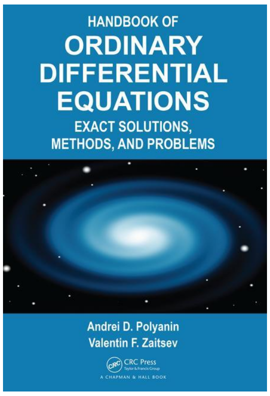

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

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Chapter 12: Discrete-Group Methods

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Chapter 12

Discrete-Group Methods

12.1 Discrete Group Method for Point Transformations

12.1.1 Classes of ODEs with Parameters. Discrete Group of Point Transformations

Consider transformations of the class of ordinary differential equations

$$y_x^{(n)} = F\big(x, y, \dots, y_x^{(n-1)}, \mathbf{a}\big), \tag{12.1.1.1}$$

whose elements are uniquely defined by a vector of essential parameters **a**.

Any set of invertible transformations

$$x = f(t, u), \quad y = g(t, u) \qquad (f_t g_u - f_u g_t \neq 0), \tag{12.1.1.2}$$

mapping each equation of class (12.1.1.1) into some (other) equation of the same class

$$u_t^{(n)} = F\big(t, u, \dots, u_t^{(n-1)}, \mathbf{b}\big), \tag{12.1.1.3}$$

and containing the identical transformation is called a *discrete point group of transformations* admitted by the class (12.1.1.1). Transformation (12.1.1.2) maps any solution of equation (12.1.1.1) to a solution of equation (12.1.1.3). Therefore, knowing the discrete group of transformations for some class of equations and having a set of solvable equations of this class, one can construct new solvable cases.

Point transformations (12.1.1.2) can be found by a direct method—namely, if one substitutes an arbitrary transformation of the form (12.1.1.2) into equation (12.1.1.1) and imposes condition (12.1.1.3), one arrives at a determining equation containing partial derivatives up to order n of the unknown functions f and g and having variable coefficients depending on x, y, $y'_x, \dots, y_x^{(n-1)}$. Since the functions f and g do not depend on the derivatives, the determining equation can be "split" with respect to the "independent" variables $y'_x, \dots, y_x^{(n-1)}$, and we obtain an overdetermined system which is nonlinear, in contrast to that obtained by the Lie method (see Section 11.1.1).

12.1.2 Illustrative Examples

Example 12.1. For second-order equations

$$y''_{xx} = F(x, y, y'_x, \mathbf{a}), \tag{12.1.2.1}$$

the substitution of (12.1.1.2) into (12.1.2.1) yields

$$
\begin{aligned}
&(f_t g_u - g_t f_u)u''_{tt} + (f_u g_{uu} - g_u f_{uu})(u'_t)^3 \\
&\quad + (f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut})(u'_t)^2 \\
&\quad + (f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut})u'_t + f_t g_{tt} - g_t f_{tt} \\
&\quad = (f_t + f_u u'_t)^3 F\Big(f, g, \frac{g_t + g_u u'_t}{f_t + f_u u'_t}, \mathbf{a}\Big).
\end{aligned}
\tag{12.1.2.2}
$$

Let us require that the transformed equation (12.1.2.2) belong to the class (12.1.2.1), i.e.,

$$
u''_{tt} = F(t, u, u'_t, \mathbf{b}). \tag{12.1.2.3}
$$

Condition (12.1.2.3) imposed on the determining equation (12.1.2.2), i.e., the replacement of u''_{tt} by the right-hand side of equation (12.1.2.3), leads us to the relation

$$
\begin{aligned}
&(f_t g_u - g_t f_u)F(t, u, u'_t, \mathbf{b}) + (f_u g_{uu} - g_u f_{uu})(u'_t)^3 \\
&\quad + (f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut})(u'_t)^2 \\
&\quad + (f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut})u'_t + f_t g_{tt} - g_t f_{tt} \\
&\quad = (f_t + f_u u'_t)^3 F\Big(f, g, \frac{g_t + g_u u'_t}{f_t + f_u u'_t}, \mathbf{a}\Big),
\end{aligned}
\tag{12.1.2.4}
$$

which contains the “independent” variable u'_t. Expanding the function F into a series in powers of u'_t, we can represent (12.1.2.4) in the form

$$
\sum_{k=0}^{\infty} P_k\big(x, y, [f], [g]\big)(u'_t)^k = 0, \tag{12.1.2.5}
$$

where the symbols $[f]$ and $[g]$ indicate dependence on the functions f, g and their partial derivatives involved in (12.1.2.4). The sum in (12.1.2.5) is finite if F is a polynomial with respect to the third variable [for a polynomial of degree $n \ge 4$, both sides of the equation must be first multiplied by $(f_t + f_u u'_t)^{n-3}$]. Condition (12.1.2.5) is satisfied if the following equations hold:

