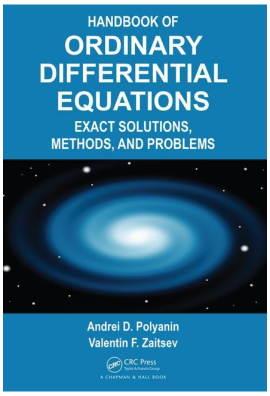

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

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Chapter 17: Higher-Order Ordinary Differential Equations

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Chapter 17

Higher-Order Ordinary Differential Equations

17.1 Linear Equations

17.1.1 Preliminary Remarks

In this chapter, we denote higher derivatives by $y_x^{(n)}$ to mean $d^n y/dx^n$.

1°. The general solution of a homogeneous linear equation of the nth-order

$$f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y'_x + f_0(x)y = 0 \qquad (1)$$

has the form:

$$y = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x). \qquad (2)$$

Here, $y_1(x)$, $y_2(x)$, ..., $y_n(x)$ make up a fundamental set of solutions (the y_k are linearly independent solutions; $y_k \not\equiv 0$); $C_1, C_2, \ldots, C_n$ are arbitrary constants.

2°. Let $y_0 = y_0(x)$ be a nontrivial particular solution of equation (1). Then the substitution

$$y = y_0(x)\int z(x)\,dx$$

leads to a linear $(n-1)$st-order equation for $z(x)$.

Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly independent particular solutions of equation (1) with $g \equiv 0$. Then the substitution

$$y = y_1 \int y_2 w\,dx - y_2 \int y_1 w\,dx$$

leads to a linear $(n-2)$nd-order equation for $w = w(x)$.

3°. Further information about higher-order linear equations can be found in Chapter 4.

17.1.2 Equations Containing Power Functions

► **Equations of the form** $f_n(x)y_x^{(n)} + f_0(x)y = g(x)$.

1. $y_x^{(6)} + ay = 0.$

1°. Solution for $a = 0$:

$$y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4 + C_6 x^5.$$

2°. Solution for $a = k^6 > 0$:

$$y = C_1 \cos kx + C_2 \sin kx + \cos\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_3 \cosh\xi + C_4 \sinh\xi\bigr) + \sin\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_5 \cosh\xi + C_6 \sinh\xi\bigr), \quad \text{where} \quad \xi = \tfrac{\sqrt{3}}{2}kx.$$

3°. Solution for $a = -k^6 < 0$:

$$y = C_1 \cosh kx + C_2 \sinh kx + \cosh\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_3 \cos\xi + C_4 \sin\xi\bigr) + \sinh\bigl(\tfrac{1}{2}kx\bigr)\bigl(C_5 \cos\xi + C_6 \sin\xi\bigr), \quad \text{where} \quad \xi = \tfrac{\sqrt{3}}{2}kx.$$

2. $\boldsymbol{y_x^{(2n)} = a^{2n}y.}$

Solution:

$$y = C_1 e^{ax} + C_2 e^{-ax} + \sum_{k=1}^{n-1} e^{\varphi_k}(A_k \cos\theta_k + B_k \sin\theta_k),$$

where $\varphi_k = ax\cos\dfrac{k\pi}{n}$, $\theta_k = ax\sin\dfrac{k\pi}{n}$; C_1, C_2, A_k, B_k $(k = 1, 2, \dots, n-1)$ are arbitrary constants.

3. $\boldsymbol{y_x^{(n)} = axy + b, \qquad a > 0.}$

Solution:

$$y = \sum_{\nu=0}^{n} C_\nu \varepsilon_\nu \int_0^\infty \exp\Bigl[\varepsilon_\nu xt - \frac{t^{n+1}}{a(n+1)}\Bigr]\,dt, \qquad \varepsilon_\nu = \exp\Bigl(\frac{2\pi\nu i}{n+1}\Bigr),$$

where $\displaystyle\sum_{\nu=0}^{n} C_\nu = \frac{b}{a}$ and $i^2 = -1$.

4. $\boldsymbol{y_x^{(n)} = ax^\beta y.}$

For specific β, see equations 17.1.2.2, 17.1.2.3 (with $b = 0$), 17.1.2.5 to 17.1.2.9, and 17.1.2.10 (with $b = 0$).

