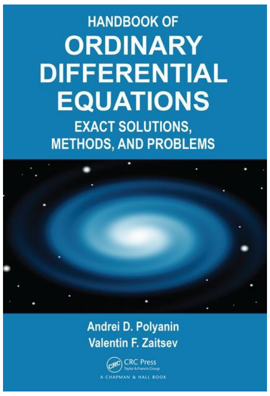

Handbook of Ordinary Differential Equations
Exact Solutions, Methods, and Problems

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Chapter 16: Fourth-Order Ordinary Differential Equations

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Chapter 16

Fourth-Order Ordinary Differential Equations

16.1 Linear Equations

16.1.1 Preliminary Remarks

1°. A nonhomogeneous linear equation of the fourth order has the form

$$f_4 y''''_{xxxx} + f_3 y'''_{xxx} + f_2 y''_{xx} + f_1 y'_x + f_0 y = g(x), \qquad f_k = f_k(x). \tag{1}$$

Let $y_0 = y_0(x)$ be a nontrivial particular solution of the corresponding homogeneous equation (with $g \equiv 0$). Then the substitution

$$y = y_0(x) \int z(x)\, dx \tag{2}$$

leads to a third-order linear equation:

$$f_4 y_0 z''' + (4 f_4 y_0' + f_3 y_0) z'' + (6 f_4 y_0'' + 3 f_3 y_0' + f_2 y_0) z' + (4 f_4 y_0''' + 3 f_3 y_0'' + 2 f_2 y_0' + f_1 y_0) z = g,$$

where the prime denotes differentiation with respect to x.

2°. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly independent particular solutions of equation (1) with $g \equiv 0$. Then the substitution

$$y = y_1 \int y_2 w\, dx - y_2 \int y_1 w\, dx$$

leads to a second-order linear equation:

$$f_4 \Delta_1 w'' + (3 f_4 \Delta_2 + f_3 \Delta_1) w' + [f_4 (3\Delta_3 + 2\varepsilon) + 2 f_3 \Delta_2 + f_2 \Delta_1] w = g,$$

where

$$\Delta_1 = y_1' y_2 - y_1 y_2', \quad \Delta_2 = y_1'' y_2 - y_1 y_2'', \quad \Delta_3 = y_1''' y_2 - y_1 y_2''', \quad \varepsilon = y_1'' y_2' - y_1' y_2''.$$

See also Sections 4.1 and 4.2.

16.1.2 Equations Containing Power Functions

► **Equations of the form** $f_4(x)y''''_{xxxx} + f_0(x)y = g(x)$.

1. $y''''_{xxxx} + ay = 0$.

1°. Solution for $a = 0$:

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3.$$

2°. Solution for $a = 4k^4 > 0$:

$$y = C_1 \cosh kx \cos kx + C_2 \cosh kx \sin kx + C_3 \sinh kx \cos kx + C_4 \sinh kx \sin kx.$$

3°. Solution for $a = -k^4 < 0$:

$$y = C_1 \cos kx + C_2 \sin kx + C_3 \cosh kx + C_4 \sinh kx.$$

2. $y''''_{xxxx} + \lambda y = ax^3 + bx^2 + cx + s, \qquad \lambda \neq 0$.

Solution: $y = \dfrac{1}{\lambda}(ax^3 + bx^2 + cx + s) + w(x)$, where $w(x)$ is the general solution of equation 16.1.2.1: $w''''_{xxxx} + \lambda w = 0$.

3. $y''''_{xxxx} = axy + b$.

This is a special case of equation 17.1.2.3 with $n = 4$.

4. $y''''_{xxxx} = ax^\beta y$.

This is a special case of equation 17.1.2.4 with $n = 4$. For $\beta = -2, -4, -6, -8$, and -9, see equations 16.1.2.5, 16.1.2.6, 16.1.2.7, 16.1.2.8, and 16.1.2.12, respectively.

The transformation $x = t^{-1}$, $y = ut^{-3}$ leads to an equation of the same form: $u''''_{tttt} = at^{-\beta-8}u$.

5. $x^2 y''''_{xxxx} = ay$.

This is a special case of equation 17.1.2.6 with $n = 2$.

6. $x^4 y''''_{xxxx} = ay$.

Solution:

$$y = C_1 x^{k_1} + C_2 x^{k_2} + C_3 x^{k_3} + C_4 x^{k_4},$$

$$k_{1,2} = \tfrac{3}{2} \pm \left(\tfrac{5}{4} + \sqrt{a+1}\right)^{1/2}, \quad k_{3,4} = \tfrac{3}{2} \pm \left(\tfrac{5}{4} - \sqrt{a+1}\right)^{1/2}.$$

7. $x^6 y''''_{xxxx} = ay$.

This is a special case of equation 17.1.2.7 with $n = 2$.

8. $x^8 y''''_{xxxx} = ay$.

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to a constant coefficient linear equation of the form 16.1.2.1: $w''''_{tttt} = aw$.

9. $(ax + b)^4 (cx + d)^4 y''''_{xxxx} = ky$.

The transformation $\xi = \ln\left|\dfrac{ax+b}{cx+d}\right|$, $w = \dfrac{y}{(cx+d)^3}$ leads to a constant coefficient linear equation.

10. $(ax^2+bx+c)^4 y''''_{xxxx} = ky.$

The transformation $\xi = \int \frac{dx}{ax^2+bx+c}$, $w = \frac{y}{(ax^2+bx+c)^{3/2}}$ leads to a constant coefficient linear equation: $w''''_{\xi\xi\xi\xi} - \frac{5}{2}Dw''_{\xi\xi} + \left(\frac{9}{16}D^2 - k\right)w = 0$, where $D = b^2 - 4ac$.

11. $(ax+b)^2(cx+d)^6 y''''_{xxxx} = ky.$

The transformation $\xi = \frac{ax+b}{cx+d}$, $w = \frac{y}{(cx+d)^3}$ leads to an equation of the form 16.1.2.5: $\xi^2 w''''_{\xi\xi\xi\xi} = k\Delta^{-4}w$, where $\Delta = ad - bc$.

12. $x^9 y''''_{xxxx} = ay + bx^4.$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the form 16.1.2.3: $4wt = atw + b$.

13. $(ax+b)^9 y''''_{xxxx} = (cx+d)y.$

The transformation $\xi = \frac{cx+d}{ax+b}$, $w = \frac{y}{(ax+b)^3}$ leads to an equation of the form 16.1.2.3: $w''''_{\xi\xi\xi\xi} = \Delta^{-4}\xi w$, where $\Delta = ad - bc$.

► Equations of the form $f_4(x)y''''_{xxxx} + f_1(x)y'_x + f_0(x)y = g(x)$.

14. $y''''_{xxxx} + ay'_x + by = 0.$

This is a special case of equation 16.1.2.41 with $a_2 = a_3 = 0$.

15. $y''''_{xxxx} + 2ay'_x - a^2x^2y = 0.$

This is a special case of equation 16.1.2.25 with $n = 1$.

16. $y''''_{xxxx} + 4axy'_x + (2a - a^2x^4)y = 0.$

This is a special case of equation 16.1.2.25 with $n = 2$.

17. $y''''_{xxxx} + (a_1x + b_1)y'_x + (a_2x + b_2)y = 0.$

This is a special case of equation 17.1.2.35 with $n = 4$.

18. $y''''_{xxxx} + ax(2b - 3a - a^2x^2)y'_x + b(2a - b + a^2x^2)y = 0.$

The substitution $w = y''_{xx} - axy'_x + by$ leads to a second-order equation of the form 14.1.2.31: $w''_{xx} + axw'_x + (2a - b + a^2x^2)w = 0$.

19. $y''''_{xxxx} + ax^k y'_x - ax^{k-1}y = bx^n.$

For $b = 0$, a particular solution is: $y_0 = x$. The substitution $z = xy'_x - y$ leads to a third-order linear equation.

20. $y''''_{xxxx} + ax^k y'_x - 2ax^{k-1}y = bx^n.$

For $b = 0$, a particular solution is: $y_0 = x^2$. The substitution $z = xy'_x - 2y$ leads to a third-order linear equation.

21. $y''''_{xxxx} + ax^k y'_x - 3ax^{k-1}y = bx^n.$

For $b = 0$, a particular solution is: $y_0 = x^3$. The substitution $z = xy'_x - 3y$ leads to a third-order linear equation: $z'''_{xxx} + ax^k z = bx^{n+1}$ (for $b = 0$, see 3.1.2.7).

22. $y''''_{xxxx} + ax^k y'_x + akx^{k-1}y = bx^n.$

Integrating yields a third-order linear equation: $y'''_{xxx} + ax^k y = \dfrac{b}{n+1}x^{n+1} + C.$

23. $y''''_{xxxx} + ax^k y'_x + a(k+3)x^{k-1}y = 0.$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the form 16.1.2.22 with $b = 0$: $w''''_{tttt} + ct^m w'_t + cmt^{m-1}w = 0$, where $c = -a$, $m = -k - 6$.

24. $y''''_{xxxx} + bx^k y'_x - a(a^3 + bx^k)y = 0.$

This is a special case of equation 16.1.6.4 with $f = bx^k$.

25. $y''''_{xxxx} + 2anx^{n-1}y'_x + a[n(n-1)x^{n-2} - ax^{2n}]y = 0.$

The substitution $w = y''_{xx} + ax^n y$ leads to a second-order equation of the form 14.1.2.7: $w''_{xx} - ax^n w = 0$.

26. $y''''_{xxxx} + (ax + b)x^k y'_x - ax^k y = 0.$

Particular solution: $y_0 = ax + b$.

27. $y''''_{xxxx} + (ax + b)x^k y'_x - 2ax^k y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

28. $y''''_{xxxx} + (ax + b)x^k y'_x - 3ax^k y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

29. $y''''_{xxxx} + (ax^k + b^3)y'_x + abx^k y = 0.$

Particular solution: $y_0 = e^{-bx}$.

30. $xy''''_{xxxx} + ax^{k+1}y'_x - [a(x+1)x^k + x + 4]y = 0.$

Particular solution: $y_0 = xe^x$.

► **Equations of the form** $f_4(x)y''''_{xxxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x).$

31. $y''''_{xxxx} + 2ay''_{xx} + a^2 y = 0.$

Solution: $y = \begin{cases} (C_1 + C_2 x)\cos(kx) + (C_3 + C_4 x)\sin(kx) & \text{if } a = k^2 > 0, \\ (C_1 + C_2 x)\exp(kx) + (C_3 + C_4 x)\exp(-kx) & \text{if } a = -k^2 < 0. \end{cases}$

32. $y''''_{xxxx} + (a + b)y''_{xx} + aby = 0.$

The case $a = b$ is given in 16.1.2.31. Let $a \neq b$.

1°. Solution for $a = \alpha^2 > 0,\ b = \beta^2 > 0$:

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

2°. Solution for $a = \alpha^2 > 0,\ b = -\beta^2 < 0$:

$$y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) + C_3 \exp(\beta x) + C_4 \exp(-\beta x).$$

3°. Solution for $a = -\alpha^2 < 0,\ b = \beta^2 > 0$:

$$y = C_1 \exp(\alpha x) + C_2 \exp(-\alpha x) + C_3 \cos(\beta x) + C_4 \sin(\beta x).$$

4°. Solution for $a = -\alpha^2 < 0,\ b = -\beta^2 < 0$:

$$y = C_1 \exp(\alpha x) + C_2 \exp(-\alpha x) + C_3 \exp(\beta x) + C_4 \exp(-\beta x).$$

33. $\boldsymbol{y''''_{xxxx} + ay''_{xx} + bx^n y'_x + bnx^{n-1} y = sx^m.}$

Integrating yields a third-order linear equation: $y'''_{xxx} + ay'_x + bx^n y = \dfrac{s}{m+1} x^{m+1} + C.$

34. $\boldsymbol{y''''_{xxxx} - 2a^2 y''_{xx} + a^4 y - \lambda(ax - b)(y''_{xx} - a^2 y) = 0.}$

This equation arises in the turbulence theory. Setting $z(x) = y''_{xx} - a^2 y$, one obtains a second-order linear equation of the form 14.1.2.12:

$$z''_{xx} - a^2 z - \lambda(ax - b)z = 0. \qquad (1)$$

Let the following boundary conditions be given:

$$y(0) = y'_x(0) = 0, \quad y(1) = y'_x(1) = 0, \qquad (2)$$

The solution of the original equation satisfying the first two conditions in (2) can represented as:

$$2ay = e^{ax} \int_0^x e^{-ax} z\, dx - e^{-ax} \int_0^x e^{ax} z\, dx.$$

To meet the last two conditions in (2), one should take the solution of (1) that satisfies the integral relations $\displaystyle\int_0^1 e^{-ax} z\, dx = \int_0^1 e^{ax} z\, dx = 0.$

35. $\boldsymbol{y''''_{xxxx} + (ax^2 + b)y''_{xx} - 2ay = 0.}$

Particular solution: $y_0 = ax^2 + b$.

36. $\boldsymbol{y''''_{xxxx} + ax^n y''_{xx} + b(ax^n - b)y = 0.}$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr),\ y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr),\ y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

The substitution $w = y''_{xx} + by$ leads to a second-order linear equation: $w''_{xx} + (ax^n - b)w = 0$.

37. $\boldsymbol{y''''_{xxxx} + ax^{n+1} y''_{xx} - 4ax^n y'_x + 6ax^{n-1} y = 0.}$

Particular solutions: $y_1 = x^2,\ \ y_2 = x^3$. The substitution $w = x^2 y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation of the form 14.1.2.7: $w''_{xx} + ax^{n+1} w = 0$.

38. $y''''_{xxxx} + 10ax^n y''_{xx} + 10anx^{n-1}y'_x + [3an(n-1)x^{n-2} + 9a^2x^{2n}]y = 0.$

This is a special case of equation 16.1.6.25 with $f = ax^n$.

39. $y''''_{xxxx} + (ax^n + b)y''_{xx} + abx^n y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \cos\bigl(x\sqrt{b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \exp\bigl(-x\sqrt{-b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-b}\bigr)$.

The substitution $w = y''_{xx} + by$ leads to a second-order linear equation of the form 14.1.2.7: $w''_{xx} + ax^n w = 0$.

40. $x^2 y''''_{xxxx} - 2(ax^2 + 6)y''_{xx} + a(ax^2 + 4)y = 0.$

Particular solutions: $y_1 = x^{-1/2} I_{1/2}\bigl(x\sqrt{a}\bigr)$, $y_2 = x^{-1/2} K_{1/2}\bigl(x\sqrt{a}\bigr)$, where $I_{1/2}(z)$ and $K_{1/2}(z)$ are modified Bessel functions.

► **Other equations.**

41. $y''''_{xxxx} + a_3 y'''_{xxx} + a_2 y''_{xx} + a_1 y'_x + a_0 y = 0.$

A fourth-order constant coefficient linear equation. For $a_0 = 0$, the substitution $w(x) = y'_x$ leads to a third-order equation. Let $a_0 \neq 0$ and $P(\lambda) = \lambda^4 + a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$ be the characteristic polynomial.

1°. Let P be factorizable, so that $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda - \lambda_3)(\lambda - \lambda_4)$, where λ_1, λ_2, λ_3, and λ_4 are real numbers. The following cases are possible:

a) λ_i are all different, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x};$$

b) $\lambda_1 = \lambda_2$; λ_3 and λ_4 are different and not equal to λ_1, then

$$y = (C_1 + C_2 x)e^{\lambda_1 x} + C_3 e^{\lambda_3 x} + C_4 e^{\lambda_4 x};$$

c) $\lambda_1 = \lambda_2 = \lambda_3 \neq \lambda_4$, then

$$y = (C_1 + C_2 x + C_3 x^2)e^{\lambda_1 x} + C_4 e^{\lambda_4 x};$$

d) $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$, then

$$y = (C_1 + C_2 x + C_3 x^2 + C_4 x^3)e^{\lambda_1 x}.$$

2°. Let $P(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2)(\lambda^2 + 2b_1\lambda + b_0)$, where λ_1 and λ_2 are real numbers, and $b_1^2 - b_0 < 0$. The following cases are possible:

a) $\lambda_1 \neq \lambda_2$, then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + e^{-b_1 x}[C_3 \cos(\mu x) + C_4 \sin(\mu x)], \quad \mu = \sqrt{b_0 - b_1^2};$$

b) $\lambda_1 = \lambda_2$, then

$$y = (C_1 + C_2 x)e^{\lambda_1 x} + e^{-b_1 x}[C_3 \cos(\mu x) + C_4 \sin(\mu x)], \quad \mu = \sqrt{b_0 - b_1^2}.$$

3°. Let us assume that $P(\lambda) = (\lambda^2 + 2b_1\lambda + b_0)(\lambda^2 + 2\beta_1\lambda + \beta_0)$, where $b_1^2 - b_0 < 0$ and $\beta_1^2 - \beta_0 < 0$. The following cases are possible:

a) $(b_1-\beta_1)^2+(b_0-\beta_0)^2\neq 0$, then

$$y=e^{-b_1x}[C_1\cos(\mu x)+C_2\sin(\mu x)]+e^{-\beta_1x}[C_3\cos(\nu x)+C_4\sin(\nu x)],$$

where $\mu=\sqrt{b_0-b_1^2}$, $\nu=\sqrt{\beta_0-\beta_1^2}$;

b) $b_1=\beta_1$, $b_0=\beta_0$, then

$$y=e^{-b_1x}[(C_1+C_2x)\cos(\mu x)+(C_3+C_4x)\sin(\mu x)],\qquad \mu=\sqrt{b_0-b_1^2}.$$

42. $y''''_{xxxx}+4axy'''_{xxx}+6a^2x^2y''_{xx}+4a^3x^3y'_x+a^4x^4y=0.$

Solution: $y=\sum\limits_{i=1}^{4}C_i\exp(\lambda_ix-\frac{1}{2}ax^2)$, where the λ_i are roots of the biquadratic equation $\lambda^4-6a\lambda^2+3a^2=0$.

43. $y''''_{xxxx}+(ax+b)y'''_{xxx}+[b(a+c)x+c]y''_{xx}+b^2cxy'_x-b^2cy=0.$

Particular solutions: $y_1=x$, $y_2=e^{-bx}$.