$$
P_k = 0, \quad k = 0,\ 1,\ 2,\ \ldots
$$

Example 12.2. Consider a special case of equation (12.1.2.1) with the right-hand side independent of the derivative y'_x:

$$
y''_{xx} = F(x, y, \mathbf{a}). \tag{12.1.2.6}
$$

Relation (12.1.2.4) has the form:

$$
\begin{aligned}
&(f_t g_u - g_t f_u)F(t, u, \mathbf{b}) + (f_u g_{uu} - g_u f_{uu})(u'_t)^3 + (f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut})(u'_t)^2 \\
&\quad + (f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut})u'_t + f_t g_{tt} - g_t f_{tt} = (f_t + f_u u'_t)^3 F(f, g, \mathbf{a}).
\end{aligned}
$$

In this case, the sum (12.1.2.5) is finite and the determining system has the form:

$$
\begin{aligned}
&f_u g_{uu} - g_u f_{uu} = f_u^3 F(f, g, \mathbf{a}), \\
&f_t g_{uu} - g_t f_{uu} + 2f_u g_{ut} - 2g_u f_{ut} = 3f_t f_u^2 F(f, g, \mathbf{a}), \\
&f_u g_{tt} - g_u f_{tt} + 2f_t g_{ut} - 2g_t f_{ut} = 3f_t^2 f_u F(f, g, \mathbf{a}), \\
&f_t g_{tt} - g_t f_{tt} + (f_t g_u - g_t f_u)F(t, u, \mathbf{b}) = f_t^3 F(f, g, \mathbf{a}).
\end{aligned}
\tag{12.1.2.7}
$$

It can be shown that for $f_t f_u g_t g_u \ne 0$, solving system (12.1.2.7) is equivalent to solving the original equation (12.1.2.6).

Consider the case $f_u = 0$. In this case, the first equation of the system holds identically and the system becomes

$$
\begin{aligned}
&f'_t g_{uu} = 0, \\
&g_u f_{tt} - 2f_t g_{ut} = 0, \\
&f_t g_{tt} - g_t f_{tt} + f_t g_u F(t, u, \mathbf{b}) = f_t^3 F(f, g, \mathbf{a}).
\end{aligned}
\tag{12.1.2.8}
$$

Since $f'_t \neq 0$, the first two equations yield

$$g(t,u) = T(t)u + \Theta(t), \quad f'_t = C[T(t)]^2. \tag{12.1.2.9}$$

Substituting (12.1.2.9) into the last equation of system (12.1.2.8) and "splitting" the resulting relation with respect to powers of the "independent" variable u, we obtain a new system of (ordinary) differential equations. Solving this system, we find the unknown functions T and Θ, and finally, the desired discrete group of transformations. In order to give calculation details, one has to know the specific structure of the function $F(x,y)$, for in the general case it was only shown that any discrete point group of transformations of equation (12.1.2.6) for $f_u = 0$ consists of Kummer–Liouville transformations (12.1.2.9).

Example 12.3. Consider the generalized Emden–Fowler equation:

$$y''_{xx} = Ax^n y^m (y'_x)^l. \tag{12.1.2.10}$$

Here, $\mathbf{a} = \{n,\ m,\ l\}$ is the vector of essential parameters, and A is an unessential parameter (it can be made equal to unity by scaling the independent variable and the unknown function).

1°. First, we note that equation (12.1.2.10) admits a discrete group of transformations determined by the hodograph transformation, i.e., by passing to the inverse function:

$$x = u, \quad y = t, \qquad \text{where} \quad u = u(t). \tag{12.1.2.11}$$

This transformation is a consequence of the invariance of equation (12.1.2.10) with respect to the transformation $x \longleftrightarrow y$, $n \longleftrightarrow m$, $l \longleftrightarrow 3-l$, $A \longleftrightarrow -A$ (note that the hodograph transformation changes the sign of the unessential parameter A). Denoting the transformation (12.1.2.11) by $\mathcal{F}$, let us schematically represent its action on the parameters of the equation as follows:

$$\{n,\ m,\ l\} \ \leftarrow - - \rightarrow \ \{m,\ n,\ 3-l\} \qquad \textit{transformation } \mathcal{F}. \tag{12.1.2.12}$$

Double application of the transformation $\mathcal{F}$ yields the original equation.