1°. Let $n \ge 2$, $\beta > -n$, and $(n+\beta)(s+1) \ne 1, 2, \dots, n-1$, where $s = 0, 1, \dots$ Then the equation has n solutions that can be represented as:

$$y_j(x) = x^{j-1} E_{n,1+\beta/n,(\beta+j-1)/n}\bigl(ax^{\beta+n}\bigr), \qquad j = 1, 2, \dots, n. \tag{1}$$

Here, $E_{n,m,l}(z)$ is a Mittag-Leffler type special function defined by:

$$E_{n,m,l}(z) = 1 + \sum_{k=1}^{\infty} b_k z^k,$$
$$b_k = \prod_{s=0}^{k-1} \frac{\Gamma\bigl(n(ms+l)+1\bigr)}{\Gamma\bigl(n(ms+l+1)+1\bigr)} = \prod_{s=0}^{k-1} \frac{1}{[n(ms+l)+1]\dots[n(ms+l)+n]}, \tag{2}$$

where $\Gamma(\xi)$ is the gamma function, l is an arbitrary number, and $m > 0$.

If $\beta \ge 0$, solutions (1) are linearly independent. Series expansions of (1) are convenient for small x.

2°. Let $n\ge 2$, $\beta<-n$, and $(n+\beta)(s+1)\ne -1,\ -2,\ \dots,\ -(n-1)$, where $s=0,\ 1,\ \dots$ Then the equation in question has n solutions that can be represented as:

$$y_j(x)=x^{j-1}E_{n,-1-\beta/n,-1-(\beta+j)/n}\bigl(a(-1)^n x^{\beta+n}\bigr),\qquad j=1,\ 2,\ \dots,\ n,\qquad (3)$$

where $E_{n,m,l}(z)$ is the Mittag-Leffler type special function defined by (2). If $\beta\le -2n$, solutions (3) are linearly independent. Series expansions of (3) are convenient for large x.

3°. The transformation $x=t^{-1}$, $y=wt^{1-n}$ leads to an equation of similar form:

$$w_t^{(n)}=a(-1)^{n+1}t^{-2n-\beta}w.$$

⊙ *Literature*: M. Saigo and A. A. Kilbas (2000).

5. $\boldsymbol{x^{2n}y_x^{(n)}=ay.}$

The transformation $x=t^{-1}$, $y=wt^{1-n}$ leads to a constant coefficient linear equation: $w_t^{(n)}=(-1)^n aw$.

6. $\boldsymbol{x^{n}y_x^{(2n)}=ay.}$

Solution:

$$y=x^{n/2}\sum_{k=1}^{n}\bigl[C_{k1}I_n\bigl(2\beta_k\sqrt{x}\,\bigr)+C_{k2}K_n\bigl(2\beta_k\sqrt{x}\,\bigr)\bigr],$$

where $I_n(z)$ and $K_n(z)$ are modified Bessel functions; $\beta_1,\ \beta_2,\ \dots,\ \beta_n$ are roots of the equation $\beta^n=\sqrt{a}$.

7. $\boldsymbol{x^{3n}y_x^{(2n)}=ay.}$

The transformation $x=t^{-1}$, $y=wt^{1-2n}$ leads to an equation of the form 17.1.2.6: $t^n w_t^{(2n)}=aw$.

8. $\boldsymbol{x^{n+1/2}y_x^{(2n+1)}=ay.}$

Solution:

$$y=x^{(2n+1)/4}\sum_{k=0}^{2n}C_k\bigl[J_{-n-1/2}\bigl(2\beta_k\sqrt{x}\,\bigr)+iJ_{n+1/2}\bigl(2\beta_k\sqrt{x}\,\bigr)\bigr],$$

where $J_m(z)$ are Bessel functions; $\beta_0,\beta_1,\dots,\beta_{2n}$ are roots of the equation $\beta^{2n+1}=-ai$; $i^2=-1$.

9. $\boldsymbol{x^{3n+3/2}y_x^{(2n+1)}=ay.}$

The transformation $x=t^{-1}$, $y=wt^{-2n}$ leads to a linear equation of the form 17.1.2.8: $t^{n+1/2}w_t^{(2n+1)}=-aw$.

10. $\boldsymbol{x^{2n+1}y_x^{(n)}=ay+bx^n.}$

The transformation $x=t^{-1}$, $y=wt^{1-n}$ leads to a linear equation of the form 17.1.2.3: $w_t^{(n)}=(-1)^n(atw+b)$.

11. $(ax+b)^{2n+1}y_x^{(n)} = (cx+d)y$.

The transformation $\xi = \dfrac{cx+d}{ax+b}$, $w = \dfrac{y}{(ax+b)^{n-1}}$ leads to an equation of the form 17.1.2.3: $w_\xi^{(n)} = \Delta^{-n}\xi w$, where $\Delta = bc - ad$.

12. $(ax+b)^n(cx+d)^n y_x^{(n)} = ky$.

1°. The transformation $\xi = \ln\dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^{n-1}}$ leads to a constant coefficient linear equation.

