big] = 0, \tag{11.3.1.2}$$

where M is an integrating factor. From this definition it is clear that $M = \frac{\partial P}{\partial y_x^{(n-1)}}$.

The algorithm of finding a first integral is similar to that of finding an admissible operator. It is necessary to prescribe the desired structure of the first integral (or the integrating factor) and substitute it into the determining equation (11.3.1.2). Subsequent splitting with respect to lower derivatives (assumed to be independent variables) leads to the determining system.

Remark 11.7. An arbitrary function of first integrals is also a first integral of the same equation. Therefore, having found a first integral depending on $(y'_x)^k$, one has to make sure that it is nontrivial; i.e., it cannot be represented as the product of first integrals depending on lower powers of the derivative.

Remark 11.8. If the equation has k functionally independent first integrals, then its order can be reduced by k by successively excluding higher derivatives (see Example 11.15).

► **Direct method.**

Rewriting equation (11.3.1.2) in expanded form, we get

$$P_x + y'_x P_y + y''_x P_{y'_x} + \dots + y_x^{(n)} P_{y_x^{(n-1)}} = -FM + y_x^{(n)} M. \tag{11.3.1.3}$$

Substituting here the value of M gives the equation

$$P_x + y'_x P_y + \dots + y_x^{(n-1)} P_{y_x^{n-2}} + F P_{y_x^{(n-1)}} = 0. \tag{11.3.1.4}$$

No general solution to this homogeneous linear partial differential equation with respect to x, y, y', …, $y^{(n-1)}$ can usually be obtained (as this equation is equivalent to the original one). However, it is quite likely that its particular solutions can be found with the splitting method by imposing certain conditions on the form of the desired first integral. For example, this can be achieved by assuming that $P(x, y, y'_x, \dots, y_x^{(n-1)})$ is linear in the highest derivative,

$$P = R\big(x, y, y'_x, \dots, y_x^{(n-2)}\big) y_x^{(n-1)} + Q\big(x, y, y'_x, \dots, y_x^{(n-2)}\big), \qquad (11.3.1.5)$$

or quadratic in the highest derivative,

$$P = R\big(x, y, y'_x, \dots, y_x^{(n-2)}\big)\big(y_x^{(n-1)}\big)^2 + Q\big(x, y, y'_x, \dots, y_x^{(n-2)}\big) y_x^{(n-1)} + S\big(x, y, y'_x, \dots, y_x^{(n-2)}\big). \qquad (11.3.1.6)$$

Substituting the structure of the first integral into (11.3.1.4) and splitting the resulting equation in powers of $y_x^{(n-2)}$, we obtain the determining system, whose solution gives us the desired first integral.

Example 11.15. The equation

$$y''''_{xxxx} = Ay^{-5/3} \qquad (11.3.1.7)$$

admits three first integrals:

$$\begin{aligned} P_1 &= y'_x y'''_{xxx} - \tfrac{1}{2}(y''_{xx})^2 + \tfrac{3}{2}Ay^{-2/3}, \\ P_2 &= xP_1 - \tfrac{3}{2}yy'''_{xxx} + \tfrac{1}{2}y'_x y''_{xx}, \\ P_3 &= xP_2 - \tfrac{1}{2}x^2P_1 + \tfrac{3}{2}yy''_{xx} - (y'_x)^2. \end{aligned}$$

Equating these expressions to independent constants C_1, C_2, C_3 and eliminating y'''_{xxx} and y''_{xx}, we obtain a first-order equation (see 4.2.1.1).

► Factorization method.

Let the nth-order equation (11.2.1.1) admit a (nonlocal) infinitesimal operator and let the factor system (11.2.3.10) have the form

$$z'_x = 0, \qquad z = H\big(x, y, y'_x, \dots, y_x^{(n-1)}\big), \qquad \frac{\partial z}{\partial y_x^{(n-1)}} \neq 0. \qquad (11.3.1.8)$$

Then the function $H\big(x, y, y'_x, \dots, y_x^{(n-1)}\big)$ is a first integral equation (11.2.1.1) and, simultaneously, a differential invariant of the admitted operator by virtue of (11.2.1.1).

► Other methods.

There are methods for finding an integrating factor for ODEs, which generalize the well-known approach to first-order equations. These use high-order Euler operators (see the paragraph on Nöther's theorem) and lead to results that are fundamentally the same as those of the direct method (see the literature for the present chapter).

11.3.2 Applications to Second-Order ODEs

For second-order equations

$$y''_{xx} = F(x, y, y'_x), \tag{11.3.2.1}$$

the determining equation (11.3.1.2) can be written in the form

$$\frac{\partial P}{\partial x} + y'_x \frac{\partial P}{\partial y} + F(x, y, y'_x) \frac{\partial P}{\partial y'_x} = 0. \tag{11.3.2.2}$$

In this case, one can solve the direct problem (find P for the given equation), as well as the inverse problem (find possible F for the given structure of the first integral).

Example 11.16. Let us find all equations of the form

$$y''_{xx} = F(x, y) \tag{11.3.2.3}$$

admitting a first integral that is quadratic with respect to the first derivative:

$$P = R(x, y)(y'_x)^2 + S(x, y)y'_x + Q(x, y).$$

Then the left-hand side of the determining equation (11.3.2.2) is a cubic polynomial with respect to y'_x. The procedure of splitting with respect to powers of y'_x yields the system of four equations:

$$\begin{aligned} &R_y = 0, \\ &R_x + S_y = 0, \\ &S_x + Q_y + 2RF = 0, \\ &Q_x + SF = 0. \end{aligned}$$

The solution of this system for F is given by:

$$\begin{aligned} F(x, y) &= R^{-3/2}\Psi(z) + \frac{1}{2}R^{-2}\Big[\Big(RR''_{xx} - \frac{1}{2}R'^2_x\Big)y - R\varphi'_x + \frac{1}{2}R'_x\varphi\Big], \\ z &= R^{-1/2}y + \frac{1}{2}\int \varphi R^{-3/2}\,dx, \end{aligned}$$

where $\Psi = \Psi(z)$, $R = R(x)$, and $\varphi = \varphi(x)$ are arbitrary functions. The class of equations obtained is essentially a solution to the inverse problem for equation (11.3.2.3), having a quadratic first integral, which is expressed as follows:

$$P = R(y'_x)^2 - (R'_x y - \varphi)y'_x + \frac{1}{4}R^{-1}(R'_x)^2y^2 - \frac{1}{2}R^{-1}R'_x\varphi y + \frac{1}{4}R^{-1}\varphi^2 - 2\int \Psi(z)dz.$$

Example 11.17. Consider the equation

$$y''_{xx} = Axy^{-1/2}.$$

Let us find its first integral, which is a cubic polynomial with respect to the first derivative:

$$P = R(x, y)(y'_x)^3 + S(x, y)(y'_x)^2 + Q(x, y)y'_x + U(x, y).$$

In this case, the left-hand side of the determining equation (11.3.2.2) is a fourth-order polynomial in y'_x, and hence the determining system consists of five equations:

$$\begin{aligned} &R_y = 0, \\ &R_x + S_y = 0, \\ &S_x + Q_y + 3Axy^{-1/2}R = 0, \\ &Q_x + U_y + 2Axy^{-1/2}S = 0, \\ &U_x + Axy^{-1/2}Q = 0. \end{aligned}$$

Solving this system, we obtain the first integral in the form:

$$P = (y'_x)^3 - 6Axy^{1/2}y'_x + 4Ay^{3/2} + 2A^2x^3.$$

The factorization method allows one to formulate, for second-order equations, a more rigorous result than in the general case. If equation (11.2.3.7) is factorized to system (11.2.3.9), then two cases of order reduction are possible:

(i) If the second (inner) equation is integrable, we obtain the classical method of order reduction, with the only difference that the application of exponential nonlocal operators provides a significant generalization of the results obtained using point operators (the generalization of the Ermakov equation is a good example).

(ii) If the first (outer) equation is integrable, we obtain a first integral of the original equation; this approach does not have classical analogues.