44. $y''''_{xxxx}=ax^ny'''_{xxx}+by'_x-abx^ny.$

Particular solutions: $y_k=\exp(\lambda_kx)$ $(k=1,\,2,\,3)$, where the λ_k are roots of the cubic equation $\lambda^3-b=0$.

45. $y''''_{xxxx}+ax^{n+3}y'''_{xxx}-3ax^{n+2}y''_{xx}+6ax^{n+1}y'_x-6ax^ny=0.$

Particular solutions: $y_1=x$, $y_2=x^2$, $y_3=x^3$. The substitution $w=x^3y'''_{xxx}-3x^2y''_{xx}+6xy'_x-6y$ leads to a first-order linear equation: $w'_x+ax^{n+3}w=0$.

46. $y''''_{xxxx}+ax^ny'''_{xxx}+bx^{m+1}y''_{xx}-2bx^my'_x+2bx^{m-1}y=0.$

Particular solutions: $y_1=x$, $y_2=x^2$. The substitution $w=x^2y''_{xx}-2xy'_x+2y$ leads to a second-order linear equation: $xw''_{xx}+(ax^{n+1}-2)w'_x+bx^{m+2}w=0$.

47. $y''''_{xxxx}+ax^ny'''_{xxx}+bx^my''_{xx}+acx^ny'_x+c(bx^m-c)y=0.$

1°. Particular solutions with $c>0$: $y_1=\cos\bigl(x\sqrt{c}\bigr)$, $y_2=\sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c<0$: $y_1=\exp\bigl(-x\sqrt{-c}\bigr)$, $y_2=\exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w=y''_{xx}+cy$ leads to a second-order linear equation: $w''_{xx}+ax^nw'_x+(bx^m-c)w=0$.

48. $y''''_{xxxx}+ax^ny'''_{xxx}+(bx^m+c)y''_{xx}+acx^ny'_x+bcx^my=0.$

1°. Particular solutions with $c>0$: $y_1=\cos\bigl(x\sqrt{c}\bigr)$, $y_2=\sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c<0$: $y_1=\exp\bigl(-x\sqrt{-c}\bigr)$, $y_2=\exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w=y''_{xx}+cy$ leads to a second-order linear equation: $w''_{xx}+ax^nw'_x+bx^mw=0$.

49. $xy''''_{xxxx}+4y'''_{xxx}+axy=0.$

The substitution $w(x)=xy$ leads to a constant coefficient linear equation of the form 16.1.2.1: $w''''_{xxxx}+aw=0$.

50. $xy''''_{xxxx} - 4ny'''_{xxx} + axy = 0, \qquad n = 1,\ 2,\ 3,\ \dots$

Solution: $y = x^{4n+3}(x^{-3}D)^n(x^{-3}w)$, where $D = \frac{d}{dx}$ and $w = w(x)$ is the general solution of a linear constant coefficient equation of the form 16.1.2.1: $w''''_{xxxx} + aw = 0$.

51. $x^2y''''_{xxxx} + 6xy'''_{xxx} + 6y''_{xx} - a^2y = 0.$

Equation of transverse vibrations of a pointed bar.

Solution: $y = \dfrac{1}{\sqrt{x}}\big[C_1J_1\big(2\sqrt{ax}\,\big) + C_2Y_1\big(2\sqrt{ax}\,\big) + C_3I_1\big(2\sqrt{ax}\,\big) + C_4K_1\big(2\sqrt{ax}\,\big)\big]$,

where $J_1(z)$ and $Y_1(z)$ are Bessel functions, and $I_1(z)$ and $K_1(z)$ are modified Bessel functions.

52. $x^2y''''_{xxxx} + 2(a+2)xy'''_{xxx} + (a+1)(a+2)y''_{xx} - b^4y = 0.$

Solution: $y = x^{-a/2}\big[C_1J_a\big(2b\sqrt{x}\,\big) + C_2Y_a\big(2b\sqrt{x}\,\big) + C_3I_a\big(2b\sqrt{x}\,\big) + C_4K_a\big(2b\sqrt{x}\,\big)\big]$, where $J_a(z)$ and $Y_a(z)$ are Bessel functions, and $I_a(z)$ and $K_a(z)$ are modified Bessel functions.

53. $x^2y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} + ax^2y = 0.$

The substitution $w(x) = x^2y$ leads to a constant coefficient linear equation of the form 16.1.2.1: $w''''_{xxxx} + aw = 0$.

54. $x^2y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = ax^3y + b.$

The substitution $w(x) = x^2y$ leads to an equation of the form 16.1.2.3: $w''''_{xxxx} = axw + b$.

55. $x^2y''''_{xxxx} + axy'''_{xxx} + (bx^{n+1} + c)y''_{xx} + (a-4)bx^ny'_x + b(c-2a+6)x^{n-1}y = 0.$

The substitution $w(x) = x^2y''_{xx} + (a-4)xy'_x + (c-2a+6)y$ leads to a second-order equation of the form 14.1.2.7: $w''_{xx} + bx^{n-1}w = 0$.

56. $x^3y''''_{xxxx} + 2x^2y'''_{xxx} - xy''_{xx} + y'_x - a^4x^3y = 0.$

Solution: $y = C_1J_0(ax) + C_2Y_0(ax) + C_3I_0(ax) + C_4K_0(ax)$, where $J_0(z)$ and $Y_0(z)$ are Bessel functions, and $I_0(z)$ and $K_0(z)$ are modified Bessel functions.

57. $x^4y''''_{xxxx} + A_3x^3y'''_{xxx} + A_2x^2y''_{xx} + A_1xy'_x + A_0y = 0.$

The Euler equation. The substitution $t = \ln|x|$ leads to a constant coefficient linear equation of the form 16.1.2.41:

$$y''''_{tttt} + (A_3 - 6)y'''_{ttt} + (11 - 3A_3 + A_2)y''_{tt} + (2A_3 - A_2 + A_1 - 6)y'_t + A_0y = 0.$$

58. $x^4y''''_{xxxx} + 2x^3y'''_{xxx} - (2a^2+1)x^2y''_{xx} + (2a^2+1)xy'_x - \big[b^4x^4 - a^2(a^2-4)\big]y = 0.$

This equation governs free transverse vibration modes of a thin round elastic plate. The equation arises from separation of variables in the two-dimensional equation

$$\Delta\Delta w - b^4w = 0,$$

where Δ is the Laplace operator written in the polar coordinate system, with x being the polar radius.

Solution: $y = C_1J_a(bx) + C_2Y_a(bx) + C_3I_a(bx) + C_4K_a(bx)$, where $J_a(z)$ and $Y_a(z)$ are Bessel functions, and $I_a(z)$ and $K_a(z)$ are modified Bessel functions. In applications, one usually sets $a = n$, where $n = 0,\ 1,\ 2,\ \dots$

⊙ The solution is specified by Popov (1998).

59. $x^4 y''''_{xxxx} - 2n(n+1)x^2 y''_{xx} + 4n(n+1)xy'_x + [ax^4 + n(n+1)(n+3)(n-2)]y = 0.$

Here, n is a positive integer and $a \neq 0$ (for $a = 0$, we have the Euler equation 16.1.2.57).

Solution: $y = x^{-n} \sum_{\nu=1}^{4} C_\nu \exp(\lambda_\nu x) P_\nu(x)$, where the λ_ν are four different roots of the equation $\lambda^4 + a = 0$, and $P_\nu(x)$ is some definite polynomial of degree $\leq 4n$.

60. $x^4 y''''_{xxxx} + 2(2-n)x^3 y'''_{xxx} + (1-n)(2-n)x^2 y''_{xx} - a^4 x^{2n} y = 0.$

Solution: $y = \sqrt{x}\,[C_1 J_{1/n}(\xi) + C_2 Y_{1/n}(\xi) + C_3 I_{1/n}(\xi) + C_4 K_{1/n}(\xi)]$, where $\xi = 2(a/n)x^{n/2}$; $J_\nu(\xi)$ and $Y_\nu(\xi)$ are Bessel functions, and $I_\nu(\xi)$ and $K_\nu(\xi)$ are modified Bessel functions.

61. $x^4 y''''_{xxxx} + 6x^3 y'''_{xxx} + [4x^4 + (7 - a^2 - b^2)x^2] y''_{xx} + x(16x^2 + 1 - a^2 - b^2)y'_x + (8x^2 + a^2 b^2)y = 0.$

Solution for $ab \neq 0$: $y = C_1 J_\mu(x)J_\nu(x) + C_2 J_\mu(x)Y_\nu(x) + C_3 Y_\mu(x)J_\nu(x) + C_4 Y_\mu(x)Y_\nu(x)$, where $J_\mu(x)$ and $Y_\mu(x)$ are Bessel functions; $\mu = \frac{1}{2}(a+b)$ and $\nu = \frac{1}{2}(a-b)$.

62. $x^8 y''''_{xxxx} + 4x^7 y'''_{xxx} = ay.$

The substitution $w(x) = xy$ leads to an equation of the form 16.1.2.8: $x^8 w''''_{xxxx} = aw$.

16.1.3 Equations Containing Exponential and Hyperbolic Functions

► **Equations with exponential functions.**

1. $y''''_{xxxx} + a^3 y'_x + be^{ax}(a^2 - be^{ax})y = 0.$

The substitution $w = y''_{xx} + ay'_x + be^{ax}y$ leads to a second-order linear equation of the form 14.1.3.10: $w''_{xx} - aw'_x + (a^2 - be^{ax})w = 0$.

2. $y''''_{xxxx} + ae^{\lambda x} y'_x + a\lambda e^{\lambda x} y = be^{\mu x}.$

Integrating yields a third-order linear equation: $y'''_{xxx} + ae^{\lambda x}y = b\mu^{-1}e^{\mu x} + C$.

3. $y''''_{xxxx} + ae^{\lambda x} y'_x - (abe^{\lambda x} + b^4)y = 0.$

Particular solution: $y_0 = e^{bx}$.

4. $y''''_{xxxx} + 2a\lambda e^{\lambda x} y'_x + a(\lambda^2 e^{\lambda x} - ae^{2\lambda x})y = 0.$

The substitution $w = y''_{xx} + ae^{\lambda x}y$ leads to a second-order linear equation of the form 14.1.3.1: $w''_{xx} - ae^{\lambda x}w = 0$.

5. $y''''_{xxxx} + (ae^{\lambda x} + b^3)y'_x + abe^{\lambda x} y = 0.$

Particular solution: $y_0 = e^{-bx}$.

6. $y''''_{xxxx} + (ax + b)e^{\lambda x} y'_x - ae^{\lambda x} y = 0.$

Particular solution: $y_0 = ax + b$.

7. $y''''_{xxxx} + (ax+b)e^{\lambda x}y'_x - 2ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

8. $y''''_{xxxx} + (ax+b)e^{\lambda x}y'_x - 3ae^{\lambda x}y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

9. $y''''_{xxxx} + ae^{\lambda x}y''_{xx} - b(ae^{\lambda x} + b)y = 0.$

1°. Particular solutions with $b > 0$: $y_1 = \exp\bigl(-x\sqrt{b}\bigr)$, $y_2 = \exp\bigl(x\sqrt{b}\bigr)$.

2°. Particular solutions with $b < 0$: $y_1 = \cos\bigl(x\sqrt{-b}\bigr)$, $y_2 = \sin\bigl(x\sqrt{-b}\bigr)$.

The substitution $w = y''_{xx} - by$ leads to a second-order linear equation of the form 14.1.3.2: $w''_{xx} + (ae^{\lambda x} + b)w = 0$.

10. $y''''_{xxxx} + (a + be^{\lambda x})y''_{xx} + abe^{\lambda x}y = 0.$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation of the form 14.1.3.1: $w''_{xx} + be^{\lambda x}w = 0$.

11. $y''''_{xxxx} + 10ae^{\lambda x}y''_{xx} + 10a\lambda e^{\lambda x}y'_x + (3a\lambda^2 e^{\lambda x} + 9a^2e^{2\lambda x})y = 0.$

This is a special case of equation 16.1.6.25 with $f(x) = ae^{\lambda x}$.

12. $y''''_{xxxx} + ay'''_{xxx} + be^{\lambda x}y'_x + abe^{\lambda x}y = 0.$

Particular solution: $y_0 = e^{-ax}$.

13. $y''''_{xxxx} = ae^{\lambda x}y'''_{xxx} + by'_x - abe^{\lambda x}y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1, 2, 3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

14. $y''''_{xxxx} + ae^{\lambda x}y'''_{xxx} + be^{\mu x}y''_{xx} + ace^{\lambda x}y'_x + c(be^{\mu x} - c)y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second-order linear equation: $w''_{xx} + ae^{\lambda x}w'_x + (be^{\mu x} - c)w = 0$.

15. $y''''_{xxxx} + ae^{\lambda x}y'''_{xxx} + (be^{\mu x} + c)y''_{xx} + ace^{\lambda x}y'_x + bce^{\mu x}y = 0.$

1°. Particular solutions with $c > 0$: $y_1 = \cos\bigl(x\sqrt{c}\bigr)$, $y_2 = \sin\bigl(x\sqrt{c}\bigr)$.

2°. Particular solutions with $c < 0$: $y_1 = \exp\bigl(-x\sqrt{-c}\bigr)$, $y_2 = \exp\bigl(x\sqrt{-c}\bigr)$.

The substitution $w = y''_{xx} + cy$ leads to a second-order equation: $w''_{xx} + ae^{\lambda x}w'_x + be^{\mu x}w = 0$.

16. $y''''_{xxxx} + ax^3e^{\lambda x}y'''_{xxx} - 3ax^2e^{\lambda x}y''_{xx} + 6axe^{\lambda x}y'_x - 6ae^{\lambda x}y = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$. The substitution $w = x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y$ leads to a first-order linear equation: $w'_x + ax^3e^{\lambda x}w = 0$.

17. $xy''''_{xxxx} + axe^{\lambda x}y'_x - [a(x+1)e^{\lambda x} + x + 4]y = 0.$

Particular solution: $y_0 = xe^x$.

18. $(ae^x + b)y''''_{xxxx} = ae^x y.$

Particular solution: $y_0 = ae^x + b$.

19. $(ax^m + be^x + c)y''''_{xxxx} = be^x y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + be^x + c$.

20. $(ax^m e^x + b)y''''_{xxxx} = by, \qquad m = 0,\ 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + be^{-x}$.

21. $y''''_{xxxx} + b\exp(\lambda x^n)\, y''_{xx} + a[b\exp(\lambda x^n) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\exp(\lambda x^n)$.

22. $y''''_{xxxx} + [a + b\exp(\lambda x^n)]y''_{xx} + ab\exp(\lambda x^n)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\exp(\lambda x^n)$.

► **Equations with hyperbolic functions.**

23. $y''''_{xxxx} + a\sinh^n(\lambda x)\, y'_x + b[a\sinh^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

24. $y''''_{xxxx} + [a\sinh^n(\lambda x) + b^3]y'_x + ab\sinh^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

25. $y''''_{xxxx} + (ax + b)\sinh^n(\lambda x)y'_x - a\sinh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

26. $y''''_{xxxx} + (ax + b)\sinh^n(\lambda x)y'_x - 2a\sinh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

27. $y''''_{xxxx} + (ax + b)\sinh^n(\lambda x)y'_x - 3a\sinh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

28. $y''''_{xxxx} + b\sinh^n(\lambda x)\, y''_{xx} + a[b\sinh^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\sinh^n(\lambda x)$.

29. $y''''_{xxxx} + [a + b\sinh^n(\lambda x)]y''_{xx} + ab\sinh^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\sinh^n(\lambda x)$.

30. $(ax^m + b\sinh x)y''''_{xxxx} = b\sinh x\, y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + b\sinh x$.

31. $y''''_{xxxx} + a\cosh^n(\lambda x)\, y'_x + b[a\cosh^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

32. $y''''_{xxxx} + [a\cosh^n(\lambda x) + b^3]y'_x + ab\cosh^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

33. $y''''_{xxxx} + (ax+b)\cosh^n(\lambda x)y'_x - a\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

34. $y''''_{xxxx} + (ax+b)\cosh^n(\lambda x)y'_x - 2a\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

35. $y''''_{xxxx} + (ax+b)\cosh^n(\lambda x)y'_x - 3a\cosh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

36. $y''''_{xxxx} + b\cosh^n(\lambda x)\,y''_{xx} + a[b\cosh^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\cosh^n(\lambda x)$.

37. $y''''_{xxxx} + [a + b\cosh^n(\lambda x)]y''_{xx} + ab\cosh^n(\lambda x)\,y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\cosh^n(\lambda x)$.

38. $(ax^m + b\cosh x)y''''_{xxxx} = b\cosh x\,y, \qquad m = 1,\ 2,\ 3.$

Particular solution: $y_0 = ax^m + b\cosh x$.

39. $y''''_{xxxx} = y + a(y'_x\cosh x - y\sinh x).$

The substitution $w = y'_x\cosh x - y\sinh x$ leads to a third-order linear equation.

40. $y''''_{xxxx} = y + a(y'_x\sinh x - y\cosh x).$

The substitution $w = y'_x\sinh x - y\cosh x$ leads to a third-order linear equation.

41. $y''''_{xxxx} + a\tanh^n(\lambda x)\,y'_x + b[a\tanh^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

42. $y''''_{xxxx} + [a\tanh^n(\lambda x) + b^3]y'_x + ab\tanh^n(\lambda x)\,y = 0.$

Particular solution: $y_0 = e^{-bx}$.

43. $y''''_{xxxx} + (ax+b)\tanh^n(\lambda x)y'_x - a\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

44. $y''''_{xxxx} + (ax+b)\tanh^n(\lambda x)y'_x - 2a\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

45. $y''''_{xxxx} + (ax+b)\tanh^n(\lambda x)y'_x - 3a\tanh^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

46. $y''''_{xxxx} + b\tanh^n(\lambda x)\,y''_{xx} + a[b\tanh^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\tanh^n(\lambda x)$.