2°. The transformation $\zeta = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^{n-1}}$ leads to the Euler equation 17.1.2.39: $\zeta^n w_\zeta^{(n)} = k\Delta^{-n}w$, where $\Delta = ad - bc$.

13. $(ax^2+bx+c)^n y_x^{(n)} = ky$.

The transformation $\xi = \displaystyle\int \frac{dx}{ax^2+bx+c}$, $w = y(ax^2+bx+c)^{\frac{1-n}{2}}$ leads to a constant coefficient linear equation.

14. $(ax+b)^n(cx+d)^{3n} y_x^{(2n)} = ky$.

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^{2n-1}}$ leads to an equation of the form 17.1.2.6: $\xi^n w_\xi^{(2n)} = k\Delta^{-2n}w$, where $\Delta = ad - bc$.

15. $(ax+b)^{n+1/2}(cx+d)^{3n+3/2} y_x^{(2n+1)} = ky$.

The transformation $\xi = \dfrac{ax+b}{cx+d}$, $w = \dfrac{y}{(cx+d)^{2n}}$ leads to an equation of the form 17.1.2.8: $\xi^{n+1/2} w_\xi^{(2n+1)} = k\Delta^{-2n-1}w$, where $\Delta = ad - bc$.

► Equations of the form $f_n(x)y_x^{(n)} + f_1(x)y_x' + f_0(x)y = g(x)$.

16. $y_x^{(n)} + ax^k y_x' + akx^{k-1}y = 0$.

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} + ax^k y = C$.

17. $y_x^{(n)} + ax^{k+1}y_x' - a(n-1)x^k y = 0$.

The substitution $z = xy_x' - (n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)} + ax^{k+1}z = 0$.

18. $y_x^{(n)} + ax^{k+1}y_x' + a(k+n)x^k y = 0$.

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of the form 17.1.2.16: $w_t^{(n)} + bt^\nu w_t' + b\nu t^{\nu-1}w = 0$, where $b = a(-1)^{n+1}$, $\nu = 1 - k - 2n$.

19. $y_x^{(n)} + (ax+b)x^k y_x' - ax^k y = 0$.

Particular solution: $y_0 = ax + b$.

20. $y_x^{(n)} + (ax+b)x^k y'_x - 2ax^k y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

21. $y_x^{(n)} + (ax+b)x^k y'_x - 3ax^k y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

22. $y_x^{(n)} + (ax+b)x^k y'_x - a(n-1)x^k y = 0.$

Particular solution: $y_0 = (ax+b)^{n-1}$. The substitution $z = (ax+b)y'_x - a(n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)} + (ax+b)x^k z = 0$.

23. $y_x^{(n)} + ax^{k+1} y'_x - amx^k y = 0, \qquad m = 1,\ 2,\ \dots,\ n-1.$

Particular solution: $y_0 = x^m$. The substitution $z = xy'_x - my$ leads to an $(n-1)$st-order linear equation:

$$D^{n-m-1}\Bigl(\frac{z_x^{(m)}}{x}\Bigr) + ax^k z = 0, \qquad \text{where} \quad D = \frac{d}{dx}.$$

► **Other equations.**

24. $y_x^{(2n)} = a^n y + bx^k (y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = bx^k$. The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order linear equation: $w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = bx^k w$.

25. $y_x^{(n)} + a_{n-1}y_x^{(n-1)} + \cdots + a_1 y'_x + a_0 y = 0.$

Constant coefficient homogeneous linear equation. To solve this equation, determine the n roots of the characteristic equation:

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0.$$

If the roots $\lambda_1, \lambda_2, \dots, \lambda_n$ are all real and different, then the general solution of the original equation is:

$$y = C_1 \exp(\lambda_1 x) + C_2 \exp(\lambda_2 x) + \cdots + C_n \exp(\lambda_n x).$$

The general case, which involves the cases of multiple and/or complex roots of the characteristic equation, is discussed in Section 4.1.1.

26. $y_x^{(n)} + ax^k y_x^{(m)} - (ab^m x^k + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

27. $y_x^{(n)} + (ax^k - b^{n-m})y_x^{(m)} - ab^m x^k y = 0.$

Particular solution: $y_0 = e^{bx}$.

28. $y_x^{(n)} + ay_x^{(n-1)} + bx^m y'_x + abx^m y = 0.$

Particular solution: $y_0 = e^{-ax}$.

29. $xy_x^{(n)} - nmy_x^{(n-1)} + axy = 0, \qquad n = 2,\ 3,\ 4,\ \dots, \qquad m = 1,\ 2,\ 3,\ \dots$

Solution:

$$y = x^{(m+1)n-1}\Bigl(x^{1-n}\frac{d}{dx}\Bigr)^m (x^{1-n}w),$$

where w is the general solution of the constant coefficient linear equation: $w_x^{(n)} + aw = 0$.