Example 11.18. The class of equations

$$y''_{xx} + \frac{\Psi\big(\sqrt{x^2+2ay}-x\big)}{\sqrt{x^2+2ay}} = 0,$$

where Ψ is an arbitrary function of its argument, admits the operator

$$X = \Big[a\partial_x + \big(\sqrt{x^2+2ay}-x\big)\partial_y\Big] \exp\int\frac{dx}{\sqrt{x^2+2ay}}.$$

Substituting its invariants

$$y'_x = u(t), \quad \sqrt{x^2+2ay}-x = t \tag{11.3.2.4}$$

yields the first-order equation

$$\dot{u}_t = \frac{\Psi(t)}{au-t},$$

which is reduced, with the transformation $au - t = -w(t)$, to an Abel equation of the second kind

$$w\dot{w}_t - w = a\Psi(t). \tag{11.3.2.5}$$

If the general solution to equation (11.3.2.5) is known, the first integral of the original equation is obtained in the (parametric) form (11.3.2.4).

11.3.3 Lie–Bäcklund Symmetries Generated by First Integrals

► Theorems on symmetries of first integrals.

1°. First, we note an important property of symmetries of differential equations: if an equation, having a first integral P, admits an operator X, the application of the operator X to the first integral P generates a first integral again (which can possibly be trivial). The following four cases are possible:

$$\begin{aligned}
&1.\quad \mathrm{X}(P) = 0,\\
&2.\quad \mathrm{X}(P) = C, \quad C = \text{const},\\
&3.\quad \mathrm{X}(P) = F(P),\\
&4.\quad \mathrm{X}(P) = P_1.
\end{aligned}$$

The second case gives a trivial first integral, while the third case gives the already known first integral; these cases are of no interest. The first case signifies that the *first integral inherits the symmetry of the original equation*, while the fourth case gives a new first integral, which is functionally independent of the already known ones. These two cases allow us to reduce the order of the original equation by two.

2°. Since any first integral of an ODE on the manifold of its solutions is a constant, the point operator admitted by the equation is essentially an infinite-dimensional Lie–Bäcklund algebra.

THEOREM 1. *Let the equation*

$$y^{(n)} = \mathcal{F}\big(x, y, y_x, \ldots, y_x^{(n-1)}\big) \tag{11.3.3.1}$$

admit a Lie algebra L_k *with basis* $\{\mathrm{X}_\alpha\}$, $\mathrm{X}_\alpha = \eta_\alpha \partial_y$, $\alpha = 1, \ldots, k$, *and have* s *independent first integrals* $\{P_\sigma\}$, $\sigma = 1, \ldots, s$ $(s \leqslant n)$. *Then, equation (11.3.3.1) admits an infinite-dimensional Lie–Bäcklund algebra with operator*

$$\mathrm{X}_B = \left(\sum_{\alpha=1}^{k} \eta_\alpha F_\alpha\right) \partial_y, \tag{11.3.3.2}$$

where F_α $(\alpha = 1, \ldots, k)$ *are arbitrary functions of* s *arguments* $P_1, \ldots, P_s$.

Example 11.19. Equation (11.3.2.5) admits a three-dimensional point Lie algebra defined by the operators

$$\mathrm{L}_3\colon\quad \mathrm{X}_1 = y'_x \partial_y, \quad \mathrm{X}_2 = \left(xy'_x - \frac{3}{2}y\right)\partial_y, \quad \mathrm{X}_3 = \left(\frac{1}{2}x^2 y'_x - \frac{3}{2}xy\right)\partial_y.$$

Hence, the equation also admits the infinite-dimensional Lie–Bäcklund algebra defined by the operator

$$\mathrm{X}_B = \left[y' F_1 + \left(xy'_x - \frac{3}{2}y\right) F_2 + \left(\frac{1}{2}x^2 y'_x - \frac{3}{2}xy\right) F_3\right] \partial_y, \tag{11.3.3.3}$$

where $F_i = F_i(P_1, P_2, P_3)$, $i = 1, 2, 3$, are arbitrary functions and P_1, P_2, and P_3 are first integrals of equation (11.3.2.5).

It follows from the theorem that the knowledge of *one* lowest symmetry and *one* first integral suffices to obtain an infinite-dimensional Lie–Bäcklund algebra.

Example 11.20. The equation $y'''_{xxx} = Axy^{-5/4}$ admits the Lie–Bäcklund algebra defined by the operator $\mathrm{X}_B = [(9xy'_x - 16y)F(P)]\partial_y$, where $P = y(y''_{xx})^2 - \frac{1}{2}(y'_x)^2 y''_{xx} - 2Axy^{-1/4}y'_x + \frac{8}{3}Ay^{3/4}$ is a first integral of the equation.

Example 11.21. The equation $y'''_{xxx} = Ay^{-1}$ admits the Lie–Bäcklund algebra defined by the operator $\mathrm{X}_B = [y'_x F_1(P) + (2xy'_x - 3y)F_2(P)]\partial_y$, where $P = yy''_{xx} - \frac{1}{2}(y'_x)^2 - Ax$ is a first integral of the equation.

Remark 11.9. Theorem 1 does not guarantee the completeness of the Lie–Bäcklund algebra obtained.

It is well known how an operator transforms when differential substitutions are used (in particular, nonlocal can arise in order reduction); however, this is not so obvious with first integrals. Some of the lowest symmetries may seem to disappear when a first integral is used. Below we show that this disappearance is only apparent. Let us look at the symmetry properties of first integrals.

THEOREM 2. *Let equation (11.3.3.1) admit the Lie–Bäcklund algebra defined by the operator (11.3.3.2). Then, for any* $P_\nu \in \{P_\sigma\}$, $\sigma = 1, \ldots, s$, *the equation*

$$P_\nu = C_\nu \tag{11.3.3.4}$$

1) has $s - 1$ *first integrals* $\{\widetilde{P}_\sigma\}$, $\sigma = 1, \ldots, s$, $\sigma \neq \nu$, *where* $\widetilde{P}_\sigma = P_\sigma|_{P_\nu = C_\nu}$;

2) admits the Lie–Bäcklund algebra defined by an operator of the form (11.3.3.2) with an arbitrariness of no less than $k - 1$ *functions of* $s - 1$ *variables* $\{\widetilde{P}_\sigma\}$.

To construct the algebra admitted by first integrals, we take advantage of the property mentioned at the beginning of this paragraph: the action of any admissible operator on a first integral gives a first integral again (which may be trivial). Let us denote $\mathrm{X}_\alpha[P_\sigma]=Q_{\alpha\sigma}$ and construct the operator

$$\mathrm{X}_B=\left(\sum_{\alpha=1}^{k}\eta_\alpha\widetilde{F}_\alpha\right)\partial_y,$$

where $\widetilde{F}_\alpha$, $\alpha=1,\,\dots,\,k$, are arbitrary functions of $s-1$ arguments $\{\widetilde{P}_\sigma\}$, $\sigma\neq\nu$. The invariance condition is written as

$$\widetilde{\mathrm{X}}_B[P_\nu-C_\nu]\Big|_{P_\nu=C_\nu}=\sum_{\alpha=1}^{k}\widetilde{F}_\alpha\mathrm{X}_\alpha[P_\nu]\Big|_{P_\nu=C_\nu}=\sum_{\alpha=1}^{k}\widetilde{F}_\alpha Q_{\alpha\nu}\Big|_{P_\nu=C_\nu}=0. \qquad (11.3.3.5)$$

Since the arguments of the functions $\widetilde{F}_\alpha$ are the first integrals $\widetilde{P}_\sigma$, and the quantities $Q_{\alpha\nu}$ are also first integrals (or constants), the last equality in (11.3.3.5) allows us to express any function $\widetilde{F}_\beta$ in terms of the others (provided that $Q_{\beta\nu}\neq 0$). The admissible operator is

$$\mathrm{X}_B=\left[\sum_{\substack{\alpha=1\\ \alpha\neq\beta}}^{k}\left(\eta_\alpha-\frac{Q_{\alpha\nu}}{Q_{\beta\nu}}\eta_\beta\right)\widetilde{F}_\alpha\right]\partial_y,\quad Q_{\beta\nu}\neq 0.$$

Remark 11.10. The arbitrariness of k functions of $s-1$ variables is achieved only if $Q_{\alpha\nu}=0$ for all $\alpha=1,\,\dots,\,k$, that is, if and only if equation (11.3.3.4) admits all $\{X_\alpha\}$.