47. $y''''_{xxxx} + [a + b\tanh^n(\lambda x)]y''_{xx} + ab\tanh^n(\lambda x)\,y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\tanh^n(\lambda x)$.

48. $y''''_{xxxx} + a\coth^n(\lambda x)\,y'_x + b[a\coth^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

49. $y''''_{xxxx} + [a\coth^n(\lambda x) + b^3]y'_x + ab\coth^n(\lambda x)\,y = 0.$

Particular solution: $y_0 = e^{-bx}$.

50. $y''''_{xxxx} + b\coth^n(\lambda x)\,y''_{xx} + a[b\coth^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\coth^n(\lambda x)$.

51. $y''''_{xxxx} + [a + b\coth^n(\lambda x)]y''_{xx} + ab\coth^n(\lambda x)\,y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\coth^n(\lambda x)$.

16.1.4 Equations Containing Logarithmic Functions

1. $y''''_{xxxx} + a\ln^k x\, y'_x - (ab\ln^k x + b^4)y = 0.$

Particular solution: $y_0 = e^{bx}$.

2. $y''''_{xxxx} + (ax + b)\ln^k(\lambda x)y'_x - a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

3. $y''''_{xxxx} + (ax + b)\ln^k(\lambda x)y'_x - 2a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

4. $y''''_{xxxx} + (ax + b)\ln^k(\lambda x)y'_x - 3a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

5. $x^2 y''''_{xxxx} + 2axy'_x - a[1 + ax^2\ln^2(bx)]y = 0.$

The substitution $w = y''_{xx} + a\ln(bx)\,y$ leads to a second-order linear equation: $w''_{xx} - a\ln(bx)w = 0$.

6. $y''''_{xxxx} + b\ln^k(\lambda x)\,y''_{xx} + a[b\ln^k(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\ln^k(\lambda x)$.

7. $y''''_{xxxx} + [a + b\ln^k(\lambda x)]y''_{xx} + ab\ln^k(\lambda x)\,y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\ln^k(\lambda x)$.

8. $y''''_{xxxx} + \ln^k(\lambda x)(x^2 y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$.

9. $y''''_{xxxx} + \ln^k(\lambda x)(x^2 y''_{xx} - 4xy'_x + 6y) = 0.$

Particular solutions: $y_1 = x^2$, $y_2 = x^3$.

10. $y''''_{xxxx} + ax^2\ln^k(\lambda x)y''_{xx} - 2a\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = x^2$.

11. $y''''_{xxxx} + ay'''_{xxx} + b\ln^k(\lambda x)y'_x + ab\ln^k(\lambda x)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

16.1.5 Equations Containing Trigonometric Functions

► **Equations with sine and cosine.**

1. $y''''_{xxxx} + a\sin^n(\lambda x)\,y'_x + b[a\sin^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

2. $y''''_{xxxx} + [a\sin^n(\lambda x) + b^3]y'_x + ab\sin^n(\lambda x)\,y = 0.$

Particular solution: $y_0 = e^{-bx}$.

3. $y''''_{xxxx} + (ax+b)\sin^n(\lambda x)y'_x - a\sin^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

4. $y''''_{xxxx} + (ax+b)\sin^n(\lambda x)y'_x - 2a\sin^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

5. $y''''_{xxxx} + (ax+b)\sin^n(\lambda x)y'_x - 3a\sin^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

6. $y''''_{xxxx} + a\sin^n(\lambda x)\,y''_{xx} + b[a\sin^n(\lambda x) - b]y = 0.$

The substitution $u = y''_{xx} + by$ leads to a second-order equation: $u''_{xx} + [a\sin^n(\lambda x) - b]u = 0$.

7. $y''''_{xxxx} + [a + b\sin^n(\lambda x)]y''_{xx} + ab\sin^n(\lambda x)\,y = 0.$

The substitution $w = y''_{xx} + ay$ leads to a second-order equation: $w''_{xx} + b\sin^n(\lambda x)\,w = 0$.

8. $y''''_{xxxx} = a\sin^n(\lambda x)\,y'''_{xxx} + by'_x - ab\sin^n(\lambda x)\,y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\ 2,\ 3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

9. $y''''_{xxxx} + a\sin^n(\lambda x)\,(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y) = 0.$

This is a special case of equation 16.1.6.33 with $f(x) = a\sin^n(\lambda x)$.

10. $x^2 y''''_{xxxx} + a\sin^n(\lambda x)\,(x^2 y''_{xx} - 4xy'_x + 6y) = 0.$

The substitution $u = x^2 y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation: $u''_{xx} + a\sin^n(\lambda x)u = 0$.

11. $(a\sin x + b)y''''_{xxxx} = a\sin x\,y.$

Particular solution: $y_0 = a\sin x + b$.

12. $(ax + b\sin x)y''''_{xxxx} = b\sin x\,y.$

Particular solution: $y_0 = ax + b\sin x$.

13. $(ax^m + b\sin x)y''''_{xxxx} = b\sin x\,y, \qquad m = 2,\ 3.$

Particular solution: $y_0 = ax^m + b\sin x$.

14. $y''''_{xxxx} + a\cos^n(\lambda x)\,y'_x + b[a\cos^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

15. $y''''_{xxxx} + [a\cos^n(\lambda x) + b^3]y'_x + ab\cos^n(\lambda x)\,y = 0.$

Particular solution: $y_0 = e^{-bx}$.

16. $y''''_{xxxx} + (ax+b)\cos^n(\lambda x)y'_x - a\cos^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

17. $y''''_{xxxx} + (ax+b)\cos^n(\lambda x)y'_x - 2a\cos^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

18. $y''''_{xxxx} + (ax+b)\cos^n(\lambda x)y'_x - 3a\cos^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

19. $y''''_{xxxx} + a\cos^n(\lambda x)\,y''_{xx} + b[a\cos^n(\lambda x) - b]y = 0.$

The substitution $u = y''_{xx} + by$ leads to a second-order equation: $u''_{xx} + [a\cos^n(\lambda x) - b]u = 0$.

20. $y''''_{xxxx} + [a + b\cos^n(\lambda x)]y''_{xx} + ab\cos^n(\lambda x)\,y = 0.$

The substitution $u = y''_{xx} + ay$ leads to a second-order equation: $u''_{xx} + b\cos^n(\lambda x)\,u = 0$.

21. $y''''_{xxxx} = a\cos^n(\lambda x)\,y'''_{xxx} + by'_x - ab\cos^n(\lambda x)\,y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\,2,\,3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

22. $y''''_{xxxx} + a\cos^n(\lambda x)\,(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y) = 0.$

This is a special case of equation 16.1.6.33 with $f(x) = a\cos^n(\lambda x)$.

23. $x^2 y''''_{xxxx} + a\cos^n(\lambda x)\,(x^2 y''_{xx} - 4xy'_x + 6y) = 0.$

The substitution $u = x^2 y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation: $u''_{xx} + a\cos^n(\lambda x)u = 0$.

24. $(a\cos x + b)y''''_{xxxx} = a\cos x\,y.$

Particular solution: $y_0 = a\cos x + b$.

25. $(ax + b\cos x)y''''_{xxxx} = b\cos x\,y.$

Particular solution: $y_0 = ax + b\cos x$.

26. $(ax^m + b\cos x)y''''_{xxxx} = b\cos x\,y, \qquad m = 2,\ 3.$

Particular solution: $y_0 = ax^m + b\cos x$.

27. $y''''_{xxxx} + 2ab\cos(bx)\,y'_x - a[b^2\sin(bx) + a\sin^2(bx)]y = 0.$

The substitution $w = y''_{xx} + a\sin(bx)\,y$ leads to a second-order linear equation of the form 14.1.6.2: $w''_{xx} - a\sin(bx)w = 0$.

28. $\sin^4 x\, y''''_{xxxx} + 2\sin^3 x \cos x\, y'''_{xxx} + \sin^2 x\,(\sin^2 x - 3)\, y''_{xx} + \sin x \cos x\,(2\sin^2 x + 3)\, y'_x + (a^4 \sin^4 x - 3)y = 0.$

Equation of a loaded rigid spherical shell. If $a^4 = 1 - \lambda^2$, the equation can be rewritten as

$$\mathbf{LL}(y) - \lambda^2 y = 0, \qquad \text{where} \quad \mathbf{L} \equiv \frac{d^2}{dx^2} + \cot x \frac{d}{dx} - \cot^2 x.$$

This equation falls into two second-order equations:

$$\mathbf{L}(y) + \lambda y = 0, \qquad \mathbf{L}(y) - \lambda y = 0,$$

which differ only in the sign of the parameter λ. The transformation $\xi = \sin^2 x$, $w = y/\sin x$ reduces the latter equations to the hypergeometric equations 2.1.2.171:

$$\xi(\xi - 1)w''_{\xi\xi} + (\tfrac{5}{2}\xi - 2)w'_\xi + \tfrac{1}{4}(1 \mp \lambda)w = 0.$$

29. $y''''_{xxxx} = y + a(y'_x \sin x - y \cos x).$

The substitution $w = y'_x \sin x - y \cos x$ leads to a third-order linear equation.

30. $y''''_{xxxx} = y + a(y'_x \cos x + y \sin x).$

The substitution $w = y'_x \cos x + y \sin x$ leads to a third-order linear equation.

► Equations with tangent and cotangent.

31. $y''''_{xxxx} + ay'_x + (a \tan x - 1)y = 0.$

Particular solution: $y_0 = \cos x$.

32. $y''''_{xxxx} + (ax + b)\tan^n(\lambda x)y'_x - a\tan^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

33. $y''''_{xxxx} + (ax + b)\tan^n(\lambda x)y'_x - 2a\tan^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

34. $y''''_{xxxx} + (ax + b)\tan^n(\lambda x)y'_x - 3a\tan^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

35. $y''''_{xxxx} + a\tan^n(\lambda x)\, y'_x + b[a\tan^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

36. $y''''_{xxxx} + [a\tan^n(\lambda x) + b^3]y'_x + ab\tan^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

37. $y''''_{xxxx} + b\tan^n(\lambda x)\, y''_{xx} + a[b\tan^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\tan^n(\lambda x)$.

38. $y''''_{xxxx} + [a + b\tan^n(\lambda x)]y''_{xx} + ab\tan^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\tan^n(\lambda x)$.

39. $y''''_{xxxx} = a\tan^n(\lambda x)\, y'''_{xxx} + by'_x - ab\tan^n(\lambda x)\, y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\ 2,\ 3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

40. $y''''_{xxxx} + ay'_x - (1 + a\cot x)y = 0.$

Particular solution: $y_0 = \sin x$.

41. $y''''_{xxxx} + (ax + b)\cot^n(\lambda x)y'_x - a\cot^n(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

42. $y''''_{xxxx} + (ax + b)\cot^n(\lambda x)y'_x - 2a\cot^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

43. $y''''_{xxxx} + (ax + b)\cot^n(\lambda x)y'_x - 3a\cot^n(\lambda x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

44. $y''''_{xxxx} + a\cot^n(\lambda x)\, y'_x + b[a\cot^n(\lambda x) - b^3]y = 0.$

Particular solution: $y_0 = e^{-bx}$.

45. $y''''_{xxxx} + [a\cot^n(\lambda x) + b^3]y'_x + ab\cot^n(\lambda x)\, y = 0.$

Particular solution: $y_0 = e^{-bx}$.

46. $y''''_{xxxx} + b\cot^n(\lambda x)\, y''_{xx} + a[b\cot^n(\lambda x) - a]y = 0.$

This is a special case of equation 16.1.6.20 with $f(x) = b\cot^n(\lambda x)$.

47. $y''''_{xxxx} + [a + b\cot^n(\lambda x)]y''_{xx} + ab\cot^n(\lambda x)\, y = 0.$

This is a special case of equation 16.1.6.21 with $f(x) = b\cot^n(\lambda x)$.

48. $y''''_{xxxx} = a\cot^n(\lambda x)\, y'''_{xxx} + by'_x - ab\cot^n(\lambda x)\, y.$

Particular solutions: $y_k = e^{\beta_k x}$ $(k = 1,\ 2,\ 3)$, where the β_k are roots of the cubic equation $\beta^3 - b = 0$.

16.1.6 Equations Containing Arbitrary Functions

► **Equations of the form** $f_4(x)y''''_{xxxx} + f_1(x)y'_x + f_0(x)y = g(x)$.

1. $y''''_{xxxx} = f(x)y.$

The transformation $x = t^{-1}$, $y = ut^{-3}$ leads to an equation of the same form: $u''''_{tttt} = t^{-8}f(1/t)u$.

2. $y''''_{xxxx} = f\Big(\dfrac{ax + b}{cx + d}\Big)\dfrac{y}{(cx + d)^8}.$

The transformation $z = \dfrac{ax + b}{cx + d}$, $u = \dfrac{y}{(cx + d)^3}$ leads to a simpler equation: $u''''_{zzzz} = \Delta^{-4}f(z)u$, where $\Delta = ad - bc$.

3. $fy''''_{xxxx} - f''''_{xxxx}y = 0, \qquad f = f(x).$

Particular solution: $y_0 = f(x)$.

4. $y''''_{xxxx} + fy'_x - a(f + a^3)y = 0, \qquad f = f(x).$

Particular solution: $y_0 = e^{ax}$.

5. $y''''_{xxxx} + (f + a^3)y'_x + afy = 0, \qquad f = f(x).$

Particular solution: $y_0 = e^{-ax}$.

6. $y''''_{xxxx} + (ax + b)f(x)y'_x - af(x)y = 0.$

Particular solution: $y_0 = ax + b$.

7. $y''''_{xxxx} + (ax + b)f(x)y'_x - 2af(x)y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

8. $y''''_{xxxx} + (ax + b)f(x)y'_x - 3af(x)y = 0.$

Particular solution: $y_0 = (ax + b)^3$. The substitution $z = (ax + b)y'_x - 3ay$ leads to a third-order linear equation: $z'''_{xxx} + (ax + b)f(x)z = 0$.

9. $y''''_{xxxx} + f(x)y'_x + f'_x(x)y = g(x).$

Integrating yields a third-order linear equation: $y'''_{xxx} + f(x)y = \int g(x)\,dx + C$.

10. $y''''_{xxxx} + 2f'_x y'_x + (f''_{xx} - f^2)y = 0, \qquad f = f(x).$

The substitution $w = y''_{xx} + f(x)y$ leads to a second-order equation: $w''_{xx} - f(x)w = 0$.

11. $xy''''_{xxxx} + xf(x)y'_x - [(x + 1)f(x) + x + 4]y = 0.$

Particular solution: $y_0 = xe^x$.

12. $y''''_{xxxx} + f(x)y'_x + g(x)y + h(x) = 0.$

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an equation of the same form:

$$w''''_{tttt} - t^{-6}f(1/t)w'_t + \left[3t^{-7}f(1/t) + t^{-8}g(1/t)\right]w + t^{-5}h(1/t) = 0.$$

13. $y''''_{xxxx} = y + f(x)(y'_x \cosh x - y \sinh x).$

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a third-order linear equation.

14. $y''''_{xxxx} = y + f(x)(y'_x \sinh x - y \cosh x).$

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a third-order linear equation.

15. $y''''_{xxxx} = y + f(x)(y'_x \sin x - y \cos x).$

The substitution $w = y'_x \sin x - y \cos x$ leads to a third-order linear equation.

16. $y''''_{xxxx} = y + f(x)(y'_x \cos x + y \sin x).$

The substitution $w = y'_x \cos x + y \sin x$ leads to a third-order linear equation.

17. $y''''_{xxxx} + fy'_x + (f \tan x - 1)y = 0, \qquad f = f(x).$

Particular solution: $y_0 = \cos x$.

18. $y''''_{xxxx} + fy'_x - (1 + f \cot x)y = 0, \qquad f = f(x).$

Particular solution: $y_0 = \sin x$.

19. $y''''_{xxxx} = \dfrac{\varphi''''_{xxxx}}{\varphi}y + f(x)\Big(y'_x - \dfrac{\varphi'_x}{\varphi}y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi}y$ leads to a third-order linear equation.

► **Equations of the form** $f_4(x)y''''_{xxxx} + f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x)$.

20. $y''''_{xxxx} + fy''_{xx} + a(f-a)y = 0, \qquad f = f(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation: $w''_{xx} + (f-a)w = 0$.

21. $y''''_{xxxx} + (f+a)y''_{xx} + afy = 0, \qquad f = f(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation: $w''_{xx} + f(x)w = 0$.

22. $y''''_{xxxx} + f(x)(x^2y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$. The substitution $z = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second-order linear equation: $xz''_{xx} - 2z'_x + x^3f(x)z = 0$.

23. $y''''_{xxxx} + f(x)(x^2y''_{xx} - 4xy'_x + 6y) = 0.$

Particular solutions: $y_1 = x^2$, $y_2 = x^3$. The substitution $w = x^2y''_{xx} - 4xy'_x + 6y$ leads to a second-order linear equation: $w''_{xx} + x^2f(x)w = 0$.

24. $y''''_{xxxx} + (ax^2 + bx + c)f(x)y''_{xx} - 2af(x)y = 0.$

Particular solution: $y_0 = ax^2 + bx + c$.

25. $y''''_{xxxx} + 10fy''_{xx} + 10f'_xy'_x + (3f''_{xx} + 9f^2)y = 0, \qquad f = f(x).$

Solution:

$$y = C_1w_1^3 + C_2w_1^2w_2 + C_3w_1w_2^2 + C_4w_2^3,$$

where w_1 and w_2 are nontrivial linearly independent solutions of the second-order linear equation: $w''_{xx} + fw = 0$.

26. $y''''_{xxxx} + (f+g)y''_{xx} + 2f'_xy'_x + (f''_{xx} + fg)y = 0, \qquad f = f(x),\ g = g(x).$

The substitution $w = y''_{xx} + fy$ leads to a second-order linear equation: $w''_{xx} + gw = 0$.

► **Other equations.**

27. $y''''_{xxxx} + f(x)y'''_{xxx} + xg(x)y'_x - 2g(x)y = 0.$

Particular solution: $y_0 = x^2$.