30. $xy_x^{(n)} + ny_x^{(n-1)} = axy + b.$

The substitution $w = xy$ leads to a constant coefficient linear equation: $w_x^{(n)} = aw + b$.

31. $xy_x^{(n)} + ny_x^{(n-1)} = ax^2y + b.$

The substitution $w = xy$ leads to an equation of the form 17.1.2.3: $w_x^{(n)} = axw + b$.

32. $xy_x^{(n)} + (n - m - 1)y_x^{(n-1)} + ax^k y'_x - amx^{k-1}y = 0.$

Particular solution: $y_0 = x^m$.

33. $xy_x^{(n)} + ax^k y_x^{(m)} - (ax^k + amx^{k-1} + x + n)y = 0.$

Particular solution: $y_0 = xe^x$.

34. $xy_x^{(n)} = \sum\limits_{\nu=0}^{n-1} [(aA_{\nu+1} - A_\nu)x + A_{\nu+1}]y_x^{(\nu)}.$

Here, $A_n = 1$, $A_0 = 0$; a and A_ν are arbitrary numbers ($\nu = 1, 2, \dots, n-1$).

Denote $f(\lambda) = \sum\limits_{\nu=0}^{n-1} A_{\nu+1}\lambda^\nu$. Let the roots $\lambda_1, \lambda_2, \dots, \lambda_{n-1}$ of the algebraic equation $f(\lambda) = 0$ be all different, and $f(a) \neq 0$. Then the solution is as follows:

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \dots + C_{n-1} e^{\lambda_{n-1} x} + C_n e^{ax} \left[x - \frac{f'_a(a)}{f(a)} \right].$$

35. $\sum\limits_{\nu=0}^{n} (a_\nu x + b_\nu) y_x^{(\nu)} = 0.$

The Laplace equation. Particular solutions:

$$y_k = \int_{\alpha_k}^{\beta_k} \frac{1}{P(t)} \exp\left[xt + \int \frac{Q(t)}{P(t)}\, dt \right] dt,$$

where $P(t) = \sum\limits_{\nu=0}^{n} a_\nu t^\nu$, $Q(t) = \sum\limits_{\nu=0}^{n} b_\nu t^\nu$; α_k and β_k are found from the condition

$$\exp\left(xt + \int \frac{Q(t)}{P(t)}\, dt \right) \bigg|_{\alpha_k}^{\beta_k} = 0.$$

In many cases, the path of integration has to be chosen on the complex plane.

36. $x^2 y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = ax^2y + b.$

The substitution $w = x^2y$ leads to a constant coefficient linear equation: $w_x^{(n)} = aw + b$.

37. $x^2 y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = ax^3y + b.$

The substitution $w = x^2y$ leads to an equation of the form 17.1.2.3: $w_x^{(n)} = axw + b$.

38. $x(x+m)y_x^{(n)} + x(ax^k - x - n)y_x^{(m)} - a(x+m)x^k y = 0.$

Particular solution: $y_0 = xe^x$.

39. $\boldsymbol{a_n x^n y_x^{(n)} + a_{n-1} x^{n-1} y_x^{(n-1)} + \cdots + a_1 x y'_x + a_0 y = 0.}$

Euler equation. If all roots λ_k $(k = 1, 2, \dots, n)$ of the algebraic equation

$$\sum_{\nu=1}^{n} a_\nu \lambda(\lambda-1)\dots(\lambda-\nu+1) = -a_0$$

are different, the general solution of the original differential equation is given by:

$$y = C_1|x|^{\lambda_1} + C_2|x|^{\lambda_2} + \cdots + C_n|x|^{\lambda_n}.$$

In the general case, the substitution $t = \ln|x|$ leads to a constant coefficient linear equation of the form 17.1.2.25:

$$\sum_{\nu=1}^{n} a_\nu D(D-1)\dots(D-\nu+1)y = -a_0 y, \qquad \text{where} \quad D = \frac{d}{dx}.$$

40. $\boldsymbol{x^{2n+1} y_x^{(n)} + n x^{2n} y_x^{(n-1)} = axy.}$

The substitution $w = xy$ leads to an equation of the form 17.1.2.5: $x^{2n} w_x^{(n)} = aw$.

41. $\boldsymbol{x^{2n+1} y_x^{(n)} + n x^{2n} y_x^{(n-1)} = ay.}$

The substitution $w = xy$ leads to an equation of the form 17.1.2.10: $x^{2n+1} w_x^{(n)} = aw$.