2°. For $l = 0$, equation (12.1.2.10) is of the class (12.1.2.5), and the last equation of system (11) becomes

$$[TT''_{tt} - 2(T'_t)^2]u + T\Theta''_{tt} - 2T'_t\Theta'_t + BT^2t^\nu u^\mu = AC^2(Tu+\Theta)^m f^n, \tag{12.1.2.13}$$

where ν, μ, and B are the parameters of the transformed equation $u''_{tt} = Bt^\nu u^\mu$, and

$$f(t) = C\int [T(t)]^2 dt.$$

Let $m,\ \mu \neq 0,\ 1,\ 2$. Then relation (12.1.2.13) is possible only if $\Theta(t) \equiv 0$. Splitting with respect to powers of u leads us to the system:

$$\begin{aligned} &TT''_{tt} - 2(T'_t)^2 = 0,\\ &Bt^\nu = AC^2T^{m+3}f^n. \end{aligned} \tag{12.1.2.14}$$

By integration we find that $T = t^{-1}$, $f = t^{-1}$ (to within unessential coefficients). Thus, we arrive at the transformation

$$x = t^{-1}, \quad y = t^{-1}u, \qquad \text{where} \quad u = u(t). \tag{12.1.2.15}$$

Denoting the transformation (12.1.2.15) by $\mathcal{H}$, let us schematically represent its action on the parameters of the equation:

$$\{n,\ m,\ 0\} \ \longleftrightarrow \ \{-n-m-3,\ m,\ 0\} \qquad \textit{transformation } \mathcal{H}. \tag{12.1.2.16}$$

Double application of the transformation $\mathcal{H}$ yields the original equation.

3°. Let $l = 0$ and $m = 2$. Then, $\mu = 2$ and the splitting procedure for equation (12.1.2.13) yields the system of three equations:

$$TT''_{tt} - 2(T'_t)^2 = 2AC^2T^6\Theta f^n,$$
$$T\Theta''_{tt} - 2T'_t\Theta'_t = AC^2T^5\Theta^2 f^n,$$
$$Bt^\nu = AC^2T^5 f^n.$$

Its solution gives us the transformation

$x = t^r, \quad y = t^k u + \alpha t^s$ — transformation of the variables, $u = u(t)$;

$\{n,\, 2,\, 0\} \longleftrightarrow \{\nu,\, 2,\, 0\}$ — transformation of the vector of essential parameters;

where we use the notation:

$$r = (8n^2 + 40n + 49)^{-1/2}, \quad k = \frac{r-1}{2}, \quad \nu = \frac{1}{2}[r(2n+5) - 5],$$
$$s = -r(n+2), \quad \alpha = \frac{(n+2)(n+3)}{A}. \qquad (12.1.2.17)$$

Example 12.4. Likewise, for the more general class of equations

$$y''_{xx} = f(x)g(y)h(y'_x)$$

we find two transformations of the variables:

$\mathcal{F} : \{f,\, g,\, h\} \quad \leftarrow - - \rightarrow \quad \{g,\, f,\, -(y'_x)^3 h(1/y'_x)\}$ — transformation (12.1.2.11);

$\mathcal{H} : \{f,\, y^m,\, 1\} \quad \longleftrightarrow \quad \{t^{-m-3} f(t^{-1}),\, y^m,\, 1\}$ — transformation (12.1.2.15).

⊙ *Literature for Section 12.1*: V. F. Zaitsev and A. D. Polyanin (1993, 1994), A. D. Polyanin and V. F. Zaitsev (2003).

12.2 Discrete Group Method Based on RF-Pairs

12.2.1 General Description of the Method. First and Second RF-Pairs

The direct method (see Section 12.1) is unsuitable for finding nonpoint transformations of second-order equations (i.e., transformations containing derivatives), since the determining equation cannot be split into equations forming an overdetermined system. Therefore, instead of searching for Bäcklund transformations in the form of arbitrary functions $x = f(t, u, u'_t)$, $y = g(t, u, u'_t)$, one uses the superposition of some "standard" transformation containing the derivative and a point transformation which can be found by the direct method. The "standard" dependence on the derivative can be introduced by means of an RF-pair, which amounts to a transformation of successively increasing and decreasing the order of the equation (this transformation is not equivalent to the identity transformation). An additional point-transformation is necessary, since the equation obtained by an RF-pair is usually outside the original class.

1°. Suppose that any equation of the original class can be solved for the independent variable x:

$$F(y, y'_x, y''_{xx}) = x.$$

Termwise differentiation of this equation with respect to x yields the following autonomous equation:

$$\frac{\partial F}{\partial y} y'_x + \frac{\partial F}{\partial y'_x} y''_{xx} + \frac{\partial F}{\partial y''_{xx}} y'''_{xxx} = 1,$$

whose order can be reduced with the substitution $y'_x = z(y)$. This pair of transformations is called a *first RF-pair*.

2°. Suppose that any equation of the original class can be solved for the dependent variable y:

$$F(x, y'_x, y''_{xx}) = y.$$

Then, termwise differentiation of this equation with respect to x brings us to the following equation which does not explicitly contain y:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y'_x} y''_{xx} + \frac{\partial F}{\partial y''_{xx}} y'''_{xxx} = y'_x.$$

The order of this equation can be reduced by means of the substitution $y'_x = z(x)$. This pair of transformations is called a *second RF-pair*.