28. $y''''_{xxxx} + f(x)y'''_{xxx} - 2a^2y''_{xx} - a^2f(x)y'_x + a^4y = 0.$

Particular solutions: $y_1 = e^{-ax}$, $y_2 = e^{ax}$.

29. $y''''_{xxxx} + fy'''_{xxx} + gy''_{xx} + afy'_x + a(g-a)y = 0, \qquad f = f(x),\ g = g(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos\bigl(x\sqrt{a}\,\bigr)$, $y_2 = \sin\bigl(x\sqrt{a}\,\bigr)$.

2°. Particular solutions with $a < 0$: $y_1 = \exp\bigl(-x\sqrt{-a}\,\bigr)$, $y_2 = \exp\bigl(x\sqrt{-a}\,\bigr)$.

The substitution $w = y''_{xx} + ay$ leads to a second-order equation: $w''_{xx} + fw'_x + (g-a)w = 0$.

30. $y''''_{xxxx} + fy'''_{xxx} + (g+a)y''_{xx} + afy'_x + agy = 0, \qquad f = f(x),\ g = g(x).$

1°. Particular solutions with $a > 0$: $y_1 = \cos(x\sqrt{a}),\ y_2 = \sin(x\sqrt{a})$.

2°. Particular solutions with $a < 0$: $y_1 = \exp(-x\sqrt{-a}),\ y_2 = \exp(x\sqrt{-a})$.

The substitution $w = y''_{xx} + ay$ leads to a second-order linear equation: $w''_{xx} + fw'_x + gw = 0$.

31. $y''''_{xxxx} + f(x)y'''_{xxx} + g(x)y''_{xx} + xh(x)y'_x - h(x)y = 0.$

Particular solution: $y_0 = x$.

32. $y''''_{xxxx} + f(x)y'''_{xxx} + g(x)(x^2y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x,\ y_2 = x^2$. The substitution $z = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second-order linear equation: $xz''_{xx} + [xf(x) - 2]z'_x + x^3g(x)z = 0$.

33. $y''''_{xxxx} + f(x)(x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y) = 0.$

Particular solutions: $y_1 = x,\ y_2 = x^2,\ y_3 = x^3$. The substitution $w = x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y$ leads to a first-order linear equation: $w'_x + x^3f(x)w = 0$.

34. $y''''_{xxxx} = f(x)y'''_{xxx} + ay'_x - af(x)y.$

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1,\ 2,\ 3)$, where the λ_k are roots of the cubic equation $\lambda^3 - a = 0$.

35. $y''''_{xxxx} = (f-a)y'''_{xxx} + (af-b)y''_{xx} + (bf-c)y'_x + cfy, \qquad f = f(x).$

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1,\ 2,\ 3)$, where the λ_k are roots of the cubic equation $\lambda^3 + a\lambda^2 + b\lambda + c = 0$.

36. $y''''_{xxxx} + (f+a)y'''_{xxx} + (af+g+axg)y''_{xx} + a^2xgy'_x - a^2gy = 0,$
$f = f(x), \quad g = g(x).$

Particular solutions: $y_1 = x,\ y_2 = e^{-ax}$.

37. $y''''_{xxxx} + (f_3+a)y'''_{xxx} + (f_2+af_3)y''_{xx} + (f_1+af_2)y'_x + af_1y = 0,$
$f_n = f_n(x) \quad (n = 1,\ 2,\ 3).$

Particular solution: $y_0 = e^{-ax}$.

38. $xy''''_{xxxx} + 4y'''_{xxx} + axy = f(x).$

The substitution $w(x) = xy$ leads to a constant coefficient nonhomogeneous linear equation: $w''''_{xxxx} + aw = f(x)$.

39. $x^2y''''_{xxxx} + axy'''_{xxx} + (x^2f+b)y''_{xx}$
$+ (a-4)xfy'_x + (b-2a+6)fy = 0, \qquad f = f(x).$

The substitution $w = x^2y''_{xx} + (a-4)xy'_x + (b-2a+6)y$ leads to a second-order linear equation: $w''_{xx} + fw = 0$.

40. $x^4y''''_{xxxx} + ax^3y'''_{xxx} + xf(x)y'_x + (a-3)f(x)y = 0.$

Particular solution: $y_0 = x^{3-a}$.

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41. $y''''_{xxxx} + 6fy'''_{xxx} + (4f'_x + 11f^2 + 10g)y''_{xx} + (f''_{xx} + 7ff'_x + 6f^3 + 30fg + 10g'_x)y'_x + 3(2f'_x g + 5fg'_x + 6f^2 g + g''_{xx} + 3g^2)y = 0.$

Here $f = f(x)$ and $g = g(x)$. Solution:

$$y = C_1 w_1^3 + C_2 w_1^2 w_2 + C_3 w_1 w_2^2 + C_4 w_2^3,$$

where w_1 and w_2 are nontrivial linearly independent solutions of the second-order linear equation: $w''_{xx} + fw'_x + gw = 0$.

42. $(fy''_{xx})''_{xx} = 0, \qquad f = f(x).$

Equation of transverse vibrations of a bar. Solution:

$$y = C_1 + C_2 x + \int_{x_0}^{x} \frac{x-t}{f(t)}(C_3 + C_4 t)\,dt.$$

16.2 Nonlinear Equations

16.2.1 Equations Containing Power Functions

► **Equations of the form** $y''''_{xxxx} = f(x, y)$.

1. $y''''_{xxxx} = Ay^{-5/3}$.

Multiply both sides of the equation by $y^{5/3}$ and differentiate the resulting expression with respect to x to obtain

$$3yy^{(5)}_x + 5y'_x y''''_{xxxx} = 0.$$

Integrating this equation three times, we arrive at the chain of equalities:

$$3yy''''_{xxxx} + 2y'_x y'''_{xxx} - (y''_{xx})^2 = 2C_2, \qquad (1)$$
$$3yy'''_{xxx} - y'_x y''_{xx} = 2C_2 x + C_1, \qquad (2)$$
$$3yy''_{xx} - 2(y'_x)^2 = C_2 x^2 + C_1 x + C_0, \qquad (3)$$

where C_0, C_1, and C_2 are arbitrary constants. By eliminating the highest derivatives from (1)–(3) with the help of the original equation, we obtain a first-order equation:

$$(2Py'_x - 3P'_x y)^2 = 9(C_1^2 - 4C_0 C_2)y^2 - 2P^3 + 54APy^{4/3},$$

where $P = C_2 x^2 + C_1 x + C_0$. The substitution $y = (P/w)^{3/2}$ leads to a separable equation, the integration of which finally yields:

$$\int \left[9(C_1^2 - 4C_0 C_2) + 54Aw - 2w^3\right]^{-1/2} \frac{dw}{w} \pm \int \frac{dx}{3P} = C_3.$$

2. $y''''_{xxxx} = Ay^m$.

This is a special case of equation 16.2.6.1 with $f(w) = Ay^m$.

1°. By integrating, we obtain

$$2y'_x y'''_{xxx} - (y''_{xx})^2 = \frac{2A}{m+1} y^{m+1} + \frac{4}{3}C,$$

where C is an arbitrary constant ($m \neq -1$). The substitution $w(y) = (y'_x)^{3/2}$ leads to a second-order equation:

$$w''_{yy} = \Big(\frac{3A}{2m+2} y^{m+1} + C\Big) w^{-5/3}.$$

The value $C = 0$ corresponds to the Emden–Fowler equation, whose integrable cases are specified in Section 2.3 for some values of m (to those cases there correspond three-parameter families of particular solutions of the original equation).

2°. Particular solution: $y = \left[\frac{8(m+1)(m+3)(3m+1)}{A(m-1)^4}\right]^{\frac{1}{m-1}} (x+C)^{\frac{4}{1-m}}$.

3. $y''''_{xxxx} = ax^{-3m-5} y^m$.

The transformation $x = t^{-1}$, $y = t^{-3} w(t)$ leads to an equation of the form 16.2.1.2: $w''''_{tttt} = aw^m$.

4. $y''''_{xxxx} = ax^{-\frac{3m+5}{2}} y^m$.

This is a special case of equation 16.2.6.5 with $f(w) = aw^m$.

5. $y''''_{xxxx} = ax^n y^m$.

Generalized homogeneous equation.

1°. The transformation $t = x^{n+4} y^{m-1}$, $u = xy'_x/y$ leads to a third-order equation.

2°. The transformation $x = z^{-1}$, $y = z^{-3} w(z)$ leads to an equation of the same form: $w''''_{zzzz} = z^{-n-3m-5} w^m$.

6. $y''''_{xxxx} = (ay + bx^k)^m, \qquad k = 0,\ 1,\ 2,\ 3.$

The substitution $aw = ay + bx^k$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = a^m w^m$.

7. $x^{3m+1}(ax+b)^4 y''''_{xxxx} = cy^m$.

This is a special case of equation 16.2.6.10 with $f(w) = cw^m$.

8. $y''''_{xxxx} = (ax^2 + bx + c)^{-\frac{3m+5}{2}} y^m$.

This is a special case of equation 16.2.6.12 with $f(w) = w^m$.

► Equations of the form $y''''_{xxxx} = f(x, y, y'_x)$.

9. $y''''_{xxxx} = ay^n y'_x + bx^k$.

By integrating, we find $y'''_{xxx} = \frac{a}{n+1} y^{n+1} + \frac{b}{k+1} x^{k+1} + C$. For $b = 0$, the order of this equation can be reduced by one with the help of the substitution $w(y) = y'_x$.

10. $y''''_{xxxx} = ax^{-4}(xy'_x - y)^n.$

The transformation $t = \ln|x|$, $w = xy'_x - y$ leads to a third-order autonomous equation: $w'''_{ttt} - 5w''_{tt} + 6w'_t = aw^n$.

11. $y''''_{xxxx} = ax^n(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = ax^n w^k$. The substitution $w = xy'_x - y$ leads to a third-order generalized homogeneous equation: $(w'_x/x)''_{xx} = ax^n w^k$.

12. $y''''_{xxxx} = ax^n(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x,w) = ax^n w^k$. The substitution $w = xy'_x - 2y$ leads to a third-order generalized homogeneous equation: $xw'''_{xxx} - w''_{xx} = ax^{n+2}w^k$.

13. $y''''_{xxxx} = ax^n(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = ax^n w^k$. The substitution $w = xy'_x - 3y$ leads to a third-order generalized homogeneous equation: $w'''_{xxx} = ax^{n+1}w^k$.

14. $y''''_{xxxx} = a^4 y + bx^n(y'_x - ay)^k.$

The substitution $u = y'_x - ay$ leads to a third-order equation: $u'''_{xxx} + au''_{xx} + a^2u'_x + a^3u = bx^n u^k$.

▶ Equations of the form $y''''_{xxxx} = f(x, y, y'_x, y''_{xx})$.

15. $y''''_{xxxx} + ay''_{xx} = by^n + c.$

This is a special case of equation 16.2.6.33 with $f(y) = by^n + c$.

16. $y''''_{xxxx} - \frac{5}{2}ay''_{xx} + \frac{9}{16}a^2 y = by^{-5/3}.$

The transformation $\xi = e^{x\sqrt{a}}$, $w(\xi) = \xi^{3/2}y$ leads to an autonomous equation of the form 16.2.1.1: $w''''_{\xi\xi\xi\xi} = a^{-2}bw^{-5/3}$.

17. $y''''_{xxxx} + ay''_{xx} + by = cyy''_{xx} - c(y'_x)^2 + k.$

1°. Particular solution:

$$y = C_1 \sinh(C_4 x) + C_2 \cosh(C_4 x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^4 + (a - cC_3)C_4^2 + b = 0,$$
$$c(C_2^2 - C_1^2)C_4^2 - bC_3 + k = 0.$$

2°. Particular solution:

$$y = C_1 \sin(C_4 x) + C_2 \cos(C_4 x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^4 - (a - cC_3)C_4^2 + b = 0,$$
$$c(C_1^2 + C_2^2)C_4^2 + bC_3 - k = 0.$$

18. $y''''_{xxxx} + ayy''_{xx} + by''_{xx} - a(y'_x)^2 + cy'_x = 0.$

Particular solution: $y = C_1 \exp(C_2 x) - \dfrac{C_2^3 + bC_2 + c}{aC_2}$.

19. $y''''_{xxxx} = ay^2 y''_{xx} - ay(y'_x)^2 + by.$

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where the constants C_1, C_2, and C_2 are related by the constraint $C_3^4 - 4aC_1C_2C_3^2 - b = 0$.

2°. Particular solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

where the constants C_1, C_2, and C_2 are related by the constraint $C_3^4 + a(C_1^2 + C_2^2)C_3^2 - b = 0$.

3°. There are also solutions $y = \pm x\sqrt{b/a} + C$ and $y = 0$.

20. $y''''_{xxxx} + a(y'_x)^n y''_{xx} = by^k + c.$

This is a special case of equation 16.2.6.34 with $f(u) = au^n$ and $g(y) = by^k + c$.

21. $y''''_{xxxx} = ax^{-2}(xy'_x - y)^n y''_{xx}.$

This is a special case of equation 16.2.6.35 with $f(w) = aw^n$.

22. $yy''''_{xxxx} - (y''_{xx})^2 = 0.$

1°. Particular solutions:

$$y = C_1 x + C_2,$$
$$y = C_1(x + C_2)^{-3/2},$$
$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$
$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x).$$

2°. Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = C_1 x + C_2.$$

The substitution $z = C_1 x + C_2$ leads to a generalized homogeneous equation.

23. $yy''''_{xxxx} - (y''_{xx})^2 + ay + b = 0.$

Particular solutions:

$$y = C_1 \exp(\lambda x) + C_2 \exp(-\lambda x) - b/a, \quad \lambda = (a^2/b)^{1/4};$$
$$y = C_1 \sin(\lambda x) + C_2 \cos(\lambda x) - b/a, \quad \lambda = (a^2/b)^{1/4}.$$

24. $yy''''_{xxxx} - (y''_{xx})^2 + ay''_{xx} = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 + ay = C_1 x + C_2$.

2°. Particular solutions:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x) - aC_3^{-2},$$
$$y = C_1 \sin(C_3 x) + C_2 \cos(C_3 x) + aC_3^{-2},$$
$$y = C_1 x + C_2.$$

25. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b.$

1°. The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient linear equation of the form 14.1.9.1: $w''_{xx} = aw + b$.

2°. Particular solutions:

$$
\begin{aligned}
&y = C_1 \exp(C_2 x) - \frac{b}{4aC_1C_2^2}\exp(-C_2 x) && \text{if } a \neq 0,\\
&y = C_1 \exp(x\sqrt{a}) - \frac{b}{4a^2C_1}(-x\sqrt{a}) + C_2 && \text{if } a > 0,\\
&y = C_1 \sin(\lambda x) + C_2 \cos(\lambda x), \quad \lambda^2 = \frac{b}{a(C_1^2 + C_2^2)} && \text{if } ab > 0,\\
&y = \frac{\sqrt{-b}}{a}\sin(x\sqrt{-a} + C_1) + C_2 && \text{if } a < 0,\ b < 0,\\
&y = \pm x\sqrt{b/a} + C_1 && \text{if } ab > 0.
\end{aligned}
$$

26. $yy''''_{xxxx} - (y''_{xx})^2 = a\big[yy''_{xx} - (y'_x)^2\big] + bx^k + c.$

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + bx^k + c$.

27. $yy''''_{xxxx} - \frac{1}{6}(y''_{xx})^2 = ax^2 + bx + c.$

Particular solution:

$$y = \tfrac{1}{24}C_1x^4 + \tfrac{1}{6}C_2x^3 + \tfrac{1}{2}C_3x^2 + C_4x + C_5,$$

where the constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints

$$
\begin{aligned}
\tfrac{1}{3}C_1C_3 - \tfrac{1}{6}C_2^2 &= a,\\
C_1C_4 - \tfrac{1}{3}C_2C_3 &= b,\\
C_1C_5 - \tfrac{1}{6}C_3^2 &= c.
\end{aligned}
$$

28. $y''''_{xxxx} = a^2y + b(y''_{xx} + ay)^k.$

The substitution $w = y''_{xx} + ay$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{xx} = aw + bw^k$.

29. $y''''_{xxxx} = ay\big[yy''_{xx} - (y'_x)^2\big]^n.$

This is a special case of equation 16.2.6.42 with $f(w) = 0$ and $g(w) = aw^n$.

▶ Equations of the form $y''''_{xxxx} = f(x, y, y'_x, y''_{xx}, y'''_{xxx})$.

30. $y''''_{xxxx} + a_3y'''_{xxx} + a_2y''_{xx} + a_1y'_x + a_0y = byy''_{xx} - b(y'_x)^2 + k.$

Particular solutions: $y = C\exp(\lambda_n x) + ka_0^{-1}$, where C is an arbitrary constant and $\lambda = \lambda_n$ are roots of the algebraic equation $\lambda^4 + a_3\lambda^3 + \Big(a_2 - \dfrac{bk}{a_0}\Big)\lambda^2 + a_1\lambda + a_0 = 0.$

31. $y''''_{xxxx} + ayy'''_{xxx} = bx^n.$

Integrating, we arrive at a third-order equation: $y'''_{xxx} + ayy''_{xx} - \dfrac{1}{2}a(y'_x)^2 = \dfrac{b}{n+1}x^{n+1} + C.$

32. $y''''_{xxxx} + ayy'''_{xxx} - ay'_x y''_{xx} = 0.$

This equation arises in hydrodynamics.

1°. Particular solutions:

$$y = C_1 x + C_2,$$
$$y = C_1 \exp(C_2 x) - a^{-1} C_2,$$
$$y = 6(ax + C_1)^{-1}.$$

2°. Integrating, we arrive at a third-order autonomous equation:

$$y'''_{xxx} + ayy''_{xx} - a(y'_x)^2 = C.$$

⊙ *Literature*: A. D. Polyanin and V. F. Zaitsev (2002).

33. $y''''_{xxxx} + ayy'''_{xxx} - ay'_x y''_{xx} = by.$

Particular solutions:

$$y = C_1 \exp(\lambda x) + C_2 \exp(-\lambda x), \quad \lambda = b^{1/4},$$
$$y = C_1 \sin(\lambda x) + C_2 \cos(\lambda x), \quad \lambda = b^{1/4}.$$

34. $y''''_{xxxx} + ayy'''_{xxx} + b(y'_x)^n y''_{xx} = cx^k + d.$

This is a special case of equation 16.2.6.50 with $f(u) = bu^n$ and $g(x) = cx^k + d$.