42. $\boldsymbol{x^n y_x^{(2n)} + 2n x^{n-1} y_x^{(2n-1)} = ay.}$

The substitution $w = xy$ leads to an equation of the form 17.1.2.6: $x^n w_x^{(2n)} = aw$.

43. $\boldsymbol{x^{3n} y_x^{(2n)} + 2n x^{3n-1} y_x^{(2n-1)} = ay.}$

The substitution $w = xy$ leads to an equation of the form 17.1.2.7: $x^{3n} w_x^{(2n)} = aw$.

44. $\boldsymbol{x^{n+1} y_x^{(2n+1)} + (2n+1) x^n y_x^{(2n)} = a\sqrt{x}\, y.}$

The substitution $w = xy$ leads to an equation of the form 17.1.2.8: $x^{n+1/2} w_x^{(2n+1)} = aw$.

45. $\boldsymbol{x^{3n+3/2} y_x^{(2n+1)} + (2n+1) x^{3n+1/2} y_x^{(2n)} = ay.}$

The substitution $w = xy$ leads to an equation of the form 17.1.2.9: $x^{3n+3/2} w_x^{(2n+1)} = aw$.

46. $\boldsymbol{P_{n-1}(x) y_x^{(n)} + P_{n-2}(x) y_x^{(n-1)} + \cdots + P_1(x) y''_{xx} + (a_1 x + b_1) y'_x - m a_1 y = 0.}$

Here, the $P_\nu(x)$ are polynomials of degree $\le \nu$, m is a positive integer, $a_1 \neq 0$.

A particular solution of this equation is the polynomial of degree m that can be written as:

$$y_0 = \sum_{k=0}^{m} \left(-\frac{1}{a_1}\right)^k [x^m I x^{-m-1} (P_{n-1} D^n + \cdots + P_1 D^2 + b_1 D)]^k x^m,$$

where $D = \dfrac{d}{dx}$, $Ix^\nu = \dfrac{x^{\nu+1}}{\nu+1}$ with $\nu \neq -1$.

⊙ *Literature*: E. Kamke (1977).

47. $[a_n x^n + P_{n-1}(x)]y_x^{(n)} + \cdots + [a_1 x + P_0(x)]y'_x + a_0 y = 0.$

Here, the $P_\nu(x)$ are polynomials of degree $\le \nu$.

Assume that for some integer $m \ge 0$:

$$\sum_{\nu=0}^{n} C_m^\nu \nu!\, a_\nu = 0, \qquad \text{where} \quad C_m^\nu = \frac{m!}{\nu!\,(m-\nu)!},$$

and m is the least of the numbers satisfying this condition. Then there exists a solution in the form of a polynomial of degree m such that no polynomial of a smaller degree satisfies the original equation.

⊙ *Literature*: E. Kamke (1977).

48. $\big[xP(\delta) - Q(\delta)\big]y = 0, \qquad \delta \equiv x\frac{d}{dx}.$

Here, $P = P(z)$ and $Q = Q(z)$ are arbitrary polynomials of degree p and q, respectively.

Suppose $Q(z+1) = (z+1)Q_1(z+1)$, where the polynomial $Q_1(z+1)$ is such that $P(z)$ and $Q_1(z+1)$ do not have common factors. Then the original equation admits a formal solution in the power series form:

$$y_0 = \sum_{n=0}^{\infty} A_n x^n, \qquad \text{where} \quad \frac{A_{n+1}}{A_n} = \frac{P(n)}{Q(n+1)}.$$

⊙ *Literature*: H. Bateman and A. Erdélyi (1953, Vol. 1).

17.1.3 Equations Containing Exponential and Hyperbolic Functions

► **Equations with exponential functions.**

1. $y_x^{(n)} + (ax+b)e^{\lambda x} y'_x - ae^{\lambda x} y = 0.$

Particular solution: $y_0 = ax + b$.

2. $y_x^{(n)} + (ax+b)e^{\lambda x} y'_x - 2ae^{\lambda x} y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

3. $y_x^{(n)} + (ax+b)e^{\lambda x} y'_x - 3ae^{\lambda x} y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

4. $y_x^{(n)} + (ax+b)e^{\lambda x} y'_x - (n-1)ae^{\lambda x} y = 0.$

Particular solution: $y_0 = (ax+b)^{n-1}$. The substitution $z = (ax+b)y'_x - a(n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)} + (ax+b)e^{\lambda x} z = 0$.

5. $y_x^{(n)} + axe^{\lambda x} y'_x - ame^{\lambda x} y = 0, \qquad m = 1,\ 2,\ \ldots,\ n-1.$

Particular solution: $y_0 = x^m$.