Example 11.22. The equation $P_1=C_1$, where P_1 is an integral of equation (11.3.2.5), so that

$$y'''_{xxx}=\frac{1}{2}(y'_x)^{-1}(y''_{xx})^2-\frac{3}{2}Ay^{-2/3}(y'_x)^{-1}+C_1(y'_x)^{-1}, \qquad (11.3.3.6)$$

admits the Lie–Bäcklund operator

$$\widetilde{\mathrm{X}}_B=\left\{y'_x\widetilde{F}_1(\widetilde{P}_2,\widetilde{P}_3)+\left[\left(xy'_x-\frac{3}{2}y\right)\widetilde{P}_2+C_1\left(\frac{1}{2}x^2y'_x-\frac{3}{2}xy\right)\right]\widetilde{F}_3(\widetilde{P}_2,\widetilde{P}_3)\right\}\partial_y,$$

where

$$\widetilde{P}_2=\frac{3}{4}y(y'_x)^{-1}(y''_{xx})^2-\frac{1}{2}y'_xy''_{xx}-\frac{9}{4}Ay^{1/3}(y'_x)^{-1}+\frac{3}{2}C_1y(y'_x)^{-1}-C_1x,$$
$$\widetilde{P}_3=\frac{3}{4}xy(y'_x)^{-1}(y''_{xx})^2-\frac{1}{2}(xy'_x-3y)y''_{xx}-\frac{9}{4}Axy^{1/3}(y'_x)^{-1}+\frac{3}{2}C_1xy(y'_x)^{-1}-\frac{3}{2}C_1x^2$$

are first integrals of equation (11.3.3.6).

► Nöther's theorem.

Definition 1. The operator

$$E_n=\sum_{i=0}^{n}(-D_x)^i\partial_{y_x^{(i)}}=\partial_y-D_x\partial_{y'_x}+D^2\partial_{y''_{xx}}-\cdots+(-D_x)^n\partial_{y_x^{(n)}}, \qquad (11.3.3.7)$$

where

$$D=\partial_x+y'\partial_y+y''\partial_{y'}+\cdots+y^{(n)}\partial_{y^{(n-1)}}+\cdots,$$

is called an Euler operator of order n.

Let us look at the functional

$$S[y(x)] = \int\limits_V L(x, y, y'_x, \dots, y_x^{(n)})\, dx. \tag{11.3.3.8}$$

Searching for extremals of a variation problem for the functional (11.3.3.8) is known to reduce to solving the Euler–Lagrange equation

$$E_n\big[L(x, y, y'_x, \dots, y_x^{(n)})\big] = 0. \tag{11.3.3.9}$$

What is of interest is the case where, among all symmetry groups admitted by the Euler–Lagrange equation (11.3.3.9), there are also groups admitted by the Lagrangian $L(x, y, y'_x, \dots, y_x^{(n)})$. Such symmetry groups are known as *variational* or *Nötherian*; they play a major role in physics and mathematics, since they are closely related to conservation laws. Obviously, the order of equation (11.3.3.9) is $2n$.

Definition 2. The differential equation

$$F(x, y, y'_x, \dots, y_x^{(n)}) = 0, \qquad \frac{\partial F}{\partial_{y_x^{(n)}}} \not\equiv 0, \tag{11.3.3.10}$$

has a variational formulation if its solutions $y = \Theta(x)$ in the domain V coincide with extremals of the functional (11.3.3.8).

THEOREM 1. *A $2n$th-order differential equation has a variational formulation if and only if it coincides with the Euler–Lagrange equation of a Lagrangian $L(x, y, y'_x, \dots, y_x^{(n)})$, so that*

$$F(x, y, y'_x, \dots, y_x^{(2n)}) = E_m[L(x, y, y'_x, \dots, y_x^{(m)})]. \tag{11.3.3.11}$$

Remark 11.11. Only even-order equations can have a variational formulation.

Definition 3. The functional $S[y(x)]$ admits an infinitesimal operator X if its Lagrangian L is invariant under transformations of the group defined by the operator. The admitted group is called variational.

THEOREM 2. *If G is a group of variational symmetries of the function (11.3.3.8), then it is also a group of symmetries of the Euler–Lagrange equation $E_n(L) = 0$. In this case, the equation is said to admit the variational (Nötherian) symmetry.*

THEOREM 3. *Every admitted Nötherian group allows one to reduce the order of the Euler–Lagrange equation by two.*

The algorithm of order reduction for the Euler–Lagrange equation using a Nötherian operator is quite clear. However, the implementation of this algorithm as applied to differential equations causes certain difficulties: first, one has to find a suitable Lagrangian $L\big(x, y, y'_x, \dots, y_x^{(n)}\big)$, but there is no algorithm for obtaining it. The algorithm allowing order reduction for the equation by two without finding a Lagrangian relies on the following theorem.

EMMY NÖTHER'S THEOREM. *A symmetry of an even-order equation is Nötherian if the coordinate of its infinitesimal operator (in canonical form) coincides, up to a constant factor, with an integrating factor of a first integral of the equation. Furthermore, this operator makes the first integral identically zero.*

► Analogues of variational symmetries. Inverse problems.

Variational symmetry is only defined for even-order equations. All attempts to introduce a Hamiltonian structure for odd-order ODEs have failed so far (in terms of integrability). Over time, the impression has been formed that there is no similar symmetry structure for odd-order equations. However, this is not so. The following simple third-order equation is a counterexample:

$$y'''_{xxx} = 2yy'_x. \tag{11.3.3.12}$$

It is autonomous and has an autonomous first integral

$$y''_{xx} = y^2 + C. \tag{11.3.3.13}$$

This means that the symmetry of equation (11.3.3.12) is perfectly analogous to variational symmetry in the sense that the first integral (11.3.3.13) inherits it and allows the order of the original equation to be reduced by two.

We will be looking for classes of third-order equations possessing an analogue of variational symmetry or, equivalently, classes of equations admitting a point operator and a first integral inheriting this symmetry. From the viewpoint of integrability, this property is a direct analogue of Nötherian symmetry. The solution of this inverse problem is directly reduced to the simultaneous solution of *three* complicated determining systems: the invariance condition for the original equation with respect to an arbitrary point symmetry, the existence condition for a first integral with a set structure, and the invariance condition for this first integral with respect to the same point symmetry. Except for the simplest cases, the implementation of this algorithm is extremely difficult. Therefore, a different strategy will be followed; specifically, we will use the principle of similarity of one-parameter point groups in the plane and solve the problem for a selected simple symmetry (e.g., for an autonomous equation admitting the operator $\mathrm{X} = \partial_x$), while indenting to extend the obtained result to an arbitrary point symmetry.

$1°$. Let us compute point groups of equivalence for some subclasses of third-order equations. Obviously, the group of equivalence for the whole class of third-order equations is the set of arbitrary invertible point transformations

$$y = f(t, u), \qquad x = g(t, u) \tag{11.3.3.14}$$

with a nonzero Jacobian

$$D = \begin{vmatrix} f_t & g_t \\ f_u & g_u \end{vmatrix} = f_t g_u - f_u g_t \not\equiv 0. \tag{11.3.3.15}$$

Assuming t in (11.3.3.14) to be the independent variable, we write out the formulas for the transformation of the derivatives:

$$y'_x = \frac{f_t + f_u \dot{u}_t}{g_t + g_u \dot{u}_t}, \tag{11.3.3.16}$$

$$\begin{aligned} y''_{xx} = \big[(g_t f_u - g_u f_t)\ddot{u}_{tt} + (g_u f_{uu} - g_{uu} f_u)(\dot{u}_t)^3 + \\ + (g_t f_{uu} - g_{uu} f_t + 2g_u f_{tu} - 2g_{tu} f_u)(\dot{u}_t)^2 + \\ + (g_u f_{tt} - g_{tt} f_u + 2g_t f_{tu} - 2g_{tu} f_t)\dot{u}_t + \\ + g_t f_{tt} - g_{tt} f_t\big](g_t + g_u \dot{u}_t)^{-3}, \end{aligned} \tag{11.3.3.17}$$