35. $y''''_{xxxx} = ay^n y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = ay^n$.

36. $xy''''_{xxxx} + 4y'''_{xxx} = ax^{-5/3}y^{-5/3}.$

The substitution $w(x) = xy$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

37. $xy''''_{xxxx} + 4y'''_{xxx} = a(xy)^k.$

The substitution $w(x) = xy$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = aw^k$.

38. $xy''''_{xxxx} + 2y'''_{xxx} = a(xy'_x - y)^k.$

The substitution $w(x) = xy'_x - y$ leads to a third-order equation: $w'''_{xxx} = aw^k$ (Section 15.2 presents its solutions for $k = -\frac{7}{2}$, $-\frac{5}{2}$, -2, $-\frac{4}{3}$, $-\frac{7}{6}$, $-\frac{1}{2}$, 0, and 1).

39. $xy''''_{xxxx} + (a+3)y'''_{xxx} = bx^n(xy'_x + ay)^k.$

The substitution $w = xy'_x + ay$ leads to a third-order generalized homogeneous equation: $w'''_{xxx} = bx^n w^k$.

40. $x^2 y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = ax^{-10/3}y^{-5/3}.$

The substitution $w(x) = x^2 y$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

41. $x^4 y''''_{xxxx} + 6x^3 y'''_{xxx} + 7x^2 y''_{xx} + xy'_x = ay^{-5/3}.$

The substitution $t = \ln|x|$ leads to an equation of the form 16.2.1.1: $y''''_{xxxx} = ay^{-5/3}$.

42. $yy''''_{xxxx} = ay'_x y'''_{xxx}.$

Having integrated this equation, we obtain the third-order equation $y'''_{xxx} = Cy^a$, whose solvable cases are specified in Section 15.2.2.

43. $\boldsymbol{yy''''_{xxxx} - y'_x y'''_{xxx} = ax^n y^2.}$

Integrating yields a third-order linear equation: $y'''_{xxx} = \left(\dfrac{a}{n+1}x^{n+1} + C\right)y$.

44. $\boldsymbol{yy''''_{xxxx} - y'_x y'''_{xxx} = ay'_x.}$

Integrating yields a third-order nonhomogeneous linear equation with constant coefficients: $y'''_{xxx} = Cy - a$.

45. $\boldsymbol{yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ax^n.}$

This is a special case of equation 16.2.6.58 with $f(x) = ax^n$.

46. $\boldsymbol{yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ay^{-10/3}.}$

The substitution $w = y^2$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = 2aw^{-5/3}$.

47. $\boldsymbol{yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ay^n.}$

The substitution $w = y^2$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = 2aw^{n/2}$.

48. $\boldsymbol{yy''''_{xxxx} + \frac{2}{3}y'_x y'''_{xxx} - \frac{1}{3}(y''_{xx})^2 = a.}$

This is a special case of equation 16.2.6.59 with $a = \frac{2}{3}$ and $f(x) = a$. Integrating the equation twice, we arrive at a second-order equation of the form 14.8.1.54:

$$3yy''_{xx} - 2(y'_x)^2 = \tfrac{3}{2}ax^2 + C_1x + C_2.$$

49. $\boldsymbol{yy''''_{xxxx} + \frac{3}{2}y'_x y'''_{xxx} + \frac{1}{2}(y''_{xx})^2 = a.}$

Integrating the equation twice, we arrive at a second-order equation of the form 14.8.1.53:

$$yy''_{xx} - \tfrac{1}{4}(y'_x)^2 = \tfrac{1}{2}ax^2 + C_1x + C_2.$$

50. $\boldsymbol{yy''''_{xxxx} + \frac{3}{2}y'_x y'''_{xxx} + \frac{1}{2}(y''_{xx})^2 = (ax+b)y^{-1/2}.}$

The transformation $x = x(t)$, $y = (x'_t)^2$ leads to a constant coefficient linear equation: $2x_t^{(5)} = ax + b$.

51. $\boldsymbol{yy''''_{xxxx} - y'_x y'''_{xxx} = ax^n yy'''_{xxx}.}$

Integrating yields a third-order linear equation: $y'''_{xxx} = C\exp\left(\dfrac{a}{n+1}x^{n+1}\right)y$.

52. $\boldsymbol{xyy''''_{xxxx} - x(y''_{xx})^2 = a.}$

Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = ax\ln|x| + C_1x + C_2.$$

53. $\boldsymbol{xyy''''_{xxxx} = xy'_x y'''_{xxx} + ayy'''_{xxx}.}$

Integrating yields a third-order linear equation of the form 15.1.2.7: $y'''_{xxx} = Cx^a y$.

54. $y^3 y''''_{xxxx} = 4y^2 y'_x y'''_{xxx} + 3y^2 (y''_{xx})^2 - 6(y'_x)^4.$

This is a special case of equation 16.2.6.67 with $f \equiv 0$.

Solution in parametric form:

$$x = \pm \int \frac{dx}{\sqrt{2\xi^4 + C_2\xi + C_1}} + C_3, \qquad y = C_4 \exp\left(\pm \int \frac{\xi\, d\xi}{\sqrt{2\xi^4 + C_2\xi + C_1}}\right).$$

55. $y''_{xx} y''''_{xxxx} = a(y'''_{xxx})^2.$

Solution: $y = \begin{cases} C_0 + C_1 x + (C_2 + C_3 x)^{\frac{3-2a}{1-a}} & \text{if } a \neq 1, \\ C_0 + C_1 x + C_2 \exp(C_3 x) & \text{if } a = 1. \end{cases}$

56. $y''_{xx} y''''_{xxxx} - \frac{1}{2}(y'''_{xxx})^2 = \alpha(x y'_x - y) + \beta y'_x + \gamma.$

Differentiating with respect to x yields

$$y''_{xx}\big[y^{(5)}_x - \alpha x - \beta\big] = 0. \tag{1}$$

Equating the second factor in (1) with zero and integrating it, we find the solution:

$$y = \alpha \frac{x^6}{6!} + \beta \frac{x^5}{5!} + C_4 x^4 + C_3 x^3 + C_2 x^2 + C_1 x + C_0.$$

The constants C_k and parameters α, β, and γ are related by the constraint

$$48\, C_2 C_4 - 18\, C_3^2 = -\alpha C_0 + \beta C_1 + \gamma,$$

obtained by means of substituting the solution into the original equation.

The other solution, which corresponds to setting the first factor in (1) to zero, is given by:

$$y = \widetilde{C}_1 x + \widetilde{C}_0, \qquad \text{where} \quad \alpha \widetilde{C}_0 - \beta \widetilde{C}_1 - \gamma = 0.$$

57. $y''''_{xxxx} = a y^k y'_x (y'''_{xxx})^s.$

This is a special case of equation 16.2.6.72 with $f(y) = a y^k$ and $g(w) = w^s$. For $k = -1$ and $s = 1$, see equation 16.2.1.42.

The first integral has the form:

$$\frac{1}{1-s}(y'''_{xxx})^{1-s} - \frac{a}{k+1} y^{k+1} = C \quad \text{if } k \neq -1,\ s \neq 1; \tag{1}$$

$$\ln|y'''_{xxx}| - \frac{a}{k+1} y^{k+1} = C \quad \text{if } k \neq -1,\ s = 1; \tag{2}$$

$$\frac{1}{1-s}(y'''_{xxx})^{1-s} - a \ln|y| = C \quad \text{if } k = -1,\ s \neq 1. \tag{3}$$

For $C = 0$, equality (1) is changing to the equation

$$y'''_{xxx} = \left[\frac{a(1-s)}{k+1}\right]^{\frac{1}{1-s}} y^{\frac{k+1}{1-s}},$$

which is discussed in Section 15.2.2 (the solutions given there generate 3-parametric families of particular solutions of the original equation for $k = (1-s)\beta - 1$, where $\beta = -\frac{7}{2}$, $-\frac{5}{2}$, -2, $-\frac{4}{3}$, $-\frac{7}{6}$, $-\frac{1}{2}$, 0, and 1).

58. $a_1(y''''_{xxxx})^2 + (a_2 y'''_{xxx} + a_3 y''_{xx} + 6a_4 y + a_5 x + a_6) y''''_{xxxx} + b_1 (y'''_{xxx})^2$
$+ (b_2 x + b_3) y'''_{xxx} - a_4 (y''_{xx})^2 + b_4 y''_{xx} + b_5 x^2 + b_6 x + b_7 = 0.$

There are particular solutions of the form $y = C_1 x^4 + C_2 x^3 + C_3 x^2 + C_4 x + C_5$, where the five constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints.

16.2.2 Equations Containing Exponential Functions

▶ Equations of the form $y''''_{xxxx} = f(x,y)$.

1. $y''''_{xxxx} = ae^{\lambda y} + b.$

This is a special case of equation 16.2.6.1 with $f(y) = ae^{\lambda y} + b$.

2. $y''''_{xxxx} = ae^{\lambda y+\beta x} + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = ae^{\lambda w} + b$.

3. $y''''_{xxxx} = ax^{-4}e^{\lambda y}.$

This is a special case of equation 16.2.6.2 with $f(y) = ae^{\lambda y}$. The substitution $t = \ln|x|$ leads to an autonomous equation.

4. $y''''_{xxxx} = ax^k e^{\lambda y}.$

This is a special case of equation 16.2.6.15 with $f(w) = aw$ and $m = k + 4$.

5. $y''''_{xxxx} = ae^{\lambda x}y^n.$

This is a special case of equation 16.2.6.14 with $f(w) = aw$ and $m = n - 1$.

6. $y''''_{xxxx} = a\exp(\lambda y + \beta x^2) + b.$

The substitution $w = y + (\beta/\lambda)x^2$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = ae^{\lambda w} + b$.

7. $y''''_{xxxx} = a(y + be^x)^{-5/3} - be^x.$

The substitution $w = y + be^x$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

8. $y''''_{xxxx} = a(y + be^x)^m - be^x.$

The substitution $w = y + be^x$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = aw^m$.

▶ Other equations.

9. $y''''_{xxxx} = ae^{\lambda y}y'_x + be^{\beta x}.$

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda}e^{\lambda y} + \dfrac{b}{\beta}e^{\beta x} + C$.

10. $y''''_{xxxx} = a^4 y + be^{\lambda x}(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x,w) = be^{\lambda x}w^k$.

11. $y''''_{xxxx} = be^{\lambda x}(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = be^{\lambda x}w^k$.

12. $y''''_{xxxx} = be^{\lambda x}(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x,w) = be^{\lambda x}w^k$.

13. $y''''_{xxxx} = be^{\lambda x}(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = be^{\lambda x}w^k$.

14. $yy''''_{xxxx} - (y''_{xx})^2 = ae^{\lambda x}.$

1°. Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = a\lambda^{-2}e^{\lambda x} + C_1 x + C_2.$$

For $C_1 = C_2 = 0$, it is an equation of the form 14.8.3.47.

2°. Particular solution: $y = C\exp(\lambda x) + \dfrac{a}{C\lambda^4}.$

15. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + be^{\lambda x} = 0.$

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay + b\lambda^{-1}e^{\lambda x} = C.$

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{aC\lambda - b}{C\lambda^4},$$
$$y = \frac{b}{2a\lambda}\exp(\lambda x) + C\exp(-\lambda x) - \frac{a}{\lambda^3}.$$

16. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + be^{\lambda x} = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1 x + C_2 + b\lambda^{-2}e^{\lambda x} = 0.$

2°. Particular solution: $y = C\exp(\lambda x) + \dfrac{aC\lambda^2 - b}{C\lambda^4}.$

17. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + be^{\lambda x}.$

1°. The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient linear equation of the form 14.1.9.1: $w''_{xx} = aw + be^{\lambda x}.$

2°. Particular solution: $y = C\exp(\lambda x) + \dfrac{b}{C\lambda^2(\lambda^2 - a)}.$

18. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + be^{\lambda x} = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1 x + C_2 + b\lambda^{-2}e^{\lambda x} = 0.$

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{aC\lambda^3 - b}{C\lambda^4},$$
$$y = \frac{b}{2a\lambda^3}\exp(\lambda x) + C\exp(-\lambda x) - \frac{a}{\lambda}.$$

19. $y''''_{xxxx} = ae^{\lambda y}y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = ae^{\lambda y}$.

20. $y''''_{xxxx} = (ae^{\lambda y}y'_x + be^{\beta x})y'''_{xxx}.$

This is a special case of equation 17.2.6.58 with $n = 4$, $f(y) = ae^{\lambda y}$, and $g(x) = be^{\beta x}$.

21. $y''''_{xxxx} - 4\lambda y'''_{xxx} + 6\lambda^2 y''_{xx} - 4\lambda^3 y'_x + \lambda^4 y = a\exp\left(\tfrac{8}{3}\lambda x\right)y^{-5/3}.$

The substitution $w(x) = ye^{-\lambda x}$ leads to an equation of the form 16.2.1.1: $w''''_{xxxx} = aw^{-5/3}$.

22. $y''''_{xxxx} - 4\lambda y'''_{xxx} + 6\lambda^2 y''_{xx} - 4\lambda^3 y'_x + \lambda^4 y = ae^{\lambda(1-m)x} y^m.$

The substitution $w(x) = ye^{-\lambda x}$ leads to an equation of the form 16.2.1.2: $w''''_{xxxx} = aw^m$.

23. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ae^{\lambda x}.$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}e^{\lambda x}$.

24. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = ae^{\lambda y} + b.$

This is a special case of equation 16.2.6.60 with $f(y) = ae^{\lambda y} + b$.

16.2.3 Equations Containing Hyperbolic Functions

► **Equations with hyperbolic sine.**

1. $y''''_{xxxx} = a\sinh^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\sinh^m(\lambda y) + b$.

2. $y''''_{xxxx} = a\sinh(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\sinh(\lambda w) + b$.

3. $y''''_{xxxx} = a(y + b\sinh x)^2 - b\sinh x.$

The substitution $w = y + b\sinh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2$.

4. $y''''_{xxxx} = ax^{-4}\sinh^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\sinh^m(\lambda y)$.

5. $y''''_{xxxx} = a\sinh(\lambda y)y'_x + b\sinh(\beta x).$

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda}\cosh(\lambda y) + \dfrac{b}{\beta}\cosh(\beta x) + C$.

6. $y''''_{xxxx} = a^4 y + b\sinh(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x,w) = b\sinh(\lambda x)w^k$.

7. $y''''_{xxxx} = b\sinh(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = b\sinh(\lambda x)w^k$.

8. $y''''_{xxxx} = b\sinh(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x,w) = b\sinh(\lambda x)w^k$.

9. $y''''_{xxxx} = b\sinh(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = b\sinh(\lambda x)w^k$.

10. $yy''''_{xxxx} - (y''_{xx})^2 = a\sinh(\lambda x).$

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = a\lambda^{-2}\sinh(\lambda x) + C_1x + C_2$.

2°. Particular solution: $y = C\sinh(\lambda x) + \dfrac{a}{C\lambda^4}$.

11. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\sinh(\lambda x) = 0.$

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay + b\lambda^{-1}\cosh(\lambda x) = C.$

2°. Particular solution: $y = \dfrac{b}{\lambda(a^2 - C^2\lambda^6)}\bigl[C\lambda^3\sinh(\lambda x) + a\cosh(\lambda x)\bigr] + C.$

12. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\sinh(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1x + C_2 + b\lambda^{-2}\sinh(\lambda x) = 0.$

2°. Particular solution: $y = C\sinh(\lambda x) + \dfrac{aC\lambda^2 - b}{C\lambda^4}.$

13. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\sinh(\lambda x) + c.$

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\sinh(\lambda x) + c.$

14. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\sinh(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1x + C_2 + b\lambda^{-2}\sinh(\lambda x) = 0.$

2°. Particular solution: $y = \dfrac{b}{\lambda^3(a^2 - C^2\lambda^2)}\bigl[C\lambda\sinh(\lambda x) + a\cosh(\lambda x)\bigr] + C.$

15. $y''''_{xxxx} = a\sinh^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\sinh^k(\lambda y)$.

16. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\sinh(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\sinh(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \Bigl[\dfrac{a}{\lambda}\cosh(\lambda x) + C\Bigr]y.$

17. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\sinh(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\sinh(\lambda x).$

18. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\sinh^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\sinh^k(\lambda y) + b.$

► Equations with hyperbolic cosine.

19. $y''''_{xxxx} = a\cosh^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\cosh^m(\lambda y) + b.$

20. $y''''_{xxxx} = a\cosh(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\cosh(\lambda w) + b.$

21. $y''''_{xxxx} = a(y + b\cosh x)^2 - b\cosh x.$

The substitution $w = y + b\cosh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2.$

22. $y''''_{xxxx} = ax^{-4}\cosh^m(\lambda y)$.

This is a special case of equation 16.2.6.2 with $f(y) = a\cosh^m(\lambda y)$.

23. $y''''_{xxxx} = a\cosh(\lambda y)y'_x + b\cosh(\beta x)$.

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda}\sinh(\lambda y) + \dfrac{b}{\beta}\sinh(\beta x) + C$.

24. $y''''_{xxxx} = a^4 y + b\cosh(\lambda x)(y'_x - ay)^k$.

This is a special case of equation 16.2.6.27 with $f(x,w) = b\cosh(\lambda x)w^k$.

25. $y''''_{xxxx} = b\cosh(\lambda x)(xy'_x - y)^k$.

This is a special case of equation 16.2.6.23 with $f(x,w) = b\cosh(\lambda x)w^k$.

26. $y''''_{xxxx} = b\cosh(\lambda x)(xy'_x - 2y)^k$.

This is a special case of equation 16.2.6.24 with $f(x,w) = b\cosh(\lambda x)w^k$.

27. $y''''_{xxxx} = b\cosh(\lambda x)(xy'_x - 3y)^k$.