Table 12.1 lists the main Bäcklund transformations for second-order differential equations, which are useful in conjunction with point transformations in searching for an equation of a given class.

TABLE 12.1
Main Bäcklund transformations for second-order differential equations

No.	Original equation	Algebraically transformed (equivalent) equation, differentiated w.r.t. x	New variables	Resulting equation
1	$F(x, y, y'_x, y''_{xx}) = 0$	$\Phi(y, y'_x, y''_{xx}) = x$	$w(y) = y'_x$	$w\frac{d}{dy}\Phi(y, w, ww'_y) = 1$
2	$F(x, y, y'_x, y''_{xx}) = 0$	$\Phi(x, y'_x, y''_{xx}) = y$	$w(x) = y'_x$	$\frac{d}{dx}\Phi(x, w, w'_x) = w$
3	$F\left(x^n y^m, x^k y^s, \frac{xy'_x}{y}, \frac{x^2 y''_{xx}}{y}\right) = 0$	$\frac{1}{x^k y^s}\Phi\left(x^n y^m, \frac{xy'_x}{y}, \frac{x^2 y''_{xx}}{y}\right) = 1$	$z = x^n y^m$, $w = \frac{xy'_x}{y}$	$z(mw+n)\frac{d\Phi}{dz} = (sw+k)\Phi$, where $\Phi = \Phi(z, w, v)$, $v = z(mw+n)w'_z + w^2 - w$
4	$F(x^n e^{\alpha y}, x^m e^{\beta y}, xy'_x, x^2 y''_{xx}) = 0$	$\frac{1}{x^m e^{\beta y}}\Phi(x^n e^{\alpha y}, xy'_x, x^2 y''_{xx}) = 1$	$z = x^n e^{\alpha y}$, $w = xy'_x$	$z(\alpha w+n)\frac{d\Phi}{dz} = (\beta w+m)\Phi$, where $\Phi = \Phi(z, w, v)$, $v = z(\alpha w+n)w'_z - w$
5	$F(e^{\alpha x} y^n, e^{\beta x} y^m, \frac{y'_x}{y}, \frac{y''_{xx}}{y}) = 0$	$\frac{1}{e^{\beta x} y^m}\Phi\left(e^{\alpha x} y^n, \frac{y'_x}{y}, \frac{y''_{xx}}{y}\right) = 1$	$z = e^{\alpha x} y^n$, $w = \frac{y'_x}{y}$	$z(nw+\alpha)\frac{d\Phi}{dz} = (mw+\beta)\Phi$, where $\Phi = \Phi(z, w, v)$, $v = z(nw+\alpha)w'_z + w^2$

Remark 12.1. To look for equations of a given class listed in Table 12.1, one can use the Bäcklund transformations in conjunction with point transformations and contact transformations described in Section 1.9; see Example 12.8 for a similar combination of transformations as well as the Legendre transformation.

12.2.2 Illustrative Examples

Example 12.5. Consider transformations of the class of generalized Emden–Fowler equations:

$$y''_{xx} = Ax^n y^m (y'_x)^l. \tag{12.2.2.1}$$

This class will be briefly denoted by the vector of essential parameters $\{n, m, l\}$. Application of the first RF-pair transforms this equation to

$$z''_{yy} = (l-1)z^{-1}(z'_y)^2 + my^{-1}z'_y + nA^{\frac{1}{n}} y^{\frac{m}{n}} z^{\frac{l-n-1}{n}} (z'_y)^{\frac{n-1}{n}}. \tag{12.2.2.2}$$

Now we have to find a point transformation that maps class (12.2.2.2) into class (12.2.2.1) (with another vector of parameters):

$$u''_{tt} = Bt^{\nu} u^{\mu} (u'_y)^{\lambda}. \tag{12.2.2.3}$$

Note that in this case, the desired transformation does not map the given class into itself as in Section 12.1, but is a mapping of the equation classes (12.2.2.2) $\longrightarrow$ (12.2.2.1). Nevertheless, the method for finding transformations

$$y = f(t,u), \quad z = g(t,u) \qquad (f_t g_u - f_u g_t \neq 0)$$

is completely the same and involves solving the determining equation:

$$\begin{aligned}&(f_t g_u - g_t f_u) B t^{\nu} u^{\mu} (u'_t)^{\lambda} + (f_u g_{uu} - g_u f_{uu})(u'_t)^3 + (f_t g_{uu} - g_t f_{uu} + 2 f_u g_{ut} - 2 g_u f_{ut})(u'_t)^2 \\ &+ (f_u g_{tt} - g_u f_{tt} + 2 f_t g_{ut} - 2 g_t f_{ut}) u'_t + f_t g_{tt} - g_t f_{tt} = \frac{l-1}{g}(f_t + f_u u'_t)(g_t + g_u u'_t)^2 \\ &+ \frac{m}{f}(f_t + f_u u'_t)^2 (g_t + g_u u'_t) + nA^{\frac{1}{n}} f^{\frac{m}{n}} g^{\frac{l-n-1}{n}} (f_t + f_u u'_t)^{\frac{2n+1}{n}} (g_t + g_u u'_t)^{\frac{n-1}{n}}.\end{aligned} \tag{12.2.2.4}$$

Following the procedure set out in Section 12.1, we omit the general case $f_t f_u g_t g_u \neq 0$ and consider transformations for which at least one of the above partial derivatives is zero.