$$
\begin{aligned}
y'''_{xxx} = \big\{&(g_t + g_u\dot{u}_t)(f_u g_t - f_t g_u)\dddot{u}_{ttt} + 3g_u(f_t g_u - f_u g_t)(\ddot{u}_{tt})^2 + \\
&+ 3\big[g_t(f_{uu}g_u - f_u g_{uu}) - g_u(f_{tu}g_u - f_u g_{tu}) - g_{uu}(f_u g_t - f_t g_u)\big](\dot{u}_t)^2\ddot{u}_{tt} + \\
&+ 3\big[g_t(f_{uu}g_t - f_t g_{uu}) - g_u(f_{tt}g_u - f_u g_{tt}) - 3g_{tu}(f_u g_t - f_t g_u)\big]\dot{u}_t\ddot{u}_{tt} + \\
&+ 3\big[g_t(f_{tu}g_t - f_t g_{tu}) - g_u(f_{tt}g_t - f_t g_{tt}) - g_{tt}(f_u g_t - f_t g_u)\big]\ddot{u}_{tt} + \\
&+ \big[g_u(f_{uuu}g_u - f_u g_{uuu}) - 3g_{uu}(f_{uu}g_u - f_u g_{uu})\big](\dot{u}_t)^5 + \\
&+ \big[g_t(f_{uuu}g_u - f_u g_{uuu}) + g_u(f_{uuu}g_t - f_t g_{uuu} + 3f_{tuu}g_u - 3f_u g_{tuu}) - \\
&- 6g_{tu}(f_{uu}g_u - f_u g_{uu}) - 3g_{uu}(f_{uu}g_t - f_t g_{uu} + 2f_{tu}g_u - 2f_u g_{tu})\big](\dot{u}_t)^4 + \\
&+ \big[g_t(f_{uuu}g_t - f_t g_{uuu} + 3f_{uut}g_u - 3f_u g_{uut}) + \\
&+ 3g_u(f_{ttu}g_u - f_u g_{ttu} + f_{tuu}g_t - f_t g_{tuu}) - 6g_{ut}(f_{uu}g_t - f_t g_{uu} + 2f_{ut}g_u - 2f_u g_{tu}) - \\
&- 3g_{uu}(f_{tt}g_u - f_u g_{tt} + 2f_{ut}g_t - 2f_t g_{ut}) - 3g_{tt}(f_{uu}g_u - f_u g_{uu})\big](\dot{u}_t)^3 + \\
&+ \big[g_u(f_{ttt}g_u - f_u g_{ttt} + 3f_{ttu}g_t - 3f_t g_{ttu}) + \\
&+ 3g_t(f_{ttu}g_u - f_u g_{ttu} + f_{tuu}g_t - f_t g_{tuu}) - 6g_{tu}(f_{tt}g_u - f_u g_{tt} + 2f_{tu}g_t - 2f_t g_{ut}) - \\
&- 3g_{tt}(f_{uu}g_t - f_t g_{uu} + 2f_{tu}g_u - 2f_u g_{tu}) - 3g_{uu}(f_{tt}g_t - f_t g_{tt})\big](\dot{u}_t)^2 + \\
&+ \big[g_u(g_t f_{ttt} - f_t g_{ttt}) + g_t(g_u f_{ttt} - f_u g_{ttt} + 3g_t f_{ttu} - 3f_t g_{ttu}) - \\
&- 3g_{tt}(g_u f_{tt} - f_u g_{tt} + 2g_t f_{tu} - 2f_t g_{tu}) - 6g_{tu}(g_t f_{tt} - f_t g_{tt})\big]\dot{u}_t + \\
&+ g_t(g_t f_{ttt} - g_{ttt}f_t) + 3g_{tt}(f_t g_{tt} - f_{tt}g_t)\big\}(g_t + g_u\dot{u}_t)^{-5}. \quad (11.3.3.18)
\end{aligned}
$$

Now, in order to find equivalence groups on a given subclass, we must find conditions for the form of the subclass to be preserved by using relations (11.3.3.16)–(11.3.3.18). First, let us find the equivalence group on the class of equations not involving the intermediate derivatives explicitly,

$$y'''_{xxx} = F(x, y). \quad (11.3.3.19)$$

So we look for transformations of the form

$$y'''_{xxx} = F(x, y) \longrightarrow \dddot{u}_{ttt} = G(t, u).$$

To this end, we require that the expression of the third derivative of the transformed variable (11.3.3.18) does not contain intermediate derivatives. It is clear that a necessary condition for that is $g_u \equiv 0$, or $g = g(t)$. By splitting expression (11.3.3.18) in powers of $\ddot{u}$ and $\dot{u}$, we obtain a determining system for the transformation elements (functions f and g):

$$
\begin{aligned}
&g_u = 0, \\
&f_{uu} = 0, \\
&g_t f_{tu} - g_{tt} f_u = 0, \\
&3g_t^2 f_{ttu} - g_t g_{ttt} f_u - 6g_t g_{tt} f_{tu} + 3g_{tt}^2 f_u = 0.
\end{aligned} \quad (11.3.3.20)
$$

Solving the first three equations gives

$$
\begin{aligned}
x &= g(t), \\
y &= Cg'(t)u + h(t),
\end{aligned} \quad (11.3.3.21)
$$

where $g(t)$ and $h(t)$ are arbitrary functions of t and C is an arbitrary constant ($C \neq 0$). In view of (11.3.3.21), the last equation in (11.3.3.20) becomes

$$2(g')^2 g''' - 3g'(g'')^2 = 0.$$

Its solution is

$$g = \frac{C_1}{t + C_2} + C_3.$$

Hence, the equivalence group for the class (11.3.3.19) consists of transformations of the form

$$x = \frac{C_1}{t + C_2} + C_3, \quad y = \frac{C_4 u}{(t + C_2)^2} + h(t), \tag{11.3.3.22}$$

where $C_1, \ldots, C_4$ are arbitrary constants and $h(t)$ is an arbitrary function of t. Very similar arguments lead to exactly the same result for the subclass of equations not involving the first derivative:

$$y'''_{xxx} = F(x, y, y''_{xx}). \tag{11.3.3.23}$$

THEOREM 1. *An arbitrary equivalence point transformation for the subclasses of third-order equations (11.3.3.19) and (11.3.3.23) has the form (11.3.3.22).*

For the subclass

$$y'''_{xxx} = F(x, y, y'_x), \tag{11.3.3.24}$$

the equivalence group is much wider and has a functional arbitrariness. The last equation in system (11.3.3.20) disappears and the solution to the system becomes (11.3.3.21).

THEOREM 2. *An arbitrary equivalence point transformation for the subclasses of third-order equations (11.3.3.24) has the form (11.3.3.21).*

2°. Let us focus on the problem, stated in the previous section, for the subclass of third-order equations not involving intermediate derivatives, i.e., subclass (11.3.3.19). In view of the known equivalence group, first integrals can be sought for the simplest autonomous third-order equation

$$y'''_{xxx} = F(y). \tag{11.3.3.25}$$

1. There is an autonomous first integral linear in the derivative y''_{xx}:

$$P = R(y, y'_x)y'_x + Q(y, y'_x). \tag{11.3.3.26}$$

THEOREM 3. *There is a nontrivial equation (11.3.3.25), with $F(y) \neq 0$, having an autonomous first integral of the form (11.3.3.26).*

Remark 11.12. Theorem 3 does not prevent equation (11.3.3.25) from having linear first integrals. A counterexample is the nontrivial equation

$$y'''_{xxx} = y^{-1}.$$

It has the linear first integral

$$P = yy''_{xx} - \frac{1}{2}(y'_x)^2 - x,$$

which is however not autonomous.

2. Now let us look, for equation (11.3.3.25), at the autonomous first integral quadratic in y''_{xx}:

$$P = R(y, y'_x)(y''_{xx})^2 + Q(y, y'_x)y''_{xx} + S(y, y'_x). \tag{11.3.3.27}$$

THEOREM 4. *The equation*

$$y'''_{xxx} = (ay^2 + by + c)^{-5/4}, \tag{11.3.3.28}$$

where a, b, and c are arbitrary constants, is the only equation from class (11.3.3.25) that has an autonomous first integral quadratic in the second derivative.

3. Autonomous first integrals of equation (11.3.3.25) cubic the second derivative.

THEOREM 5. *The equation*

$$y'''_{xxx} = (ay + b)^{-5/2} \tag{11.3.3.29}$$

is the only equation from class (11.3.3.25) that has a first integral cubic in the second derivative.