This is a special case of equation 16.2.6.25 with $f(x,w) = b\cosh(\lambda x)w^k$.

28. $yy''''_{xxxx} - (y''_{xx})^2 = a\cosh(\lambda x)$.

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = a\lambda^{-2}\cosh(\lambda x) + C_1 x + C_2$.

2°. Particular solution: $y = C\cosh(\lambda x) + \dfrac{a}{C\lambda^4}$.

29. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\cosh(\lambda x) = 0$.

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay + b\lambda^{-1}\sinh(\lambda x) = C$.

2°. Particular solution: $y = \dfrac{b}{\lambda(a^2 - C^2\lambda^6)}\bigl[C\lambda^3\cosh(\lambda x) + a\sinh(\lambda x)\bigr] + C$.

30. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\cosh(\lambda x) = 0$.

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1 x + C_2 + b\lambda^{-2}\cosh(\lambda x) = 0$.

2°. Particular solution: $y = C\cosh(\lambda x) + \dfrac{aC\lambda^2 - b}{C\lambda^4}$.

31. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\cosh(\lambda x) + c$.

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\cosh(\lambda x) + c$.

32. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\cosh(\lambda x) = 0$.

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1 x + C_2 + b\lambda^{-2}\cosh(\lambda x) = 0$.

2°. Particular solution: $y = \dfrac{b}{\lambda^3(a^2 - C^2\lambda^2)}\bigl[C\lambda\cosh(\lambda x) + a\sinh(\lambda x)\bigr] + C$.

33. $y''''_{xxxx} = a\cosh^k(\lambda y)y'_x y'''_{xxx}$.

This is a special case of equation 16.2.6.51 with $f(y) = a\cosh^k(\lambda y)$.

34. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\cosh(\lambda x)y^2$.

This is a special case of equation 16.2.6.57 with $f(x) = a\cosh(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \left[\frac{a}{\lambda}\sinh(\lambda x) + C\right]y$.

35. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cosh(\lambda x)$.

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\cosh(\lambda x)$.

36. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cosh^k(\lambda y) + b$.

This is a special case of equation 16.2.6.60 with $f(y) = a\cosh^k(\lambda y) + b$.

► Equations with hyperbolic tangent.

37. $y''''_{xxxx} = a\tanh^m(\lambda y) + b$.

This is a special case of equation 16.2.6.1 with $f(y) = a\tanh^m(\lambda y) + b$.

38. $y''''_{xxxx} = a\tanh(\lambda y + \beta x) + b$.

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\tanh(\lambda w) + b$.

39. $y''''_{xxxx} = ax^{-4}\tanh^m(\lambda y)$.

This is a special case of equation 16.2.6.2 with $f(y) = a\tanh^m(\lambda y)$.

40. $y''''_{xxxx} = a\tanh(\lambda y)y'_x + b\tanh(\beta x)$.

This is a special case of equation 16.2.6.21 with $f(y) = a\tanh(\lambda y)$ and $g(x) = b\tanh(\beta x)$.

41. $y''''_{xxxx} = a^4y + b\tanh(\lambda x)(y'_x - ay)^k$.

This is a special case of equation 16.2.6.27 with $f(x, w) = b\tanh(\lambda x)w^k$.

42. $y''''_{xxxx} = b\tanh(\lambda x)(xy'_x - y)^k$.

This is a special case of equation 16.2.6.23 with $f(x, w) = b\tanh(\lambda x)w^k$.

43. $y''''_{xxxx} = b\tanh(\lambda x)(xy'_x - 2y)^k$.

This is a special case of equation 16.2.6.24 with $f(x, w) = b\tanh(\lambda x)w^k$.

44. $y''''_{xxxx} = b\tanh(\lambda x)(xy'_x - 3y)^k$.

This is a special case of equation 16.2.6.25 with $f(x, w) = b\tanh(\lambda x)w^k$.

45. $y''''_{xxxx} = y + a(y'_x - y\tanh x)^k$.

This is a special case of equation 17.2.6.32 with $f(x, u) = au^k$ and $\varphi(x) = \cosh x$.

46. $y''''_{xxxx} = a\tanh^k(\lambda y)y'_x y'''_{xxx}$.

This is a special case of equation 16.2.6.51 with $f(y) = a\tanh^k(\lambda y)$.

47. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\tanh(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\tanh(\lambda x)$.

48. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tanh^k(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\tanh^k(\lambda x) + b$.

49. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tanh^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\tanh^k(\lambda y) + b$.

► **Equations with hyperbolic cotangent.**

50. $y''''_{xxxx} = a\coth^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\coth^m(\lambda y) + b$.

51. $y''''_{xxxx} = a\coth(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\coth(\lambda w) + b$.

52. $y''''_{xxxx} = ax^{-4}\coth^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\coth^m(\lambda y)$.

53. $y''''_{xxxx} = a\coth(\lambda y)y'_x + b\coth(\beta x).$

This is a special case of equation 16.2.6.21 with $f(y) = a\coth(\lambda y)$ and $g(x) = b\coth(\beta x)$.

54. $y''''_{xxxx} = a^4 y + b\coth(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b\coth(\lambda x)w^k$.

55. $y''''_{xxxx} = b\coth(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b\coth(\lambda x)w^k$.

56. $y''''_{xxxx} = b\coth(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b\coth(\lambda x)w^k$.

57. $y''''_{xxxx} = b\coth(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x, w) = b\coth(\lambda x)w^k$.

58. $y''''_{xxxx} = y + a(y'_x - y\coth x)^k.$

This is a special case of equation 17.2.6.32 with $f(x, u) = au^k$ and $\varphi(x) = \sinh x$.

59. $y''''_{xxxx} = a\coth^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\coth^k(\lambda y)$.

60. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\coth(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\coth(\lambda x)$.

61. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\coth^k(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\coth^k(\lambda x) + b$.

62. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\coth^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\coth^k(\lambda y) + b$.

16.2.4 Equations Containing Logarithmic Functions

▶ **Equations of the form** $y''''_{xxxx} = f(x, y)$.

1. $y''''_{xxxx} = a \ln^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a \ln^m(\lambda y) + b$.

2. $y''''_{xxxx} = a \ln(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a \ln(\lambda w) + b$.

3. $y''''_{xxxx} = a \ln(\lambda y + \beta x^2) + b.$

The substitution $w = y + (\beta/\lambda)x^2$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a \ln(\lambda w) + b$.

4. $y''''_{xxxx} = ax^{-4} \ln^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a \ln^m(\lambda y)$.

5. $y''''_{xxxx} = ay(\lambda x + m \ln y).$

This is a special case of equation 16.2.6.14 with $f(w) = a \ln w$.

6. $y''''_{xxxx} = ax^{-4}(\lambda y + m \ln x).$

This is a special case of equation 16.2.6.15 with $f(w) = a \ln w$.

7. $y''''_{xxxx} = ax^{-3}(\ln y - \ln x).$

This is a special case of equation 16.2.6.3 with $f(w) = a \ln w$.

8. $y''''_{xxxx} = ax^{-5}(\ln y - 3 \ln x).$

This is a special case of equation 16.2.6.4 with $f(w) = a \ln w$.

9. $y''''_{xxxx} = ax^{-5/2}(2 \ln y - 3 \ln x).$

This is a special case of equation 16.2.6.5 with $f(w) = 2a \ln w$.

10. $y''''_{xxxx} = ax^{n-4}(\ln y - n \ln x).$

This is a special case of equation 16.2.6.6 with $f(w) = a \ln w$ and $k = -n$.

▶ **Other equations.**

11. $y''''_{xxxx} = a^4 y + b \ln(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b \ln(\lambda x) w^k$.

12. $y''''_{xxxx} = b \ln(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b \ln(\lambda x) w^k$.

13. $y''''_{xxxx} = b \ln(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b \ln(\lambda x) w^k$.

14. $y''''_{xxxx} = b\ln(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = b\ln(\lambda x)w^k$.

15. $yy''''_{xxxx} - (y''_{xx})^2 = a\ln(\lambda x).$

This is a special case of equation 16.2.6.36 with $f(x) = a\ln(\lambda x)$.

16. $xy''''_{xxxx} + 4y'''_{xxx} = a(\ln x + \ln y).$

The substitution $w(x) = xy$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = a\ln w$.

17. $y''''_{xxxx} = a\ln(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\ln(\lambda y)$.

18. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\ln(\lambda x)y^2.$

Integrating yields a third-order linear equation: $y'''_{xxx} = [ax\ln(\lambda x) - ax + C]y$.

19. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\ln^m(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\ln^m(\lambda x) + b$.

20. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\ln^m(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\ln^m(\lambda y) + b$.

16.2.5 Equations Containing Trigonometric Functions

► **Equations with sine.**

1. $y''''_{xxxx} = a\sin^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\sin^m(\lambda y) + b$.

2. $y''''_{xxxx} = a\sin(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\sin(\lambda w) + b$.

3. $y''''_{xxxx} = a(y + b\sin x)^2 - b\sin x.$

The substitution $w = y + b\sin x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2$.

4. $y''''_{xxxx} = ax^{-4}\sin^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\sin^m(\lambda y)$.

5. $y''''_{xxxx} = a\sin(\lambda y)y'_x + b\sin(\beta x).$

Integrating yields a third-order equation: $y'''_{xxx} = -\dfrac{a}{\lambda}\cos(\lambda y) - \dfrac{b}{\beta}\cos(\beta x) + C$.

6. $y''''_{xxxx} = a^4 y + b\sin(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x,w) = b\sin(\lambda x)w^k$.

7. $y''''_{xxxx} = b\sin(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = b\sin(\lambda x)w^k$.

8. $y''''_{xxxx} = b\sin(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x,w) = b\sin(\lambda x)w^k$.

9. $y''''_{xxxx} = b\sin(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = b\sin(\lambda x)w^k$.

10. $yy''''_{xxxx} - (y''_{xx})^2 = a\sin(\lambda x).$

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = -a\lambda^{-2}\sin(\lambda x) + C_1x + C_2$.

2°. Particular solution: $y = C\sin(\lambda x) + \dfrac{a}{C\lambda^4}$.

11. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\sin(\lambda x) = 0.$

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_xy''_{xx} - ay - b\lambda^{-1}\cos(\lambda x) = C$.

2°. Particular solution: $y = -\dfrac{b}{\lambda(a^2 + C^2\lambda^6)}\bigl[a\cos(\lambda x) + C\lambda^3\sin(\lambda x)\bigr] + C$.

12. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\sin(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1x + C_2 - b\lambda^{-2}\sin(\lambda x) = 0$.

2°. Particular solution: $y = C\sin(\lambda x) - \dfrac{b + aC\lambda^2}{C\lambda^4}$.

13. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\sin(\lambda x) + c.$

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\sin(\lambda x) + c$.

14. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\sin(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1x + C_2 - b\lambda^{-2}\sin(\lambda x) = 0$.

2°. Particular solution: $y = \dfrac{b}{\lambda^3(a^2 + C^2\lambda^2)}\bigl[a\cos(\lambda x) - C\lambda\sin(\lambda x)\bigr] + C$.

15. $y''''_{xxxx} = a\sin^k(\lambda y)y'_xy'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\sin^k(\lambda y)$.

16. $yy''''_{xxxx} - y'_xy'''_{xxx} = a\sin(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\sin(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \Bigl[C - \dfrac{a}{\lambda}\cos(\lambda x)\Bigr]y$.

17. $yy''''_{xxxx} + 4y'_xy'''_{xxx} + 3(y''_{xx})^2 = a\sin(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\sin(\lambda x)$.

18. $yy''''_{xxxx} + 4y'_xy'''_{xxx} + 3(y''_{xx})^2 = a\sin^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\sin^k(\lambda y) + b$.

► **Equations with cosine.**

19. $y''''_{xxxx} = a\cos^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\cos^m(\lambda y) + b$.

20. $y''''_{xxxx} = a\cos(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\cos(\lambda w) + b$.

21. $y''''_{xxxx} = a(y + b\cos x)^2 - b\cos x.$

The substitution $w = y + b\cos x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = aw^2$.

22. $y''''_{xxxx} = ax^{-4}\cos^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\cos^m(\lambda y)$.

23. $y''''_{xxxx} = a\cos(\lambda y)y'_x + b\cos(\beta x).$

Integrating yields a third-order equation: $y'''_{xxx} = \dfrac{a}{\lambda}\sin(\lambda y) + \dfrac{b}{\beta}\sin(\beta x) + C.$

24. $y''''_{xxxx} = a^4 y + b\cos(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x, w) = b\cos(\lambda x)w^k$.

25. $y''''_{xxxx} = b\cos(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x, w) = b\cos(\lambda x)w^k$.

26. $y''''_{xxxx} = b\cos(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x, w) = b\cos(\lambda x)w^k$.

27. $y''''_{xxxx} = b\cos(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x, w) = b\cos(\lambda x)w^k$.

28. $yy''''_{xxxx} - (y''_{xx})^2 = a\cos(\lambda x).$

1°. Integrating the equation twice, we arrive at a second-order equation: $yy''_{xx} - (y'_x)^2 = -a\lambda^{-2}\cos(\lambda x) + C_1 x + C_2$.

2°. Particular solution: $y = C\cos(\lambda x) + \dfrac{a}{C\lambda^4}$.

29. $yy''''_{xxxx} - (y''_{xx})^2 - ay'_x + b\cos(\lambda x) = 0.$

1°. Integrating yields a third-order equation: $yy'''_{xxx} - y'_x y''_{xx} - ay + b\lambda^{-1}\sin(\lambda x) = C$.

2°. Particular solution: $y = \dfrac{b}{\lambda(a^2 + C^2\lambda^6)}\bigl[a\sin(\lambda x) - C\lambda^3\cos(\lambda x)\bigr] + C.$

30. $yy''''_{xxxx} - (y''_{xx})^2 - ay''_{xx} + b\cos(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay + C_1 x + C_2 - b\lambda^{-2}\cos(\lambda x) = 0$.

2°. Particular solution: $y = C\cos(\lambda x) - \dfrac{aC\lambda^2 + b}{C\lambda^4}.$

31. $yy''''_{xxxx} - (y''_{xx})^2 = a[yy''_{xx} - (y'_x)^2] + b\cos(\lambda x) + c.$

The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order constant coefficient nonhomogeneous linear equation of the form 14.1.9.1: $w''_{xx} = aw + b\cos(\lambda x) + c$.

32. $yy''''_{xxxx} - ay'''_{xxx} - (y''_{xx})^2 + b\cos(\lambda x) = 0.$

1°. Integrating the equation two times, we obtain a second-order equation: $yy''_{xx} - (y'_x)^2 - ay'_x + C_1x + C_2 - b\lambda^{-2}\cos(\lambda x) = 0$.

2°. Particular solution: $y = -\dfrac{b}{\lambda^3(a^2 + C^2\lambda^2)}\bigl[C\lambda\cos(\lambda x) + a\sin(\lambda x)\bigr] + C$.

33. $y''''_{xxxx} = a\cos^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\cos^k(\lambda y)$.

34. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\cos(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\cos(\lambda x)$. Integrating yields a third-order linear equation: $y'''_{xxx} = \Bigl[\dfrac{a}{\lambda}\sin(\lambda x) + C\Bigr]y$.

35. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cos(\lambda x).$

Solution: $y^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-4}\cos(\lambda x)$.

36. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cos^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\cos^k(\lambda y) + b$.

► **Equations with tangent.**

37. $y''''_{xxxx} = a\tan^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\tan^m(\lambda y) + b$.

38. $y''''_{xxxx} = a\tan(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\tan(\lambda w) + b$.

39. $y''''_{xxxx} = ax^{-4}\tan^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\tan^m(\lambda y)$.

40. $y''''_{xxxx} = a\tan(\lambda y)y'_x + b\tan(\beta x).$

This is a special case of equation 16.2.6.21 with $f(y) = a\tan(\lambda y)$ and $g(x) = b\tan(\beta x)$.

41. $y''''_{xxxx} = a^4y + b\tan(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x,w) = b\tan(\lambda x)w^k$.

42. $y''''_{xxxx} = b\tan(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = b\tan(\lambda x)w^k$.

43. $y''''_{xxxx} = b\tan(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x,w) = b\tan(\lambda x)w^k$.

44. $y''''_{xxxx} = b\tan(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = b\tan(\lambda x)w^k$.

45. $y''''_{xxxx} = y + a(y'_x + y\tan x)^k.$

This is a special case of equation 17.2.6.32 with $f(x,u) = au^k$ and $\varphi(x) = \cos x$.

46. $y''''_{xxxx} = a\tan^k(\lambda y)y'_x y'''_{xxx}.$

This is a special case of equation 16.2.6.51 with $f(y) = a\tan^k(\lambda y)$.

47. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\tan(\lambda x)y^2.$

This is a special case of equation 16.2.6.57 with $f(x) = a\tan(\lambda x)$.

48. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tan^k(\lambda x) + b.$

This is a special case of equation 16.2.6.58 with $f(x) = a\tan^k(\lambda x) + b$.

49. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\tan^k(\lambda y) + b.$

This is a special case of equation 16.2.6.60 with $f(y) = a\tan^k(\lambda y) + b$.

▶ Equations with cotangent.

50. $y''''_{xxxx} = a\cot^m(\lambda y) + b.$

This is a special case of equation 16.2.6.1 with $f(y) = a\cot^m(\lambda y) + b$.

51. $y''''_{xxxx} = a\cot(\lambda y + \beta x) + b.$

The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = a\cot(\lambda w) + b$.

52. $y''''_{xxxx} = ax^{-4}\cot^m(\lambda y).$

This is a special case of equation 16.2.6.2 with $f(y) = a\cot^m(\lambda y)$.

53. $y''''_{xxxx} = a\cot(\lambda y)y'_x + b\cot(\beta x).$

This is a special case of equation 16.2.6.21 with $f(y) = a\cot(\lambda y)$ and $g(x) = b\cot(\beta x)$.

54. $y''''_{xxxx} = a^4 y + b\cot(\lambda x)(y'_x - ay)^k.$

This is a special case of equation 16.2.6.27 with $f(x,w) = b\cot(\lambda x)w^k$.