1°. *Case* $f_u = 0$, $g_t = 0$. Equation (12.2.2.4) has the form

$$\begin{aligned}B f_t g_u t^{\nu} u^{\mu} (u'_t)^{\lambda} + f_t g_{uu} (u'_t)^2 - g_u f_{tt} u'_t &= \frac{l-1}{g} f_t (g_u)^2 (u_t)^2 \\ &+ \frac{m}{f}(f_t)^2 g_u u'_t + nA^{\frac{1}{n}} f^{\frac{m}{n}} g^{\frac{l-n-1}{n}} (f_t)^{\frac{2n+1}{n}} (g_u)^{\frac{n-1}{n}} (u_t)^{\frac{n-1}{n}}.\end{aligned}$$

and for $n \neq 0, -1$, $\lambda \neq 1, 2$ can easily be solved by splitting,

$$f = t^{\frac{1}{m+1}}, \qquad g = u^{\frac{1}{l-2}}.$$

As a result, using an RF-pair, we obtain:

$$\begin{aligned}&x = (u'_t)^{\frac{1}{n}}, \quad y = t^{\frac{1}{m+1}}, \quad y'_x = u^{\frac{1}{2-l}} && \text{transformation of variables;} \\ &\{n,\, m,\, l\} \longmapsto \left\{-\frac{m}{m+1},\, \frac{1}{l-2},\, \frac{n-1}{n}\right\} && \text{transformation of parameters,}\end{aligned} \tag{12.2.2.5}$$

where $u = u(t)$.

2°. *Case* $f_t = 0$, $g_u = 0$. Similar calculations bring us to the formulas:

$$\begin{aligned}&x = (u'_t)^{-\frac{1}{n}}, \quad y = u^{\frac{1}{m+1}}, \quad y'_x = t^{\frac{1}{2-l}} && \text{transformation of variables;} \\ &\{n,\, m,\, l\} \longmapsto \left\{\frac{1}{1-l},\, -\frac{n}{n+1},\, \frac{2m+1}{m}\right\} && \text{transformation of parameters.}\end{aligned} \tag{12.2.2.6}$$

Transformation (12.2.2.6) can be obtained by successive application of transformation (12.2.2.5) and the hodograph transformation $\mathcal{F}$ (see Example 12.3, Item 1°).

The inverse transformations have a similar structure. For instance, the inverse of transformation (12.2.2.5) can be written (after changing notation) as follows:

$$x = u^{\frac{1}{n+1}}, \quad y = (u'_t)^{-\frac{1}{m}}, \quad y'_x = t^{\frac{1}{1-l}}, \qquad \text{where} \quad u = u(t). \tag{12.2.2.7}$$

Denoting the transformation (12.2.2.7) by $\mathcal{G}$, let us schematically represent its action on the parameters of the equation:

$$\{n,\, m,\, l\} \longmapsto \left\{\frac{1}{1-l},\, -\frac{n}{n+1},\, \frac{2m+1}{m}\right\} \qquad \textit{transformation } \mathcal{G}. \tag{12.2.2.8}$$

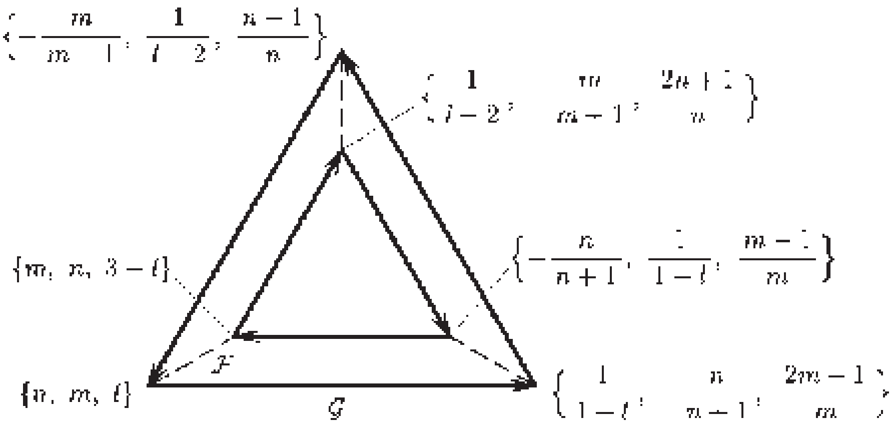

Figure 12.1: Parameters of the original and the transformed equations of the form (12.2.2.1) are obtained by superposition of the transformations $\mathcal{G}$ and $\mathcal{F}$.