3°. Let us focus on inverse problems for the subclass (11.3.3.24). In this case, there are equations that have a linear first integral. An example is equation (11.3.3.24) with

$$F = \frac{R''(y'_x)^3 - 2S'y'_x}{2R}, \tag{11.3.3.30}$$

where R and S are arbitrary functions of y. The first integral is given by

$$P = Ry''_{xx} - \frac{1}{2}R'(y'_x)^2 + S.$$

We will now look for subclasses having a quadratic first integral (11.3.3.27). Applying the direct method results in the determining system

$$\begin{aligned} &R_{y'_x} = 0, \\ &R_y y'_x + Q_{y'_x} = 0, \\ &Q_y y'_x + S_{y'_x} = -2RF, \\ &S_y y'_x = -QF. \end{aligned} \tag{11.3.3.31}$$

From the third and forth equations of system (11.3.3.31), we obtain the consistency condition

$$\left(\frac{T}{y'_x} - \frac{1}{2}R_y y'_x\right) F_{y'_x} - 2RF_y = \left(\frac{T}{(y'_x)^2} + \frac{5}{2}R_y\right) F - \frac{1}{2}R_{yyy}(y'_x)^3 + T_{yy}y'_x, \tag{11.3.3.32}$$

which is a linear nonhomogeneous partial differential equation for $F(y, y')$. Solving this equation leads to the following statement.

THEOREM 6. *The subclass of equations*

$$y'''_{xxx} = R^{-3/2}y'_x\Phi(u) + \frac{2RR'' - (R')^2}{8R^2}(y'_x)^3 - \frac{2RT' - R'T}{4R^2}y'_x, \tag{11.3.3.33}$$

where $u = R^{-1/2}(y'_x)^2 + \int TR^{-3/2}\,dy$, R and T are arbitrary functions of y, and Φ is an arbitrary function of u, is the only subclass of equations of class (11.3.3.24) having a quadratic first integral in y''_{xx}:

$$P = R(y'')^2 + \left[-\frac{1}{2}R'(y'_x)^2 + T(y)\right] y''_{xx} + \frac{1}{16}\frac{(R')^2}{R}(y'_x)^4 - \int \Phi(u)\,du - \frac{R'T}{4R}(y'_x)^2 + \frac{1}{4}\frac{T^2}{R}. \tag{11.3.3.34}$$

Formula (11.3.3.33) gives *all* right-hand sides of equation (11.3.3.24) that possess the given property; in the special case $T \equiv 0$, it becomes much simpler and

$$F = R^{-5/4}\Phi_1\left(R^{-1/4}y'_x\right) + \frac{2RR'' - (R')^2}{8R^2}(y'_x)^3. \tag{11.3.3.35}$$

Remark 11.13. Of course, formula (11.3.3.33) contains (11.3.3.30), in which case the quadratic first integral is a quadratic form of the linear first integral.

⊙ *Literature for Section 11.3*: P. J. Olver (1986), G. W. Bluman and S. C. Anco (2002), A. D. Polyanin and V. F. Zaitsev (2003), N. H. Ibragimov (2010), V. F. Zaitsev and H. N. Huan (2013, 2014), V. F. Zaitsev and L. V. Linchuk (2014, 2015).

11.4 Underdetermined Equations

11.4.1 Preliminary Remarks

Consider the differential relation

$$y^{(n)}_x = F\big(x, y, y'_x, \dots, w, w'_x, \dots, w^{(n)}_x\big), \tag{11.4.1.1}$$

where $y(x)$ and $w(x)$ are some (unknown) smooth functions of the independent variable x. Relation (11.4.1.1) can be treated as an underdetermined differential equation or as a differential constraint between y and w.

Underdetermined ordinary differential equations and systems of such equations arise when one searches for exact solutions to nonlinear partial differential equations with the methods of generalized or functional separation of variables as the original PDEs are reduced to an underdetermined system of ODEs. Monge seems to have been the first to consider such systems when he was working on his geometric theory of PDEs (this is why such equations are sometimes referred to as *Monge equations*). Below is an example that illustrates such an ODE resulting from seeking generalized separable solutions to unsteady Navier–Stokes equations.

Example 11.23. Consider the first-order equation

$$yw'_x - wy'_x + k(y^2 + w^2) = 0 \tag{11.4.1.2}$$

which relates y and w. We change to the new variables

$$y = \rho\cos\xi, \quad w = -\rho\sin\xi,$$

where $\rho = \rho(x)$ and $\xi = \xi(x)$. As a result, we get the simple equation $\xi'_x = k$. It follows that

$$y = \rho(x)\cos(kx + C), \quad w = -\rho(x)\sin(kx + C), \tag{11.4.1.3}$$

where $\rho = \rho(x)$ is an arbitrary function and C is an arbitrary constant.

Interestingly, if $w = w(x)$ in (11.4.1.2) was treated as a given function, the equation would be a Riccati equation for $y = y(x)$, whose general solution would be much more difficult to obtain. In this case, the solution is given either in implicit form or by two relations

$$y^2 + w^2(x) = \rho^2(x), \quad w(x)/y = -\tan(kx + C),$$

which follow from (11.4.1.3).

11.4.2 Factorization Principle

Let X be a linear operator of the form

$$\mathrm{X} = \Phi\left(x, y, w, y'_x, w'_x, \ldots\right) \frac{\partial}{\partial y} + \Psi\left(x, y, w, y'_x, w'_x, \ldots\right) \frac{\partial}{\partial w}. \tag{11.4.2.1}$$

Operator (11.4.2.1) is a Lie–Bäcklund operator in the space of variables (x, y, w). The functions Φ and Ψ, dependent on arbitrarily high-order derivatives, are called the coordinates of this operator.

The operator

$$\underset{k}{\mathrm{X}} = \sum_{i=0}^{k} \left[D_x^i(\Phi) \frac{\partial}{\partial y_x^{(i)}} + D_x^i(\Psi) \frac{\partial}{\partial w^{(i)}} \right], \tag{11.4.2.2}$$

is called the kth prolongation of operator (11.4.2.1). Here, D_x is the total derivative operator defined by the formal series

$$D_x = \frac{\partial}{\partial x} + \sum_{i=0}^{\infty} y_x^{(i+1)} \frac{\partial}{\partial y_x^{(i)}} + \sum_{i=0}^{\infty} w_x^{(i+1)} \frac{\partial}{\partial w_x^{(i)}}.$$

The Lie–Bäcklund operator (11.4.2.1) is admitted by equation (11.4.1.1) if

$$\underset{n}{\mathrm{X}} \left[y_x^{(n)} - F\big(x, y, w, y'_x, w'_x, \ldots, y_x^{(n-1)}, w_x^{(n-1)}, w_x^{(n)}\big) \right] \Big|_{[y_x^{(n)} = F]} = 0. \tag{11.4.2.3}$$

The transformation $J_k = J_k(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)})$ is called a kth-order differential invariant of operator (11.4.2.1) by virtue of equation (11.4.1.1) if

$$\underset{k}{\mathrm{X}}[J_k] \big|_{[y_x^{(n)} = F]} = 0 \tag{11.4.2.4}$$

and $\left|\partial J_k / \partial y_x^{(k)}\right| + \left|\partial J_k / \partial w_x^{(k)}\right| \neq 0$.

Let the underdetermined differential equation (11.4.1.1) admit operator (11.4.2.1) and let $\mathbf{J}_k$ denote the set of functionally independent invariants of order not higher that k of operator (11.4.2.1) by virtue of equation (11.4.1.1). The universal invariant is $x = J_0 \in \mathbf{J}_k$ for any k.

By definition, equation (11.4.1.1) is factorized to the system

$$\begin{aligned} &G\big(x, z_1, z'_{1_x}, \ldots, z_{1_x}^{(m_1)}, z_2, z'_{2_x}, \ldots, z_{2_x}^{(m_2)}\big) = 0, \\ &z_i = H_i\big(x, y, y'_x, \ldots, y_x^{(r_1)}, w, w'_x, \ldots, w_x^{(r_2)}\big), \quad i = 1, 2, \end{aligned} \tag{11.4.2.5}$$

with $r_1, r_2, m_1, m_2 < n$ or the system

$$\begin{aligned} &G\big(x, z, z'_x, \ldots, z_x^{(m)}\big) = 0, \\ &z = H\big(x, y, y'_x, \ldots, y_x^{(n-m)}, w, w'_x, \ldots, w_x^{(n-m)}\big), \end{aligned} \tag{11.4.2.6}$$

with $0 < m < n$, if system (11.4.2.5) or (11.4.2.6) is a consequence of equation (11.4.1.1) (in the sense that if $y = y(x)$ and $w = w(x)$ satisfy equation (11.4.1.1), they also satisfy system (11.4.2.5) or (11.4.2.6)). These systems are called factor systems.