55. $y''''_{xxxx} = b\cot(\lambda x)(xy'_x - y)^k.$

This is a special case of equation 16.2.6.23 with $f(x,w) = b\cot(\lambda x)w^k$.

56. $y''''_{xxxx} = b\cot(\lambda x)(xy'_x - 2y)^k.$

This is a special case of equation 16.2.6.24 with $f(x,w) = b\cot(\lambda x)w^k$.

57. $y''''_{xxxx} = b\cot(\lambda x)(xy'_x - 3y)^k.$

This is a special case of equation 16.2.6.25 with $f(x,w) = b\cot(\lambda x)w^k$.

58. $y''''_{xxxx} = y + a(y'_x - y\cot x)^k.$

This is a special case of equation 17.2.6.32 with $f(x,u) = au^k$ and $\varphi(x) = \sin x$.

59. $y''''_{xxxx} = a\cot^k(\lambda y)y'_x y'''_{xxx}$.

This is a special case of equation 16.2.6.51 with $f(y) = a\cot^k(\lambda y)$.

60. $yy''''_{xxxx} - y'_x y'''_{xxx} = a\cot(\lambda x)y^2$.

This is a special case of equation 16.2.6.57 with $f(x) = a\cot(\lambda x)$.

61. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cot^k(\lambda x) + b$.

This is a special case of equation 16.2.6.58 with $f(x) = a\cot^k(\lambda x) + b$.

62. $yy''''_{xxxx} + 4y'_x y'''_{xxx} + 3(y''_{xx})^2 = a\cot^k(\lambda y) + b$.

This is a special case of equation 16.2.6.60 with $f(y) = a\cot^k(\lambda y) + b$.

16.2.6 Equations Containing Arbitrary Functions

► **Equations of the form** $y''''_{xxxx} = f(x, y)$.

1. $y''''_{xxxx} = f(y)$.

Autonomous equation. By integrating, we obtain $2y'_x y'''_{xxx} - (y''_{xx})^2 = 2\int f(y)\,dy + 2C$. The substitution $w(y) = |y'_x|^{3/2}$ leads to a second-order equation:

$$2w_{yy}'' = \tfrac{3}{2}\Big[\int f(y)\,dy + C\Big]w^{-5/3}.$$

2. $y''''_{xxxx} = x^{-4}f(y)$.

This is a special case of equation 16.2.6.55 with $a_1 = a_2 = a_3 = 0$. The substitution $t = \ln|x|$ leads to an autonomous equation.

3. $y''''_{xxxx} = x^{-3}f(y/x)$.

Homogeneous equation. The transformation $t = \ln x$, $w = y/x$ leads to an autonomous equation of the form 16.2.6.79.

4. $y''''_{xxxx} = x^{-5}f(yx^{-3})$.

The transformation $x = t^{-1}$, $y = wt^{-3}$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{tttt} = f(w)$.

5. $y''''_{xxxx} = x^{-5/2}f(yx^{-3/2})$.

The transformation $x = e^t$, $y = x^{3/2}w$ leads to an autonomous equation of the form 16.2.6.33: $w''''_{tttt} - \frac{5}{2}w''_{tt} = -\frac{9}{16}w + f(w)$.

6. $y''''_{xxxx} = x^{-k-4}f(x^k y)$.

Generalized homogeneous equation.

1°. The transformation $t = \ln x$, $z = x^k y$ leads to an autonomous equation.

2°. The transformation $z = x^k y$, $w = xy'_x/y$ leads to a third-order equation.

7. $y''''_{xxxx} = yx^{-4} f(x^k y^m)$.

Generalized homogeneous equation. The transformation $z = x^k y^m$, $w = xy'_x/y$ leads to a third-order equation.

8. $y''''_{xxxx} = f(y + a_3x^3 + a_2x^2 + a_1x + a_0)$.

The substitution $w = y + a_3x^3 + a_2x^2 + a_1x + a_0$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

9. $y''''_{xxxx} = f(y + ax^4)$.

The substitution $w = y + ax^4$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w) + 24a$.

10. $x(ax + b)^4 y''''_{xxxx} = f(yx^{-3})$.

The transformation $\xi = \ln\left|\dfrac{ax+b}{x}\right|$, $w = \dfrac{y}{x^3}$ leads to an autonomous equation of the form 16.2.6.79.

11. $y''''_{xxxx} = (ax + by + c)^{-3} f\left(\dfrac{ax + by + c}{\alpha x + \beta y + \gamma}\right)$.

This is a special case of equation 17.2.6.19 with $n = 4$.

12. $y''''_{xxxx} = (ax^2 + bx + c)^{-5/2} f\left(\dfrac{y}{(ax^2 + bx + c)^{3/2}}\right)$.

1°. The transformation $\xi = \displaystyle\int \frac{dx}{ax^2 + bx + c}$, $w = \dfrac{y}{(ax^2 + bx + c)^{3/2}}$ leads to an autonomous equation of the form 16.2.6.33 with respect to $w = w(\xi)$:

$$w''''_{\xi\xi\xi\xi} - \tfrac{5}{2}\Delta w''_{\xi\xi} + \tfrac{9}{16}\Delta^2 w = f(w), \qquad \text{where} \quad \Delta = b^2 - 4ac.$$

Therefore, having integrated the latter equation, we obtain

$$w'_\xi w'''_{\xi\xi\xi} - \tfrac{1}{2}(w''_{\xi\xi})^2 - \tfrac{5}{4}\Delta(w'_\xi)^2 = -\tfrac{9}{32}\Delta^2 w^2 + \int f(w)\,dw + C.$$

The substitution $z(w) = |w'_\xi|^{3/2}$ leads to a second-order equation:

$$z''_{ww} = \tfrac{15}{8}\Delta z^{-1/3} + \tfrac{3}{2}\Big[-\tfrac{9}{32}\Delta^2 w^2 + \int f(w)\,dw + C\Big] z^{-5/3}.$$

2°. The first integral of the original equation has the form:

$$(Py'_x - \tfrac{3}{2}P'_x y)y'''_{xxx} - \tfrac{1}{2}P(y''_{xx})^2 + \tfrac{1}{2}P'_x y'_x y''_{xx} + 3ayy''_{xx} - 2a(y'_x)^2 = \int f(w)\,dw + C,$$

where $P = ax^2 + bx + c$, $w = yP^{-3/2}$.

13. $y''''_{xxxx} = e^{\lambda x} f(ye^{-\lambda x})$.

This is a special case of equation 16.2.6.47 with $a = b = c = 0$. The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation.

14. $y''''_{xxxx} = yf(e^{\lambda x} y^m)$.

The transformation $z = e^{\lambda x} y^m$, $w(z) = y'_x/y$ leads to a third-order equation.

15. $y''''_{xxxx} = x^{-4} f(x^m e^{\lambda y})$.

The transformation $z = x^m e^{\lambda y}$, $w(z) = xy'_x$ leads to a third-order equation.

16. $y''''_{xxxx} = f(y + ae^x) - ae^x$.

The substitution $w = y + ae^x$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

17. $y''''_{xxxx} = f(y + a\cosh x) - a\cosh x$.

The substitution $w = y + a\cosh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

18. $y''''_{xxxx} = f(y + a\sinh x) - a\sinh x$.

The substitution $w = y + a\sinh x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

19. $y''''_{xxxx} = f(y + a\cos x) - a\cos x$.

The substitution $w = y + a\cos x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

20. $y''''_{xxxx} = f(y + a\sin x) - a\sin x$.

The substitution $w = y + a\sin x$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

► Equations of the form $y''''_{xxxx} = f(x, y, y'_x)$.

21. $y''''_{xxxx} = f(y)y'_x + g(x)$.

By integrating, we find $y'''_{xxx} = \int f(y)\,dy + \int g(x)\,dx + C$. For $g(x) \equiv 0$, the order of this equation can reduced by one with the help of the substitution $w(y) = y'_x$.

22. $y''''_{xxxx} = x^{-4} f(xy'_x - y)$.

The transformation $t = \ln|x|$, $w = xy'_x - y$ leads to a third-order autonomous equation: $w'''_{ttt} - 5w''_{tt} + 6w'_t = f(w)$.

23. $y''''_{xxxx} = f(x, xy'_x - y)$.

The substitution $w = xy'_x - y$ leads to a third-order equation: $(w'_x/x)''_{xx} = f(x, w)$.

24. $y''''_{xxxx} = f(x, xy'_x - 2y)$.

The substitution $w = xy'_x - 2y$ leads to a third-order equation: $xw'''_{xxx} - w''_{xx} = x^2 f(x, w)$.

25. $y''''_{xxxx} = f(x, xy'_x - 3y)$.

The substitution $w = xy'_x - 3y$ leads to a third-order equation: $w'''_{xxx} = xf(x, w)$.

26. $y''''_{xxxx} = yx^{-4} f(xy'_x/y)$.

The transformation $z = xy'_x/y$, $w = x^2 y''_{xx}/y$ leads to a second-order equation.

27. $y''''_{xxxx} = a^4 y + f(x, y'_x - ay)$.

The substitution $w = y'_x - ay$ leads to a third-order equation: $w'''_{xxx} + aw''_{xx} + a^2 w'_x + a^3 w = f(x, w)$.

28. $y''''_{xxxx} = f(x, y'_x \sinh x - y \cosh x) + y.$

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a third-order equation.

29. $y''''_{xxxx} = f(x, y'_x \cosh x - y \sinh x) + y.$

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a third-order equation.

30. $y''''_{xxxx} = f(x, y'_x \sin x - y \cos x) + y.$

The substitution $w = y'_x \sin x - y \cos x$ leads to a third-order equation.

31. $y''''_{xxxx} = f(x, y'_x \cos x + y \sin x) + y.$

The substitution $w = y'_x \cos x + y \sin x$ leads to a third-order equation.

32. $y''''_{xxxx} = \dfrac{\varphi''''_{xxxx}}{\varphi} y + f\Big(x, y'_x - \dfrac{\varphi'_x}{\varphi} y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi} y$ leads to a third-order equation.

► **Equations of the form** $y''''_{xxxx} = f(x, y, y'_x, y''_{xx})$.

33. $y''''_{xxxx} + a y''_{xx} = f(y).$

Having integrated this equation, we obtain $2y'_x y'''_{xxx} - (y''_{xx})^2 + a(y'_x)^2 = 2\int f(y)\,dy + 2C$, where C is an arbitrary constant. The substitution $w(y) = |y'_x|^{3/2}$ leads to a second-order equation:

$$w''_{yy} = -\tfrac{3}{4} a w^{-1/3} + \tfrac{3}{2}\Big[\int f(y)\,dy + C\Big] w^{-5/3}.$$

34. $y''''_{xxxx} + f(y'_x) y''_{xx} = g(y).$

Having integrated this equation, we obtain a third-order autonomous equation:

$$2y'_x y'''_{xxx} - (y''_{xx})^2 + 2F(y'_x) = 2\int g(y)\,dy + 2C, \qquad \text{where} \quad F(u) = \int u f(u)\,du.$$

The substitution $w(y) = y'_x$ leads to a second-order equation.

35. $y''''_{xxxx} = x^{-2} f(x y'_x - y) y''_{xx}.$

The transformation $t = \ln|x|$, $w = x y'_x - y$ leads to a third-order equation:

$$w'''_{ttt} - 5w''_{tt} + 6w'_t = f(w) w'_t.$$

Integrating it, we obtain a second-order autonomous equation:

$$w''_{tt} - 5w'_t + 6w = \int f(w)\,dw + C.$$

The substitution $z(w) = \tfrac{1}{5} w'_t$ leads to an Abel equation of the second kind:

$$z z'_w - z = \tfrac{1}{25}\Big[-6w + \int f(w)\,dw + C\Big]$$

(see Section 13.3.1).

36. $yy''''_{xxxx} - (y''_{xx})^2 = f(x).$

Integrating the equation twice, we arrive at a second-order equation:

$$yy''_{xx} - (y'_x)^2 = \int_0^x (x-t)f(t)\,dt + C_1x + C_2.$$

37. $yy''''_{xxxx} - (y''_{xx})^2 = f(x)[yy''_{xx} - (y'_x)^2] + g(x).$

This is a special case of equation 16.2.6.93. The substitution $w(x) = yy''_{xx} - (y'_x)^2$ leads to a second-order linear equation: $w''_{xx} = f(x)w + g(x)$.

38. $y''''_{xxxx} = a^2y + f(y''_{xx} + ay).$

The substitution $w = y''_{xx} + ay$ leads to a second-order autonomous equation of the form 14.9.1.1: $w''_{xx} = aw + f(w)$.

39. $y''''_{xxxx} = f(y, y''_{xx}).$

The substitution $w(y) = \pm(y'_x)^2$ leads to a third-order equation: $ww'''_{yyy} + \frac{1}{2}w'_yw''_{yy} = 2f\left(y, \pm\frac{1}{2}w'_y\right)$.

40. $y''''_{xxxx} = a^2y + f(x, y''_{xx} + ay).$

The substitution $w = y''_{xx} + ay$ leads to a second-order equation: $w''_{xx} = aw + f(x, w)$.

41. $y''''_{xxxx} = yf\left(yy''_{xx} - y'^2_x\right).$

This is a special case of equation 16.2.6.42.

42. $y''''_{xxxx} = y''_{xx}f\left(yy''_{xx} - y'^2_x\right) + yg\left(yy''_{xx} - y'^2_x\right).$

1°. Particular solution:

$$y = C_1\exp(C_3x) + C_2\exp(-C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$C_3^4 - C_3^2f(4C_1C_2C_3^2) - g(4C_1C_2C_3^2) = 0.$$

2°. Particular solution:

$$y = C_1\cos(C_3x) + C_2\sin(C_3x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$C_3^4 + C_3^2f(-C_1^2C_3^2 - C_2^2C_3^2) - g(-C_1^2C_3^2 - C_2^2C_3^2) = 0.$$

43. $y''''_{xxxx} = x^mf(x^2y''_{xx} - 2xy'_x + 2y).$

The substitution $w = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation: $xw''_{xx} - 2w'_x = x^{m+3}f(w)$. For $m = -4$, the substitution $z(w) = \frac{1}{3}xw'_x$ leads to an Abel equation of the second kind: $zz'_w - z = \frac{1}{9}f(w)$ (see Section 13.3.1).

44. $y''''_{xxxx} = f(x, xy'_x - y, y''_{xx}).$

The substitution $w(x) = xy'_x - y$ leads to a third-order equation.

45. $y''''_{xxxx} = f(x, x^2 y''_{xx} - 2xy'_x + 2y)$.

The substitution $w = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation: $xw''_{xx} - 2w'_x = x^3 f(x, w)$.

46. $y''''_{xxxx} = y'_x f\left(\dfrac{y''_{xx}}{y'_x},\ y'_x - y\dfrac{y''_{xx}}{y'_x}\right)$.

Particular solution: $y = C_1 \exp(C_2 x) + C_3$, where C_1 is an arbitrary constant and the constants C_2 and C_2 are related by the constraint $C_2^3 = f(C_2, -C_2 C_3)$.

► Equations of the form $y''''_{xxxx} = f(x, y, y'_x, y''_{xx}, y'''_{xxx})$.

47. $y''''_{xxxx} + ay'''_{xxx} + by''_{xx} + cy'_x = e^{\lambda x} f(ye^{-\lambda x})$.

The substitution $w(x) = ye^{-\lambda x}$ leads to an autonomous equation:

$$w''''_{xxxx} + (4\lambda + a)w'''_{xxx} + (6\lambda^2 + 3a\lambda + b)w''_{xx} + (4\lambda^3 + 3a\lambda^2 + 2b\lambda + c)w'_x + (\lambda^4 + a\lambda^3 + b\lambda^2 + c\lambda)w = f(w),$$

which can be reduced to a third-order equation by means of the substitution $z(w) = w'_x$. For $a = -4\lambda$ and $c = 8\lambda^3 - 2b\lambda$, the above equation coincides, up to notation, with equation 16.2.6.33 and can be reduced to a second-order equation.

48. $y''''_{xxxx} + ayy'''_{xxx} = f(x)$.

Integrating, we arrive at a third-order equation: $y'''_{xxx} + ayy''_{xx} - \frac{1}{2}a(y'_x)^2 = \int f(x)\,dx + C$.

49. $y''''_{xxxx} + ayy'''_{xxx} - ay'_x y''_{xx} = f(x)$.

Integrating, we arrive at a third-order equation: $y'''_{xxx} + ayy''_{xx} - a(y'_x)^2 = \int f(x)\,dx + C$.

50. $y''''_{xxxx} + ayy'''_{xxx} + f(y'_x)y''_{xx} = g(x)$.

Integrating, we arrive at a third-order equation:

$$y'''_{xxx} + ayy''_{xx} - \tfrac{1}{2}a(y'_x)^2 + F(y'_x) = \int g(x)\,dx + C, \qquad \text{where} \quad F(u) = \int f(u)\,du.$$

51. $y''''_{xxxx} = f(y)y'_x y'''_{xxx}$.

Integrating, we arrive at a third-order autonomous equation of the form 15.5.1.1: $y'''_{xxx} = C\exp\left[\int f(y)\,dy\right]$.

52. $xy''''_{xxxx} + 4y'''_{xxx} = f(xy)$.

The substitution $w(x) = xy$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

53. $xy''''_{xxxx} + (a+3)y'''_{xxx} = f(x, xy'_x + ay)$.

The substitution $w = xy'_x + ay$ leads to a third-order equation: $w'''_{xxx} = f(x, w)$.

54. $x^2 y''''_{xxxx} + 8xy'''_{xxx} + 12y''_{xx} = f(x^2 y)$.

The substitution $w(x) = x^2 y$ leads to an autonomous equation of the form 16.2.6.1: $w''''_{xxxx} = f(w)$.

55. $x^4 y''''_{xxxx} + a_3 x^3 y'''_{xxx} + a_2 x^2 y''_{xx} + a_1 x y'_x = f(y)$.