Applying the transformation $\mathcal{G}$ three times, we obtain the original equation.

It can be shown that all transformations which can be found from equation (12.2.2.4), without additional restrictions on the parameters of the original and the transformed equations, are obtained by superposition of the transformations $\mathcal{G}$ and $\mathcal{F}$ (see Example 12.3, Item 1°), which form a group of order 6. The parameters of these equations are given in Figure 12.1.

Example 12.6. Suppose that $l = 0$ in equation (12.2.2.1). Then, on the class of Emden–Fowler equations

$$y''_{xx} = Ax^n y^m \qquad \text{(briefly denoted by } \{n, m, 0\}\text{)}, \tag{12.2.2.9}$$

one can define the transformation $\mathcal{H}$ (see Example 12.3, Item 2°). Therefore, in this case, the group considered in the previous example is prolonged to a group of order 12 (see Figure 12.2).

This prolongation takes place each time the third component of the parameter vector becomes equal to zero. This happens, for instance, if $n = 1$ in equation (12.2.2.9). In this case, the order of the group is equal to 24 (see Figure 12.3).

Example 12.7. The class of second-order equations

$$y''_{xx} = f(x)g(y)h(y'_x) \tag{12.2.2.10}$$

admits a discrete group of transformations similar to that for the generalized Emden–Fowler equation. Most simply, this group can be obtained by inverting the transformation (12.2.2.6). Thus, we seek the parameters of the transformation as functions of a single variable,

$$x = \varphi(u'_t), \quad y = \psi(u), \quad y'_x = \chi(t).$$

Introducing a point generator $\mathcal{F}$ (see Example 12.3, Item 1°), we find a discrete group of transformations relating the equations shown in Figure 12.4. The functions $f_1(x_1)$, $g_1(y_1)$, $h_1(y'_{x_1})$ determine the original equation, while the corresponding functions for the transformed equations, $f_k(x_k)$, $g_k(y_k)$, $h_k(y'_{x_k})$ with $k = 2$, 3, are determined by the parametric formulas:

$$\begin{aligned} f_2(x_2) &= w_1, & x_2 &= \int \frac{dw_1}{h_1(w_1)}, \\ g_2(y_2) &= \frac{1}{f_1(x_1)}, & y_2 &= \int f_1(x_1)\,dx, \\ h_2(w_2) &= -\frac{1}{[g_1(y_1)]^3}\frac{dg_1}{dy_1}, & w_2 &= \frac{1}{g_1(y_1)} \end{aligned} \tag{12.2.2.11}$$