6. $y_x^{(2n)} = a^n y + be^{\lambda x}(y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = be^{\lambda x}$. The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order linear equation: $w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = be^{\lambda x}w$.

7. $y_x^{(n)} + (ae^{\lambda x} - b^{n-m})y_x^{(m)} - ab^m e^{\lambda x} y = 0.$

Particular solution: $y_0 = e^{bx}$.

8. $y_x^{(n)} + ay_x^{(n-1)} + be^{\lambda x} y'_x + abe^{\lambda x} y = 0.$

Particular solution: $y_0 = e^{-ax}$.

9. $y_x^{(n)} + ae^{\lambda x} y_x^{(m)} - (ab^m e^{\lambda x} + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

10. $y_x^{(n)} = \sum\limits_{k=0}^{n-1} (A_{k+1} e^{\lambda x} + bA_{k+1} - A_k) y_x^{(k)}.$

Here, $A_n = 1$, $A_0 = 0$; b and A_k are arbitrary numbers ($k = 1, 2, \dots, n-1$).

Particular solutions: $y_m = e^{\mu_m x}$, where the μ_m are roots of the polynomial equation $\sum\limits_{k=0}^{n-1} A_{k+1}\mu^k = 0$.

11. $xy_x^{(n)} + axe^{\lambda x} y_x^{(m)} - [a(x+m)e^{\lambda x} + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

12. $xy_x^{(n)} + (n-m-1)y_x^{(n-1)} + axe^{\lambda x} y'_x - ame^{\lambda x} y = 0.$

Particular solution: $y_0 = x^m$.

13. $x(x+m)y_x^{(n)} + x(ae^{\lambda x} - x - n)y_x^{(m)} - a(x+m)e^{\lambda x} y = 0.$

Particular solution: $y_0 = xe^x$.

14. $(ax^m + be^x + c)y_x^{(n)} = be^x y, \qquad m = 0,\ 1,\ \dots,\ n-1.$

Particular solution: $y_0 = ax^m + be^x + c$.

15. $(ax^m e^x + b)y_x^{(n)} = (-1)^n by, \qquad m = 0,\ 1,\ \dots,\ n-1.$

Particular solution: $y_0 = ax^m + be^{-x}$.

16. $\Bigl(ae^x + \sum\limits_{k=0}^{n-1} b_k x^k\Bigr) y_x^{(n)} = ae^x y.$

Particular solution: $y_0 = ae^x + \sum\limits_{k=0}^{n-1} b_k x^k$.

► Equations with hyperbolic functions.

17. $y_x^{(2n)} = a^n y + b \sinh^k(\lambda x)(y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\sinh^k(\lambda x)$.

18. $y_x^{(n)} + a\sinh^k x\, y_x^{(m)} - (ab^m \sinh^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

19. $y_x^{(n)} + (a\sinh^k x - b^{n-m})y_x^{(m)} - ab^m \sinh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

20. $y_x^{(n)} + (ax+b)\sinh^m(\lambda x)y'_x - a\sinh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax+b.$

21. $xy_x^{(n)} + ax\sinh^k x\, y_x^{(m)} - [a(x+m)\sinh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x.$

22. $y_x^{(2n)} = a^n y + b\cosh^k(\lambda x)(y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\cosh^k(\lambda x)$.

23. $y_x^{(n)} + a\cosh^k x\, y_x^{(m)} - (ab^m\cosh^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}.$

24. $y_x^{(n)} + (a\cosh^k x - b^{n-m})y_x^{(m)} - ab^m\cosh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}.$

25. $y_x^{(n)} + (ax+b)\cosh^m(\lambda x)y'_x - a\cosh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax+b.$

26. $xy_x^{(n)} + ax\cosh^k x\, y_x^{(m)} - [a(x+m)\cosh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x.$

27. $y_x^{(2n)} = y + a(y'_x\cosh x - y\sinh x).$

The substitution $w = y'_x\cosh x - y\sinh x$ leads to a $(2n-1)$st-order linear equation.

28. $y_x^{(2n)} = y + a(y'_x\sinh x - y\cosh x).$

The substitution $w = y'_x\sinh x - y\cosh x$ leads to a $(2n-1)$st-order linear equation.