A factor system is a kind of Russian nesting doll in which the first equation is either an ordinary differential equation of order $< n$ (system (11.4.2.6)) or an underdetermined differential equation of a reduced order (system (11.4.2.5)). The remaining equations of system (11.4.2.5) and (11.4.2.6) also have a simpler structure than the original equation.

THEOREM 1. *Let the underdetermined differential equation (11.4.1.1) admit a Lie–Bäcklund operator (11.4.2.1) having a low invariant*

$$z = H\big(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)}\big) \in \mathbf{J}_k \quad (k \leqslant n).$$

1. *If* $D_x^{n-k}(z)\big|_{[y_x^{(n)}=F]} \in \mathbf{J}_{n-1}$, *then equation (11.4.1.1) is factorized to the system of two equations*
$$z = H\big(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)}\big),$$
$$z_x^{(n-k)} = G\big(x, z, \ldots, z_x^{(n-k-1)}\big).$$

2. *If* $D_x^{n-k}(z)\big|_{[y_x^{(n)}=F]} \in \mathbf{J}_n \setminus \mathbf{J}_{n-1}$ *and* $z^* = H^*\big(x, y, w, \ldots, y_x^{(n)}, w_x^{(n)}\big)$ *is such that* $z^* \in \mathbf{J}_n \setminus \mathbf{J}_{n-1}$ *and the mappings* $x, z^{(0)}, \ldots, z_x^{(n-k)}, z^*$ *are functionally independent, then equation (11.4.1.1) reduces to the system of three equations*
$$z = H\big(x, y, w, \ldots, y_x^{(k)}, w_x^{(k)}\big),$$
$$z^* = H^*\big(x, y, w, \ldots, y_x^{(n)}, w_x^{(n)}\big),$$
$$z^* = G\big(x, z, \ldots, z_x^{(n-k)}\big).$$

THEOREM 2. *Let the underdetermined differential equation (11.4.1.1) admit a formal operator (11.4.2.1) having two low invariants*

$$z_i = H_i\big(x, y, w, \ldots, y_x^{(k_i)}, w_x^{(k_i)}\big) \in \mathbf{J}_{k_i}, \quad i = 1, 2,$$

of order $k_i \leqslant n$. *Then*

1) *if* $D_x^{n-k_1}(z_1)\big|_{[y_x^{(n)}=F]} \in \mathbf{J}_{n-1}\big|_{[y_x^{(n)}=F]}$, *then equation (11.4.1.1) can be represented as the factor system*
$$z_1 = H_1\big(x, y, w, \ldots, y_x^{(k_1)}, w_x^{(k_1)}\big),$$
$$z_2 = H_2\big(x, y, w, \ldots, y_x^{(k_2)}, w_x^{(k_2)}\big),$$
$$z_1^{(n-k_1)} = G\big(x, z_1, \ldots, z_{1_x}^{(n-k_1-1)}, z_2, \ldots, z_{2_x}^{(n-k_2-1)}\big),$$
for $k_2 < n$,
$$z_1 = H_1\big(x, y, w, \ldots, y_x^{(k_1)}, w_x^{(k_1)}\big),$$
$$z_1^{(n-k_1)} = G\big(x, z_1, \ldots, z_{1_x}^{(n-k_1-1)}\big),$$
for $k_2 = n$;

2) *if* $D_x^{n-k_1}(z_1)\big|_{[y^{(n)}=F]}$ *and* $D_x^{n-k_2}(z_2)\big|_{[y^{(n)}=F]} \in \mathbf{J}_n \setminus \mathbf{J}_{n-1}$, *then equation (11.4.1.1) is factorized to the system*
$$z_1 = H_1\big(x, y, w, \ldots, y_x^{(k_1)}, w_x^{(k_1)}\big),$$
$$z_2 = H_2\big(x, y, w, \ldots, y_x^{(k_2)}, w_x^{(k_2)}\big),$$
$$z_1^{(n-k_1)} = G\big(x, z_1, \ldots, z_{1_x}^{(n-k_1-1)}, z_2, \ldots, z_{2_x}^{(n-k_2)}\big).$$

THEOREM 3. *Let the underdetermined differential equation of order* n *(11.4.1.1)*

1) *reduce to the ordinary differential equation*

$$z_1^{(n-k_1)} = G\big(x, z_1, z'_{1_x}, \dots, z_{1_x}^{(m_1)}\big)$$

with the substitution

$$z_1 = H_1\big(x, y(x), w(x), \dots, y_x^{(k_1)}(x), w_x^{(k_1)}(x)\big), \quad k_1 < n,$$

where $0 \leqslant m_1 < n - k_1$*;*

2) *reduce to the underdetermined differential equation*

$$z_{1_x}^{(n-k_1)} = G\big(x, z_1, z'_{1_x}, \dots, z_{1_x}^{(m_1)}, z_2, z'_{2_x}, \dots, z_{2_x}^{(m_2)}\big)$$

with a substitution of the form

$$z_1 = H_1\big(x, y(x), w(x), \dots, y_x^{(k_1)}(x), w_x^{(k_1)}(x)\big), \qquad k_1 < n,$$
$$z_2 = H_2\big(x, y(x), w(x), \dots, y_x^{(k_2)}(x), w_x^{(k_2)}(x)\big), \qquad k_2 < n,$$
$$\frac{\partial z_1}{\partial y_x^{(k_1)}} \neq 0, \quad -1 < m_1 < n - k_1, \quad 0 \leqslant m_2 \leqslant n - k_2.$$

Then the original equation admits a formal operator (11.4.2.1) such that all z_i $(i = 1, 2)$ *are its invariants:* $z_i \in \mathbf{J}_{k_i}$.

11.4.3 Some Technical Elements. Examples

Consider the first-order underdetermined differential equation linear in the derivatives

$$y'_x + G(x, y, w) w'_x + F(x, y, w) = 0 \qquad (11.4.3.1)$$

and let us look for a point infinitesimal operator admissible by equation (11.4.3.1) and having the form

$$\mathrm{X} = \xi(x, y, w)\,\partial_x + \eta(x, y, w)\,\partial_y + \zeta(x, y, w)\,\partial_w. \qquad (11.4.3.2)$$

Note that, unlike first-order ODEs, the invariance condition for underdetermined differential equations can be split into a system, because the independent variable w'_x arises.

The determining system consists of three equations, with the first one (the coefficient of $(w'_x)^2$) satisfied identically and the other two expressed as

$$\eta_w - \eta_y G + G(\zeta_w - \zeta_y G) + F(\xi_w - \xi_y G) + \xi G_x + \eta G_y + \zeta G_w = 0,$$
$$\eta_x - \eta_y F + G(\zeta_x - \zeta_y F) + F(\xi_x - \xi_y F) + \xi F_x + \eta F_y + \zeta F_w = 0.$$

Let us require that the functions $I_1 = x$ and $I_2 = z = H(x, y, w)$ are invariants of operator (11.4.3.2), so that the second invariant is independent of derivatives. To this end, we rewrite operator (11.4.3.2) in the canonical form

$$\widetilde{\mathrm{X}} = (\eta - \xi y'_x)\,\partial_y + (\zeta - \xi w'_x)\,\partial_w$$

and solve the characteristic equation

$$\frac{dy}{\eta - \xi y'_x} = \frac{dw}{\zeta - \xi w'_x}$$

or, by virtue of the original equation (11.4.3.1),

$$\frac{dy}{\eta + \xi(Gw'_x + F)} = \frac{dw}{\zeta - \xi w'_x}.$$

This equation must have the integral $H(x, y, w) = C$, so that $H_y\,dy + H_w\,dw = 0$. As a result, the characteristic equation becomes

$$H_w(\xi w'_x - \zeta) + H_y[\eta + \xi(Gw'_x + F)] = 0.$$

After splitting with respect to the independent variable w'_x, this equation gives two conditions

$$H_w = GH_y, \qquad \eta = \xi F - \zeta G.$$

Substituting the second condition in both determining equations gives another condition relating F and G:

$$F_w + FG_y - F_y G - G_x = 0.$$

This allows us to uniquely determine the coefficients of equation (11.4.3.1) in terms of the invariant H:

$$F = \frac{H_x}{H_y} + \frac{1}{H_y}\Phi(x, H), \qquad G = \frac{H_w}{H_y}. \tag{11.4.3.3}$$

Thus, equation (11.4.3.1) is factorized to the system

$$\begin{aligned} &z'_x + \Phi(x, z) = 0,\\ &z = H(x, y, w), \end{aligned} \tag{11.4.3.4}$$

if the coefficients of equation (11.4.3.1) satisfy conditions (11.4.3.3); furthermore, as one can easily see, these conditions are necessary and sufficient.