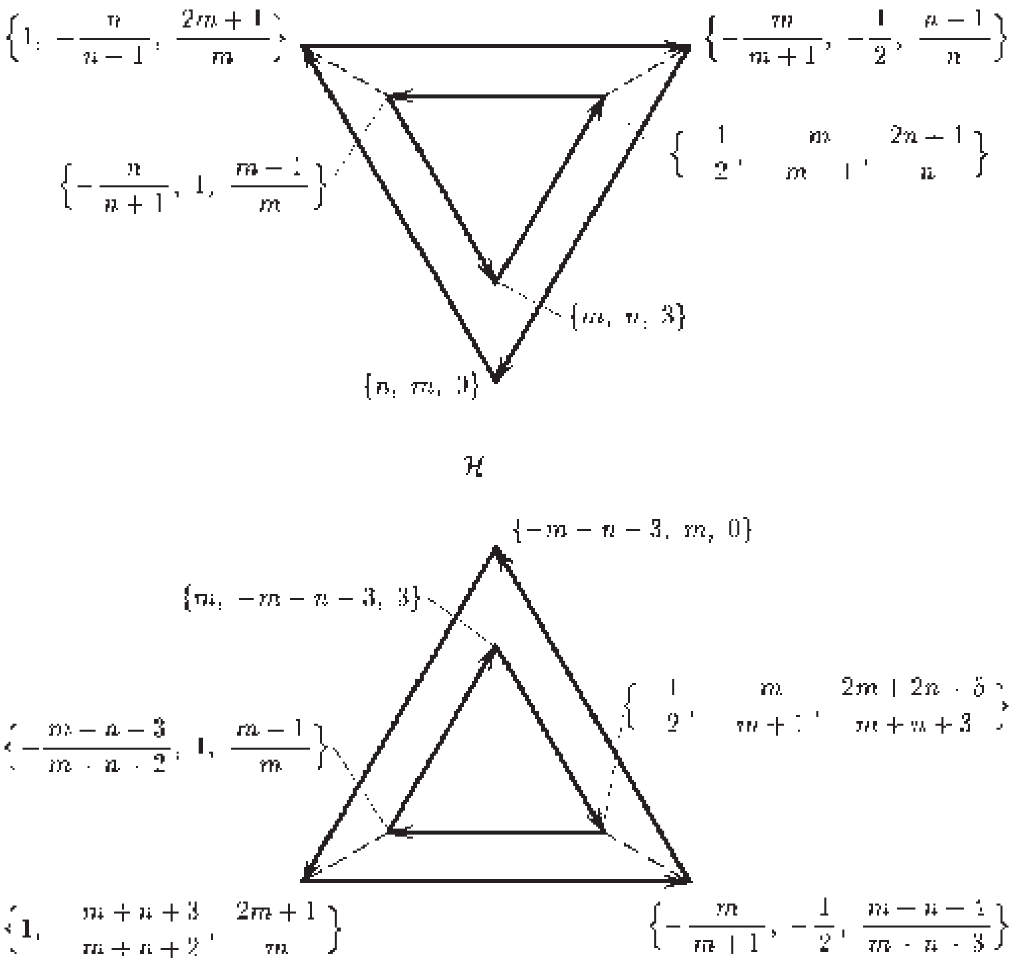

Figure 12.2: Parameters of the original equation (12.2.2.9) and the transformed generalized Emden–Fowler equations of the form (12.2.2.1) are obtained by superposition of the transformations $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{H}$.

and

$$
\begin{aligned}
f_3(x_3) &= \frac{1}{g_1(y_1)}, & x_3 &= \int g_1(y_1)\,dy_1,\\
g_3(y_3) &= \frac{1}{w_1}, & y_3 &= \int \frac{w_1\,dw_1}{h_1(w_1)},\\
h_3(w_3) &= \frac{df_1}{dx_1}, & w_3 &= f_1(x_1),
\end{aligned}
\qquad (12.2.2.12)
$$

where $w_k = y'_{x_k}$, $k = 1, 2, 3$.

The above example enables us to eliminate “singular points” of the group of transformations defined by (12.2.2.7) for $n = -1$, $m = -1$, $l = 1, 2$. For these values of the parameters, the form (12.2.2.10) and the transformations (12.2.2.11), (12.2.2.12) should be used.

⊙ *Literature for Section 12.2*: V. F. Zaitsev and A. D. Polyanin (1993, 1994), A. D. Polyanin and V. F. Zaitsev (2003), V. F. Zaitsev, L. V. Linchuk, and A. V. Flegontov (2014).

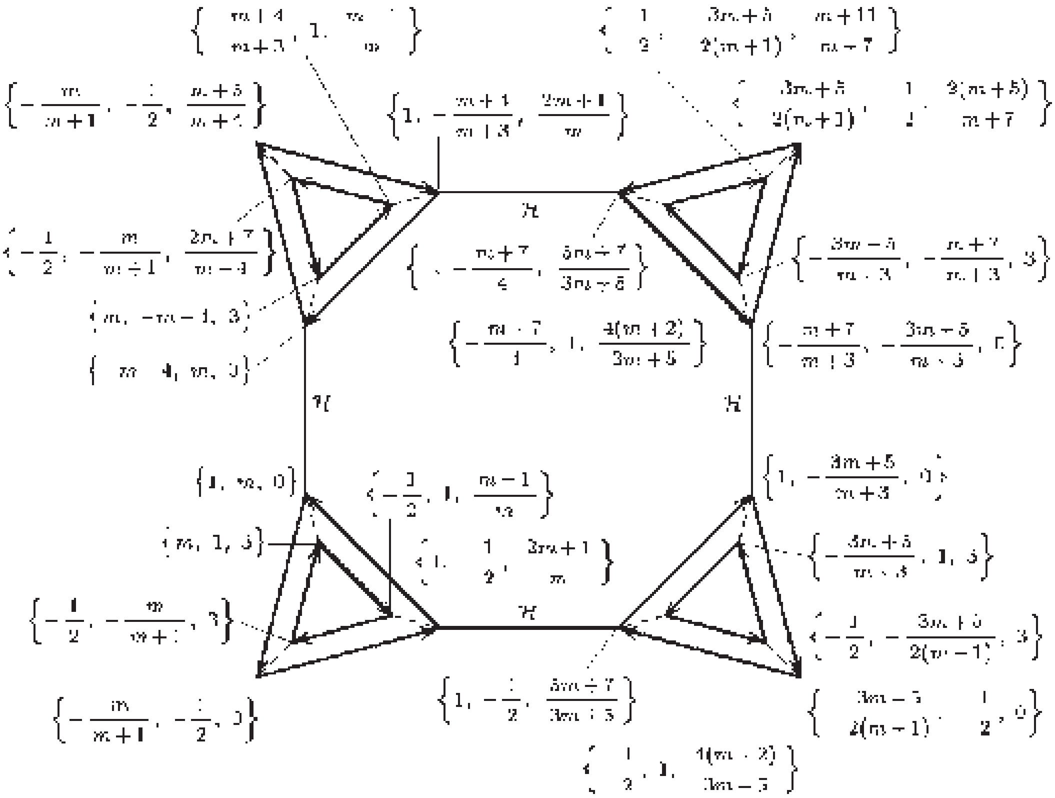

Figure 12.3: Parameters of the original equation (12.2.2.9) with $n = 1$ and the transformed Emden–Fowler equations of the form (12.2.2.1) are obtained by superposition of the transformations $\mathcal{G}$, $\mathcal{F}$, and $\mathcal{H}$.