29. $y_x^{(n)} + a\tanh^k x\, y_x^{(m)} - (ab^m\tanh^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}.$

30. $y_x^{(n)} + (a\tanh^k x - b^{n-m})y_x^{(m)} - ab^m\tanh^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}.$

31. $y_x^{(n)} + (ax+b)\tanh^m(\lambda x)y'_x - a\tanh^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax+b.$

32. $xy_x^{(n)} + ax\tanh^k x\, y_x^{(m)} - [a(x+m)\tanh^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x.$

33. $y_x^{(n)} + a\coth^k x\, y_x^{(m)} - (ab^m\coth^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}.$

34. $y_x^{(n)} + (a\coth^k x - b^{n-m})y_x^{(m)} - ab^m\coth^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}.$

35. $y_x^{(n)} + (ax+b)\coth^m(\lambda x)\, y'_x - a\coth^m(\lambda x)\, y = 0.$

Particular solution: $y_0 = ax+b.$

36. $xy_x^{(n)} + ax\coth^k x\, y_x^{(m)} - [a(x+m)\coth^k x + x + n]y = 0.$

Particular solution: $y_0 = xe^x.$

17.1.4 Equations Containing Logarithmic Functions

1. $y_x^{(2n)} = a^n y + b \ln x (y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b \ln x$.

2. $y_x^{(n)} + a \ln^k x \, y_x^{(m)} - (ab^m \ln^k x + b^n) y = 0.$

Particular solution: $y_0 = e^{bx}$.

3. $y_x^{(n)} + (a \ln^k x - b^{n-m}) y_x^{(m)} - ab^m \ln^k x \, y = 0.$

Particular solution: $y_0 = e^{bx}$.

4. $y_x^{(n)} + a y_x^{(n-1)} + b \ln^k(\lambda x) y'_x + ab \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}$.

5. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - a \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

6. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - 2a \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = (ax + b)^2$.

7. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - 3a \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = (ax + b)^3$.

8. $y_x^{(n)} + (ax + b) \ln^k(\lambda x) y'_x - a(n-1) \ln^k(\lambda x) y = 0.$

Particular solution: $y_0 = (ax + b)^{n-1}$.

9. $y_x^{(n)} + ax \ln^k(\lambda x) y'_x - am \ln^k(\lambda x) y = 0, \qquad m = 1, 2, \ldots, n-1.$

Particular solution: $y_0 = x^m$.

10. $x y_x^{(n)} + ax \ln^k(\lambda x) y_x^{(m)} - [a(x + m) \ln^k(\lambda x) + x + n] y = 0.$

Particular solution: $y_0 = x e^x$.

17.1.5 Equations Containing Trigonometric Functions

► **Equations with sine and cosine.**

1. $y_x^{(n)} + a \sin^k x \, y_x^{(m)} - (ab^m \sin^k x + b^n) y = 0.$

Particular solution: $y_0 = e^{bx}$.

2. $y_x^{(n)} + (a \sin^k x - b^{n-m}) y_x^{(m)} - ab^m \sin^k x \, y = 0.$

Particular solution: $y_0 = e^{bx}$.

3. $y_x^{(n)} + a y_x^{(n-1)} + b \sin^m(\lambda x) y'_x + ab \sin^m(\lambda x) y = 0.$

Particular solution: $y_0 = e^{-ax}$.

4. $y_x^{(n)} + (ax + b) \sin^m(\lambda x) y'_x - a \sin^m(\lambda x) y = 0.$

Particular solution: $y_0 = ax + b$.

5. $y_x^{(n)} + (ax+b)\sin^m(\lambda x)y'_x - 2a\sin^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

6. $y_x^{(n)} + (ax+b)\sin^m(\lambda x)y'_x - 3a\sin^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

7. $y_x^{(2n)} = a^n y + b\sin^k(\lambda x)(y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\sin^k(\lambda x)$.

8. $xy_x^{(n)} + ax\sin^k(\lambda x)y_x^{(m)} - [a(x+m)\sin^k(\lambda x) + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

9. $(ax^m + b\sin x)y_x^{(n)} = b\sin\left(x + \tfrac{1}{2}\pi n\right)y, \qquad m = 0,\ 1,\ \dots,\ n-1.$

Particular solution: $y_0 = ax^m + b\sin x$.

10. $\Bigl(a\sin x + \sum\limits_{k=0}^{n-1} b_k x^k\Bigr)y_x^{(n)} = a\sin\left(x + \tfrac{1}{2}\pi n\right)y.$

Particular solution: $y_0 = a\sin x + \sum\limits_{k=0}^{n-1} b_k x^k$.

11. $y_x^{(n)} + a\cos^k x\, y_x^{(m)} - (ab^m\cos^k x + b^n)y = 0.$

Particular solution: $y_0 = e^{bx}$.

12. $y_x^{(n)} + (a\cos^k x - b^{n-m})y_x^{(m)} - ab^m\cos^k x\, y = 0.$

Particular solution: $y_0 = e^{bx}$.

13. $y_x^{(n)} + ay_x^{(n-1)} + b\cos^m(\lambda x)y'_x + ab\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = e^{-ax}$.