If the first equation of system (11.4.3.4) has been integrated, the underdetermined differential equation (11.4.3.1) reduces to a functional (not differential) equation, the second equation of system (11.4.3.4).

It should be stressed once again that the obtained result is absolutely *independent* of whether there are additional constraints between the variables y and w. One should only bear in mind that if there is such a constraint, the quantity w' must be replaced with its value in terms of y and its derivatives.

Example 11.24. The underdetermined first-order differential equation

$$y'_x + G(x)w'_x + f(x)y + g(x)w + h(x) = 0$$

with $g(x) = G'(x) + f(x)G(x)$ admits the infinitesimal operator

$$\mathrm{X} = G(x)\partial_y - \partial_w$$

and is factorized to the system

$$\begin{aligned} &z'_x + f(x)z + h(x) = 0,\\ &z = y + G(x)w. \end{aligned}$$

11.4.4 On Second-Order Equations

This section will look thoroughly into some of the results following from the general theorems 1 to 3 as applied to second-order underdetermined differential equations.

THEOREM 4. *For the canonical infinitesimal operator*

$$\hat{\mathrm{X}} = \bigl[\eta_1(x,y,w) - \xi(x,y,w)y'_x\bigr]\partial_y + \bigl[\eta_2(x,y,w) - \xi(x,y,w)w'_x\bigr]\partial_w, \tag{11.4.4.1}$$

admitted by the underdetermined second-order differential equation

$$y''_{xx} = F(x,y,w,y'_x,w'_x,w''_{xx}), \tag{11.4.4.2}$$

to possess first-order differential invariants, it is necessary that

1) *for* $\xi \neq 0$, *the equation be linear in the highest derivative of* w, *so that*

$$F = f_1 w''_{xx} + f_2, \quad F_{w''_{xx}} \neq 0, \quad f_i = f_i(x,y,w,y'_x,w'_x), \quad i = 1,2, \tag{11.4.4.3}$$

or

$$F = g_1 w'_x + g_2, \quad F_{w'_x} \neq 0, \quad g_i = g_i(x,y,w,y'_x), \quad i = 1,2, \tag{11.4.4.4}$$

with the last condition being also sufficient for the class of equations

$$y''_{xx} = F(x,y,w,y'_x,w'_x),$$

if the relation

$$\begin{aligned}\eta_1 g_{1x} + (2\eta_1 - \xi y'_x)y'_x g_{1y} + \eta_2 y'_x g_{1w} - \eta_2 {g_1}^2 +{}\\ + \Bigl[\eta_{1x} + (\eta_{1y} - \xi_x)y'_x - \xi_y {y'_x}^2\Bigr] y'_x g_{1y'_x} + \Bigl(\eta_{1x} + \eta_{2w}y'_x + \xi_y {y'_x}^2\Bigr) g_1 +{}\\ + (\eta_1 - \xi y'_x)(g_{1y'_x} g_2 - g_1 g_{2y'_x} - g_{2w}) - \xi g_1 g_2 = 0\end{aligned} \tag{11.4.4.5}$$

holds;

2) *for* $\xi = 0$, *such invariants always exist.*

However, as mentioned previously, what is important is not only the existence of differential invariants but also the dimensionality of the invariant basis admitted by the operator, since it affects the structure of the system to which the original equation is reduced.

If the coordinate ξ in the operator (11.4.4.1) is zero, the invariant basis consists of two universal invariants, including $J_0 = x$, and one first-order differential invariant. If the equation has the form (11.4.4.4) and condition (11.4.4.5) holds, the dimensionality of the invariant basis admitted by operator (11.4.4.1) equals two, as the basis consists of one invariant of the zeroth order $J_0 = x$ and one first-order differential invariant. This case is remarkable because the factorization of the underdetermined differential equation (11.4.4.1) reduces it to a first-order ordinary differential equation. If the structure of the original equation satisfies condition (11.4.4.4), the invariant basis contains one universal invariant $J_0 = x$ and no more than two first-order differential invariants; in addition, the following theorem holds.

THEOREM 5. *For the canonical infinitesimal operator (11.4.4.1), with $\xi \neq 0$, admitted by equation (11.4.4.2) to have two different first-order differential invariants, it is necessary and sufficient that the right-hand side of equation (11.4.4.2) have the form (11.4.4.3) and the functions f_1 and f_2 satisfy the relations*

$$f_1 = \frac{\eta_1 - \xi y'_x}{\eta_2 - \xi w'_x},$$
$$f_1 f_{2y'_x} + f_{2w'_x} - 2D(f_1)\Big|_{y''_{xx}=f_1 w''_{xx}+f_2} = 0.$$

Significant restrictions on the structure of invariants of point operators lead one to consider the Lie–Bäcklund operator; however, the algorithm for finding an admissible operator becomes more complicated. In general, such an operator can be written as

$$\mathrm{X} = \exp\left(\int \zeta_1\, dx\right)\partial_y + \exp\left(\int \zeta_2\, dx\right)\partial_w, \qquad (11.4.4.6)$$

where ζ_1 and ζ_2 can depend on x, y, and w and their derivatives of any order. For simplicity, we will give the case $\zeta_i = \zeta_i(x, y, w, y'_x, w'_x)$, $i = 1, 2$, a detailed consideration. Let us write out the determining equation for the underdetermined second-order differential equation

$$y''_{xx} = F(x, y, w, y'_x, w'_x, w''_{xx}) \qquad (11.4.4.7)$$

and operator (11.4.4.6):

$$\begin{aligned}\big[\zeta_{1x} + y'_x\zeta_{1y} + w'\zeta_{1w} + +F\zeta_{1y'_x} + w''_{xx}\zeta_{1w'_x} + {\zeta_1}^2 - \zeta_1 F_{y'_x} - F_y\big] - \\ - \big[(\zeta_{2x} + y'_x\zeta_{2y} + w'_x\zeta_{2w} + F\zeta_{2y'_x} + w''_{xx}\zeta_{2w'_x} + {\zeta_2}^2)F_{w''_{xx}} + \\ + F_{w'_x}\zeta_2 + F_w\big]\exp\left\{\int(\zeta_2 - \zeta_1)\, dx\right\} = 0. \qquad (11.4.4.8)\end{aligned}$$

In splitting equation (11.4.4.8), we must take into account the structure of the nonlocal factor $\exp\left\{\int(\zeta_2 - \zeta_1)\, dx\right\}$. If the integrand is a total derivative of some function, the further reasoning is similar to that used in constructing the determining system for a point operator. Otherwise, if the integrand is not a total derivative, equation (11.4.4.8) should first be split with respect to the nonlocal variable. The invariant basis admitted by the operators found in the latter case can be chosen so that each invariant depends on either x, y, y' or x, w, w'. The structure of the invariants of the basis affects the type of factorization of equation (11.4.4.7).

Example 11.25. The underdetermined second-order differential equation

$$\begin{aligned}y''_{xx} = C{w'_x}^2 + (\psi_1 y + \psi_2)y'_x + (\chi_1 w + \chi_2)w'_x + \tfrac{1}{2}({\psi_1}' + \psi_1\alpha - \psi_1\psi_2)y^2 + \\ + (\alpha' + \alpha^2 - \psi_2\alpha)y + h(x, w), \qquad (11.4.4.9)\end{aligned}$$

where ψ_1, ψ_2, χ_1, χ_2, and α are sufficiently smooth functions of x and $C \in \mathbb{R}$, admits, under the condition that $2Cw'_x + \chi_1 w + \chi_2 \neq 0$, the Lie–Bäcklund operator

$$\mathrm{X} = \exp\left[\int(\psi_1 y + \alpha)dx\right]\partial_y + \exp\left[-\int \frac{\chi_2 w'_x + h_{2w}}{2Cw'_x + \chi_2 w + \chi_3}\, dx\right]\partial_w.$$

Apart from the universal invariant x, this operator has two low invariants (two first-order differential invariants) using which, one can factorize the original equation to the system

$$\begin{aligned} z_1 &= y'_x - \frac{1}{2}\psi_1 y^2 - \alpha y, \\ z_2 &= C{w'_x}^2 + \chi_1 w w'_x + \chi_2 w' + h, \\ {z_1}'_x &= (\psi_2 - \alpha) z_1 + z_2. \end{aligned} \tag{11.4.4.10}$$

The outer equation in this system is an underdetermined first-order differential equation.