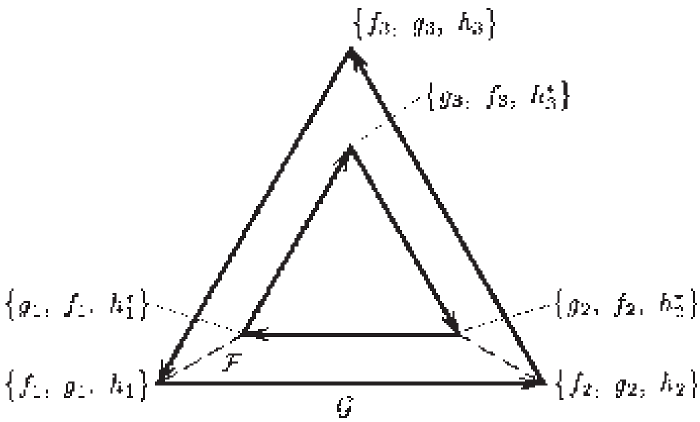

Figure 12.4: Parameters of the original and the transformed equations of the form (12.2.2.10) are obtained by superposition of the transformations $\mathcal{G}$ and $\mathcal{F}$.

12.3 Discrete Group Method Based on the Inclusion Method

To construct generators admitted by a certain class of ODEs, one can take advantage of the inclusion method. It was used above (Example 12.7) to extend the group admitted by the class of generalized Emden–Fowler equations to a wider class (12.2.2.10). Let us look at the reverse situation: suppose we study a class of equations DE_1, admitting a discrete group G_1, which is enclosed in another class, DE_2, admitting a known group G_2 such that part of its generators are not contained in G_1. Then, there is a possibility (not guaranteed) that some combination of the generators of G_2 will be closed on class DE_1.

Example 12.8. It is clear that the class of generalized Emden–Fowler equations is enclosed in the four-parameter class

$$y''_{xx} = Ax^n y^m (y'_x)^l \left(xy'_x - y\right)^k,$$

whose element will be denoted $[n, m, l, k]$. The generators $\mathcal{F}$ and $\mathcal{H}$ are closed on this class, but the generator $\mathcal{G}$ is not. However, an additional generator $\mathcal{L}$ can be introduced using the tangential Legendre transformation:

$$\begin{aligned}
&\mathcal{F}\colon \quad x = u, \quad y = t, && [n, m, l, k] \longrightarrow [m, n, 3-l-k, k],\\
&\mathcal{H}\colon \quad x = 1/t, \quad y = u/t, && [n, m, l, k] \longrightarrow [-n-m-3, m, k, l],\\
&\mathcal{L}\colon \quad x = u'_t, \quad y = tu'_t - u, && [n, m, l, k] \longrightarrow [-l, -k, -n, -m].
\end{aligned}$$

The structure of the group becomes obvious if we use the minimal group code (minimal basis of the group) and introduce the new generator $\mathcal{P} = \mathcal{HL}$:

$$\mathcal{P}\colon \quad x = \frac{1}{u'_t}, \quad y = \frac{tu'_t - u}{u'_t}, \qquad [n, m, l, k] \longrightarrow [-k, -l, n+m+3, -m].$$

Then $\mathcal{P}^6 = E$ and we obtain a group of order 12.

Obviously, for $k=0$ and $m+n+3=0$, the graph of the group will have two new vertices, which correspond to generalized Emden–Fowler equations, with the transformation $\mathcal{P}^3 \equiv \mathcal{Q}$ defining a new partial generator on the class concerned:

$$\mathcal{Q}\colon \quad x = -\frac{u'_t}{tu'_t - u}, \quad y = \frac{1}{tu'_t - u}, \qquad \{-n-m-3, m, l\} \longrightarrow \{-l, l-3, m+3\}.$$

⊙ *Literature for Section 12.3*: V. F. Zaitsev and A. D. Polyanin (1993, 1994), A. D. Polyanin and V. F. Zaitsev (2003), O. V. Zaitsev and Z. N. Khakimova (2014), V. F. Zaitsev, L. V. Linchuk, and A. V. Flegontov (2014).