The substitution $t = \ln|x|$ leads to an autonomous equation:

$$y''''_{tttt} + (a_3 - 6)y'''_{ttt} + (11 - 3a_3 + a_2)y''_{tt} + (2a_3 - a_2 + a_1 - 6)y'_t = f(y), \qquad (1)$$

the order of which can be lowered with the help of the substitution $w(y) = y'_t$. For $a_3 = 6$ and $a_1 = a_2 - 6$, equation (1) coincides, up to notation, with equation 16.2.6.33 and can be reduced to a second-order equation.

56. $x^4 y''''_{xxxx} + a x^3 y'''_{xxx} + b x^2 y''_{xx} + c x y'_x + s y = x^{-k} f(y x^k)$.

The transformation $t = \ln x$, $w = y x^k$ leads to an autonomous equation of the form 16.2.6.79.

57. $y y''''_{xxxx} - y'_x y'''_{xxx} = f(x) y^2$.

Integrating yields a third-order linear equation: $y'''_{xxx} = \Big[\int f(x)\,dx + C\Big] y$.

58. $y y''''_{xxxx} + 4 y'_x y'''_{xxx} + 3(y''_{xx})^2 = f(x)$.

Solution: $y^2 = C_3 x^3 + C_2 x^2 + C_1 x + C_0 + \frac{1}{3}\int_{x_0}^{x} (x - t)^3 f(t)\,dt$.

59. $y y''''_{xxxx} + a y'_x y'''_{xxx} + (a - 1)(y''_{xx})^2 = f(x)$.

Integrating the equation two times, we obtain a second-order equation:

$$y y''_{xx} + \frac{a - 2}{2}(y'_x)^2 = C_1 x + C_0 + \int_{x_0}^{x} (x - t) f(t)\,dt.$$

60. $y y''''_{xxxx} + 4 y'_x y'''_{xxx} + 3(y''_{xx})^2 = f(y)$.

The substitution $w = y^2$ leads to an equation of the form 16.2.6.1: $w''''_{xxxx} = 2f\big(\pm\sqrt{w}\,\big)$.

61. $y y''''_{xxxx} - y'_x y'''_{xxx} = f(x) y y'''_{xxx}$.

Integrating yields a third-order linear equation: $y'''_{xxx} = C \exp\Big[\int f(x)\,dx\Big] y$.

62. $y y''''_{xxxx} + y'_x y'''_{xxx} = f(x) y y'''_{xxx}$.

Integrating yields a third-order equation of the form 15.5.1.2: $y y'''_{xxx} = C \exp\Big[\int f(x)\,dx\Big]$.

63. $y y''''_{xxxx} + (f - 1) y'_x y'''_{xxx} + f g y y'_x + g'_x y^2 = 0, \qquad f = f(x), \quad g = g(x)$.

The functions that solve the third-order linear equation $y'''_{xxx} + g(x) y = 0$ are solutions of the given equation.

64. $y y''''_{xxxx} + (4 y'_x + f y) y'''_{xxx} + 3(y''_{xx})^2 + 3 f y'_x y''_{xx} + g(x) = 0, \qquad f = f(x)$.

The substitution $w = (y y'_x)''_{xx}$ leads to a first-order linear equation: $w'_x + f w + g = 0$.

Solution:

$$y^2 = C_2 x^2 + C_1 x + C_0 + \int_{x_0}^{x} (x - t)^2 w(t)\,dt,$$

where $w(x) = e^{-F(x)}\Big[C_3 - \int e^{F(x)} g(x)\,dx\Big]$, $F(x) = \int f(x)\,dx$; x_0 is an arbitrary number.

65. $yy''''_{xxxx} + (4y'_x + fy)y'''_{xxx} + 3(y''_{xx})^2 + (3fy'_x + gy)y''_{xx} + g(y'_x)^2 + hyy'_x + s = 0.$

Here, $f = f(x)$, $g = g(x)$, $h = h(x)$, $s = s(x)$. The substitution $w = yy'_x$ leads to a third-order nonhomogeneous linear equation: $w'''_{xxx} + fw''_{xx} + gw'_x + hw + s = 0$.

66. $(y + ax + b)y''''_{xxxx} + 4(y'_x + a)y'''_{xxx} + 3(y''_{xx})^2 = f(x).$

Solution: $(y + ax + b)^2 = C_3x^3 + C_2x^2 + C_1x + C_0 + \frac{1}{3}\int_{x_0}^{x}(x - t)^3 f(t)\,dt.$

67. $yy''''_{xxxx} = 4y'_x y'''_{xxx} + 3(y''_{xx})^2 - 6\dfrac{(y'_x)^4}{y^2} + [yy''_{xx} - (y'_x)^2]f\Big(\dfrac{y'_x}{y}\Big).$

The transformation $\xi = \dfrac{y'_x}{y}$, $w = \dfrac{y''_{xx}}{y} - \Big(\dfrac{y'_x}{y}\Big)^2$ leads to a second-order linear equation with respect to w^2: $(w^2)''_{\xi\xi} = 24\xi^2 + 2f(\xi)$. Integrating it, we obtain

$$w^2 = C_2\xi + C_1 + 2\xi^4 + 2\int_{\xi_0}^{\xi}(\xi - t)f(t)\,dt.$$

Taking into account that $\xi'_x = w$, $y'_x = \xi y$, $y'_\xi = \xi y/w$, we find the solution in parametric form:

$$x = \int \frac{d\xi}{w} + C_3, \qquad y = C_4 \exp\Big(\int \frac{\xi\,d\xi}{w}\Big),$$

where $w = \pm\Big[C_2\xi + C_1 + 2\xi^4 + 2\int_{\xi_0}^{\xi}(\xi - t)f(t)\,dt\Big]^{1/2}$.

68. $y^2y''''_{xxxx} - 2yy'_x y'''_{xxx} + f(x)y^2y'''_{xxx} + 2(y'_x)^2y''_{xx} - f(x)yy'_x y''_{xx} + 2f'_x(x)y^2y''_{xx} + 2f(x)(y'_x)^3 + [f^2(x) - 2f'_x(x)]y(y'_x)^2 + f''_{xx}(x)y^2y'_x = 0.$

The solution satisfies the second-order linear equation $y''_{xx} + f(x)y'_x - z(x, C_1, C_2)y = 0$, where $z = z(x, C_1, C_2)$ is the Weierstrass elliptic function determined by the second-order autonomous equation $z''_{xx} + z^2 = 0$.

69. $y^2y''''_{xxxx} - 2yy'_x y'''_{xxx} + f(x)y^2y'''_{xxx} + 2(y'_x)^2y''_{xx} - f(x)yy'_x y''_{xx} + 2f'_x(x)y^2y''_{xx} + 2f(x)(y'_x)^3 + [f^2(x) - 2f'_x(x)]y(y'_x)^2 + f''_{xx}(x)y^2y'_x = Axy^3.$

The solution satisfies the second-order linear equation $y''_{xx} + f(x)y'_x - z(x, C_1, C_2)y = 0$, where $z = z(x, C_1, C_2)$ is the solution of the first Painlevé transcendent $z''_{xx} + z^2 = Ax$.

70. $y^2y''''_{xxxx} - 2yy'_x y'''_{xxx} + [a + f(x)]y^2y'''_{xxx} - y(y''_{xx})^2 - [3a + f(x)]yy'_x y''_{xx} + 2(y'_x)^2y''_{xx} + [af(x) + g(x)]y^2y''_{xx} + 2a(y'_x)^3 - af(x)y(y'_x)^2 + ag(x)y^2y'_x = h(x)y^3.$

The solution satisfies the second-order linear equation $y''_{xx} + ay'_x - z(x, C_1, C_2)y = 0$, where $z = z(x, C_1, C_2)$ is the solution of the second-order linear equation $z''_{xx} + f(x)z'_x + g(x)z = h(x)$.

71. $y''_{xx}y''''_{xxxx} - 3(y'''_{xxx})^2 = f(xy'_x - y)(y''_{xx})^5.$

The Legendre transformation $x = u'_t$, $y = tu'_t - u$ leads to an equation of the form 16.2.6.1: $u''''_{tttt} = -f(u)$.

72. $y''''_{xxxx} = f(y)y'_x g(y'''_{xxx})$.

Integrating yields a third-order autonomous equation:

$$\int \frac{dw}{g(w)} = \int f(y)\,dy + C, \qquad \text{where} \quad w = y'''_{xxx},$$

the order of which can be lowered by means of the substitution $z(y) = y'_x$.

73. $xy''''_{xxxx} + 2y'''_{xxx} = (xy''_{xx})^{-5} f\Big(\dfrac{xy''_{xx}}{\sqrt{xy'_x - y}}\Big)$.

The substitution $w(x) = xy'_x - y$ leads to a third-order equation of the form 15.5.2.27:

$$w'''_{xxx} = w^{-5/2} F\Big(\frac{w'_x}{\sqrt{w}}\Big), \qquad \text{where} \quad F(\xi) = \xi^{-5} f(\xi).$$

74. $x^2 y''''_{xxxx} + 2xy'''_{xxx} = f(x^2 y''_{xx} - 2xy'_x + 2y)g(x^2 y'''_{xxx})$.

The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation of the form 14.9.4.36: $w''_{xx} = f(w)g(w'_x)$.

75. $y''''_{xxxx} = f(x)g(x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y)$.

The substitution $w(x) = x^3 y'''_{xxx} - 3x^2 y''_{xx} + 6xy'_x - 6y$ leads to a first-order separable equation: $w'_x = x^3 f(x)g(w)$.

► **Other equations.**

76. $yy''''_{xxxx} - \frac{1}{6}(y''_{xx})^2 = x^2 f_1(y''''_{xxxx}) + xf_2(y''''_{xxxx}) + f_3(y''''_{xxxx})$.

Particular solution:

$$y = \tfrac{1}{24}C_1x^4 + \tfrac{1}{6}C_2x^3 + \tfrac{1}{2}C_3x^2 + C_4x + C_5,$$

where the constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints

$$\begin{aligned} \tfrac{1}{3}C_1C_3 - \tfrac{1}{6}C_2^2 &= f_1(C_1), \\ C_1C_4 - \tfrac{1}{3}C_2C_3 &= f_2(C_1), \\ C_1C_5 - \tfrac{1}{6}C_3^2 &= f_3(C_1). \end{aligned}$$

77. $yy''''_{xxxx} - \frac{1}{6}(y''_{xx})^2 = y'''_{xxx} f_1(y''''_{xxxx}) + y''_{xx} f_2(y''''_{xxxx})$
$\qquad + x^2 f_3(y''''_{xxxx}) + xf_4(y''''_{xxxx}) + f_5(y''''_{xxxx})$.

Particular solution:

$$y = \tfrac{1}{24}C_1x^4 + \tfrac{1}{6}C_2x^3 + \tfrac{1}{2}C_3x^2 + C_4x + C_5,$$

where the constants C_1, C_2, C_3, C_4, and C_5 are related by three constraints

$$\begin{aligned} \tfrac{1}{3}C_1C_3 - \tfrac{1}{6}C_2^2 &= \tfrac{1}{2}C_1f_2(C_1) + f_3(C_1), \\ C_1C_4 - \tfrac{1}{3}C_2C_3 &= C_1f_1(C_1) + C_2f_2(C_1) + f_4(C_1), \\ C_1C_5 - \tfrac{1}{6}C_3^2 &= C_2f_1(C_1) + C_3f_2(C_1) + f_5(C_1). \end{aligned}$$

78. $y''''_{xxxx} = F(x,\, y'_x,\, y''_{xx},\, y'''_{xxx})$.

The substitution $w(x) = y'_x$ leads to a third-order equation: $w'''_{xxx} = F(x,\, w,\, w'_x,\, w''_{xx})$.

79. $y''''_{xxxx} = F(y,\ y'_x,\ y''_{xx},\ y'''_{xxx}).$

Autonomous equation. The substitution $w(y) = (y'_x)^2$ leads to a third-order equation:

$$ww'''_{yyy} + \tfrac{1}{2}w'_y w''_{yy} = 2F(y,\ \pm\sqrt{w},\ \tfrac{1}{2}w'_y,\ \pm\tfrac{1}{2}\sqrt{w}\,w''_{yy}).$$

80. $y''''_{xxxx} = x^{-3}F(y/x,\ y'_x,\ xy''_{xx},\ x^2y'''_{xxx}).$

Homogeneous equation. The transformation $t = \ln x,\ w = y/x$ leads to an autonomous equation of the form 16.2.6.79.

81. $y''''_{xxxx} = x^{-k-4}F(x^k y,\ x^{k+1}y'_x,\ x^{k+2}y''_{xx},\ x^{k+3}y'''_{xxx}).$

Generalized homogeneous equation. The transformation $t = \ln x,\ w = x^k y$ leads to an autonomous equation of the form 16.2.6.79.

82. $y''''_{xxxx} = F(x,\ xy'_x - y,\ y''_{xx},\ y'''_{xxx}).$

This is a special case of equation 17.2.6.78 with $n = 4$. The substitution $w = xy'_x - y$ leads to a third-order equation.

83. $y''''_{xxxx} = F(x,\ xy'_x - 2y,\ y'''_{xxx}).$

The substitution $w = xy'_x - 2y$ leads to a third-order equation: $\zeta'_x = F(x,\ w,\ \zeta)$, where $\zeta = w''_{xx}/x$.

84. $y''''_{xxxx} = F(x,\ x^2y''_{xx} - 2xy'_x + 2y,\ y'''_{xxx}).$

The substitution $w(x) = x^2y''_{xx} - 2xy'_x + 2y$ leads to a second-order equation: $(x^{-2}w'_x)'_x = F(x, w, x^{-2}w'_x)$.

85. $y''''_{xxxx} = yF\left(\dfrac{y'_x}{y},\ \dfrac{y''_{xx}}{y},\ \dfrac{y'''_{xxx}}{y}\right).$

The transformation $\xi = \dfrac{y'_x}{y},\ w = \dfrac{y''_{xx}}{y} - \left(\dfrac{y'_x}{y}\right)^2$ leads to a second-order equation:

$$w^2w''_{\xi\xi} + w(w'_\xi)^2 + 4\xi ww'_\xi + 3w^2 + 6\xi^2 w + \xi^4 = F(\xi,\ w + \xi^2,\ ww'_\xi + 3\xi w + \xi^3).$$

86. $y''''_{xxxx} = yx^{-4}F\left(x^k y^m,\ \dfrac{xy'_x}{y},\ \dfrac{x^2y''_{xx}}{y},\ \dfrac{x^3y'''_{xxx}}{y}\right).$

Generalized homogeneous equation. The transformation $t = x^k y^m,\ z = \dfrac{xy'_x}{y}$ leads to a third-order equation.

87. $y''''_{xxxx} = yx^{-4}F\left(\dfrac{xy'_x}{y},\ \dfrac{x^2y''_{xx}}{y},\ \dfrac{x^3y'''_{xxx}}{y}\right).$

The transformation $z = \dfrac{xy'_x}{y},\ w = \dfrac{x^2y''_{xx}}{y}$ leads to a second-order equation.

88. $y''''_{xxxx} = y'_x F\left(\dfrac{y''_{xx}}{y'_x},\ y'_x - y\dfrac{y''_{xx}}{y'_x},\ \dfrac{y'''_{xxx}}{y'_x}\right).$

Autonomous equation. This is a special case of equation 16.2.6.79.

Particular solution:

$$y = C_1\exp(C_2x) + C_3,$$

where C_1 is an arbitrary constant and the constants C_2 and C_2 are related by the constraint $C_2^3 = F(C_2, -C_2C_3, C_2^2)$.

89. $y''''_{xxxx} = e^{-\alpha x} F\bigl(e^{\alpha x} y,\ e^{\alpha x} y'_x,\ e^{\alpha x} y''_{xx},\ e^{\alpha x} y'''_{xxx}\bigr).$

This equation is *invariant under "translation–dilatation" transformation.* The substitution $u = e^{\alpha x} y$ leads to an autonomous equation of the form 16.2.6.79.

90. $y''''_{xxxx} = x^{-4} F(x^m e^{\alpha y},\ x y'_x,\ x^2 y''_{xx},\ x^3 y'''_{xxx}).$

The transformation $z = x^m e^{\alpha y}$, $w = x y'_x$ leads to a third-order equation.

91. $y''''_{xxxx} = y F\Bigl(e^{\alpha x} y^m,\ \dfrac{y'_x}{y},\ \dfrac{y''_{xx}}{y},\ \dfrac{y'''_{xxx}}{y}\Bigr).$

The transformation $z = e^{\alpha x} y^m$, $w = y'_x / y$ leads to a third-order equation.

92. $F(y''_{xx}/y,\ y y''_{xx} - y'^2_x,\ y'''_{xxx}/y'_x,\ y''''_{xxxx}/y) = 0.$

Autonomous equation. This is a special case of equation 16.2.6.79.

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where C_1, C_2, and C_2 are related by the constraint $F\bigl(C_3^2,\ 4C_1 C_2 C_3^2,\ C_3^2,\ C_3^4\bigr) = 0.$

2°. Particular solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

with C_1, C_2, and C_2 related by the constraint $F\bigl(-C_3^2, -(C_1^2 + C_2^2) C_3^2, -C_3^2, C_3^4\bigr) = 0.$

93. $F\bigl(x,\ y y''_{xx} - (y'_x)^2,\ y y'''_{xxx} - y'_x y''_{xx},\ y y''''_{xxxx} - (y''_{xx})^2\bigr) = 0.$

The substitution $w(x) = y y''_{xx} - (y'_x)^2$ leads to a second-order equation of the form $F(x, w, w'_x, w''_{xx}) = 0$.

94. $F\Bigl(\dfrac{y''''_{xxxx}}{y'_x},\ y\dfrac{y''''_{xxxx}}{y'_x} - y'''_{xxx}\Bigr) = 0.$

A solution of this equation is any function that solves the third-order linear equation:

$$y'''_{xxx} = C_1 y + C_2,$$

where the constants C_1 and C_2 are related by the constraint $F(C_1, -C_2) = 0$.

95. $F(x,\ y''_{xx} + a y,\ y''''_{xxxx} - a^2 y,\ y''''_{xxxx} + a y''_{xx}) = 0.$

The substitution $u = y''_{xx} + a y$ leads to a second-order equation: $F(x, u, u''_{xx} - a u, u''_{xx}) = 0$.