THEOREM 6. *Equation (11.4.4.7) is factorized to the system*

$$\begin{aligned} u &= J_1^1(x, y, y'_x), \\ v &= J_1^2(x, w, w'_x), \\ G&(x, u, v, v', u') = 0, \end{aligned}$$

where

$$\frac{\partial J_1^1(x, y, y'_x)}{\partial y'_x} \neq 0, \qquad \frac{\partial J_1^2(x, w, w'_x)}{\partial w'_x} \neq 0,$$

if and only if it admits operator (11.4.4.6) whose structural components ζ_1 and ζ_2 satisfy the system

$$\begin{aligned} &\zeta_{1x} + y'\zeta_{1y} + F\zeta_{1y'_x} + (\zeta_1 - F_{y'_x})\zeta_1 - F_y = 0, \\ &F_{w''_{xx}}\zeta_{2x} + F_{w''_{xx}} w'_x \zeta_{2w} + F_{w''_{xx}} w''_{xx} \zeta_{2w'_x} + (F_{w'_x} + F_{w''_{xx}}\zeta_2)\zeta_2 + F_w = 0. \end{aligned}$$

The nature of the admitted operator itself may suggest that, on the manifold in question, the operator has only one first-order differential invariant and one universal invariant x. Then, according to Item 1 of Theorem 1, the original equation reduces to a system of two equations. It follows that the outer equation is surely a first-order ordinary differential equation; on solving this equation, we are guaranteed to reduce the order of the original equation by one.

Example 11.26. The underdetermined second-order differential equation

$$y y''_{xx} + w''_{xx} + (y'_x)^2 + (w'_x)^2 + (y y'_x + w) w'_x = 0, \tag{11.4.4.11}$$

admits the nonlocal operator

$$\mathrm{X} = \frac{\partial}{\partial w} - \left[y^{-1} \int (y y'_x + w'_x + w)\, dx \right] \frac{\partial}{\partial y}.$$

The factor system has the form

$$\begin{aligned} z &= e^w (y y'_x + w'_x + w - 1), \\ z'_x &= 0. \end{aligned}$$

Integrating the last equation yields a first integral of the original equation:

$$e^w (y y'_x + w'_x + w - 1) = C. \tag{11.4.4.12}$$

It follows that any underdetermined second-order differential equation (11.4.4.11) can be reduced to the underdetermined first-order equation (11.4.4.12).

Example 11.27. Let us look at the underdetermined differential equation

$$y''_{xx} = C(y'_x)^2 + (\psi_1 y - C\psi_2 w + \psi_3)y'_x + \psi_2 w'_x + H_1 w + H_0,$$

where $C \neq 0$, ψ_1, ψ_2, and ψ_3 are sufficiently smooth functions of x, $\psi_2 \neq 0$, and H_1 and H_0 are given by

$$\begin{aligned} H_0 &= -\frac{1}{C^2{\alpha_1}^2}\Big\{\alpha_2 \exp(Cy) + \big[C(\psi_1 y + \psi_3) + \psi_1\big](C\alpha_3 + {\alpha_1}')\alpha_1 \\ &\quad + \big[C({\psi_1}' y + {\psi_3}') + {\psi_1}'\big]{\alpha_1}^2 + C({\alpha_1}''\alpha_1 + C{\alpha_1}'\alpha_3 + C\alpha_1{\alpha_3}' + C^2{\alpha_3}^2)\Big\}, \\ H_1 &= \frac{\psi_2{\alpha_1}' + {\psi_2}'\alpha_1 + C\psi_2\alpha_3}{\alpha_1}, \end{aligned}$$

with $\alpha_i = \alpha_i(x)$, $i = 1, 2, 3$, and $\alpha_1 \neq 0$. Using the classical algorithm for solving a direct problem, we find a family of admissible point operators whose canonical form is

$$\hat{\mathrm{X}} = \big(\eta_1 - \xi y'_x\big)\partial_y + \big(\eta_2 - \xi w'_x\big)\partial_w,$$

and the coordinates are

$$\begin{aligned} \xi &= \alpha_1, \\ \eta_1 &= g\exp(Cy) + \alpha_3, \\ \eta_2 &= \frac{(g\exp(Cy) + \alpha_3)C\psi_2 w - N - (H_1 w + H_0)\alpha_1}{\psi_2}, \end{aligned}$$

where N is given by

$$\begin{aligned} N = \frac{1}{\alpha_1}\big\{\big[(\psi_1 y + \psi_3)\alpha_1 g + {\alpha_1}' g - \alpha_1 g' + Cg\alpha_3\big]\exp(Cy) \\ + (\psi_1 y + \psi_3)\alpha_1\alpha_3 + {\alpha_1}'\alpha_3 - \alpha_1{\alpha_3}' + C{\alpha_3}^2\big\}, \end{aligned}$$

and $g = g(x)$. The basis of the zeroth- and first-order invariants consists (regardless of g and, in particular, for $g \equiv 0$) of two functions: the universal invariant x and one differential invariant. Therefore, the equation is factorized to the system

$$\begin{aligned} z &= \frac{C^2\alpha_1(y'_x - \psi_2 w) + C\psi_1\alpha_1 y + (\psi_1 + C\psi_3)\alpha_1 + C(C\alpha_3 + \alpha'_1)}{C^2\alpha_1\exp(Cy)}, \\ z'_x &+ \frac{\alpha'_1 + C\alpha_3}{\alpha_1}z + \frac{\alpha_2}{C^2\alpha_1^2} = 0. \end{aligned}$$

The second equation involves only one dependent variable, z, with respect to which it is a first-order linear differential equation, which is always solvable.

The admissible Lie algebra is infinite-dimensional, consisting of a one-dimensional subalgebra L_1 and an infinite-dimensional subalgebra L_∞ defined by the operators

$$\begin{aligned} \hat{\mathrm{X}}_1 = \big(\alpha_3 - \alpha_1 y'_x\big)\partial_y + \bigg[\frac{(C\psi_2 w - \psi_1 y - \psi_3)\alpha_1\alpha_3 - \alpha'_1\alpha_3 + \alpha_1\alpha'_3 - C\alpha_3^2}{\psi_2\alpha_1} - \\ - \frac{(H_1 w + H_0)\alpha_1}{\psi_2} - \alpha_1 w'_x\bigg]\partial_w, \end{aligned}$$

and

$$\hat{\mathrm{X}}_\infty = g\exp(Cy)\partial_y + \frac{\big[(C\psi_2 w - \psi_1 y - \psi_3)\alpha_1 g - \alpha'_1 g + \alpha_1 g' - Cg\alpha_3\big]\exp(Cy)}{\psi_2\alpha_1}\partial_w.$$

The factorization obtained using the operator $\hat{\mathrm{X}}_1$ is specified above. The invariant basis of the second operator, $\hat{\mathrm{X}}_\infty$, consists, unlike $\hat{\mathrm{X}}_1$, of three invariants: two universal invariants and one differential invariant. The system to which the original equation is reduced has the form

$$z_1 = \frac{Cgy'_x + g'}{Cg\exp(Cy)},$$
$$z_2 = \frac{C\alpha_1 g(\psi_1 y - C\psi_2 w + \psi_3) + \psi_1\alpha_1 g + C^2\alpha_3 g - C\alpha_1 g' + C\alpha'_1 g}{C^2\psi_2\alpha_1 g\exp(Cy)},$$
$$z'_1 + \psi_2 z'_2 + \frac{C\psi_2\alpha_3 + \psi_2\alpha'_1 + \psi'_2\alpha_1}{\alpha_1} z_1 + \frac{C\alpha_3 + \alpha'_1}{\alpha_1} z_2 + \frac{\alpha_2}{C^2\alpha_1^2} = 0.$$

With the change of variable $z = z_1 + \psi_2 z_2$, we can reduce the original equation to a first-order ordinary differential equation, which is obtained by using the operator $\hat{\mathrm{X}}_1$.

⊙ *Literature for Section 11.4*: L. V. Linchuk (2001), V. I. Elkin (2009, 2010), A. D. Polyanin and V. F. Zaitsev (2012), V. F. Zaitsev and L. V. Linchuk (2014), A. D. Polyanin and A. I. Zhurov (2016c).