14. $y_x^{(n)} + (ax+b)\cos^m(\lambda x)y'_x - a\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = ax + b$.

15. $y_x^{(n)} + (ax+b)\cos^m(\lambda x)y'_x - 2a\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

16. $y_x^{(n)} + (ax+b)\cos^m(\lambda x)y'_x - 3a\cos^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

17. $y_x^{(2n)} = a^n y + b\cos^k(\lambda x)(y''_{xx} - ay).$

This is a special case of equation 17.1.6.18 with $f(x) = b\cos^k(\lambda x)$.

18. $xy_x^{(n)} + ax\cos^k(\lambda x)y_x^{(m)} - [a(x+m)\cos^k(\lambda x) + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

19. $(ax^m + b\cos x)y_x^{(n)} = b\cos\left(x + \tfrac{1}{2}\pi n\right)y, \qquad m = 0,\ 1,\ \dots,\ n-1.$

Particular solution: $y_0 = ax^m + b\cos x$.

20. $\Big(a\cos x+\sum\limits_{k=0}^{n-1}b_kx^k\Big)y_x^{(n)}=a\cos\big(x+\tfrac{1}{2}\pi n\big)y.$

Particular solution: $y_0=a\cos x+\sum\limits_{k=0}^{n-1}b_kx^k$.

21. $y_x^{(2n)}=(-1)^ny+a(y_x'\sin x-y\cos x).$

The substitution $w=y_x'\sin x-y\cos x$ leads to a $(2n-1)$st-order linear equation.

22. $y_x^{(2n)}=(-1)^ny+a(y_x'\cos x+y\sin x).$

The substitution $w=y_x'\cos x+y\sin x$ leads to a $(2n-1)$st-order linear equation.

► **Equations with tangent and cotangent.**

23. $y_x^{(n)}+a\tan^k x\,y_x^{(m)}-(ab^m\tan^k x+b^n)y=0.$

Particular solution: $y_0=e^{bx}$.

24. $y_x^{(n)}+(a\tan^k x-b^{n-m})y_x^{(m)}-ab^m\tan^k x\,y=0.$

Particular solution: $y_0=e^{bx}$.

25. $y_x^{(n)}+ay_x^{(n-1)}+b\tan^m(\lambda x)y_x'+ab\tan^m(\lambda x)y=0.$

Particular solution: $y_0=e^{-ax}$.

26. $y_x^{(n)}+(ax+b)\tan^m(\lambda x)y_x'-a\tan^m(\lambda x)y=0.$

Particular solution: $y_0=ax+b$.

27. $y_x^{(n)}+(ax+b)\tan^m(\lambda x)y_x'-2a\tan^m(\lambda x)y=0.$

Particular solution: $y_0=(ax+b)^2$.

28. $y_x^{(n)}+(ax+b)\tan^m(\lambda x)y_x'-3a\tan^m(\lambda x)y=0.$

Particular solution: $y_0=(ax+b)^3$.

29. $xy_x^{(n)}+ax\tan^k(\lambda x)y_x^{(m)}-[a(x+m)\tan^k(\lambda x)+x+n]y=0.$

Particular solution: $y_0=xe^x$.

30. $y_x^{(n)}+a\cot^k x\,y_x^{(m)}-(ab^m\cot^k x+b^n)y=0.$

Particular solution: $y_0=e^{bx}$.

31. $y_x^{(n)}+(a\cot^k x-b^{n-m})y_x^{(m)}-ab^m\cot^k x\,y=0.$

Particular solution: $y_0=e^{bx}$.

32. $y_x^{(n)}+ay_x^{(n-1)}+b\cot^m(\lambda x)y_x'+ab\cot^m(\lambda x)y=0.$

Particular solution: $y_0=e^{-ax}$.

33. $y_x^{(n)}+(ax+b)\cot^m(\lambda x)y_x'-a\cot^m(\lambda x)y=0.$

Particular solution: $y_0=ax+b$.

34. $y_x^{(n)} + (ax+b)\cot^m(\lambda x)y'_x - 2a\cot^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

35. $y_x^{(n)} + (ax+b)\cot^m(\lambda x)y'_x - 3a\cot^m(\lambda x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

36. $xy_x^{(n)} + ax\cot^k(\lambda x)y_x^{(m)} - [a(x+m)\cot^k(\lambda x) + x + n]y = 0.$

Particular solution: $y_0 = xe^x$.

17.1.6 Equations Containing Arbitrary Functions

► **Equations of the form** $f_n(x)y_x^{(n)} + f_1(x)y'_x + f_0(x)y = g(x)$.

1. $y_x^{(n)} = f(x).$

Solution: $y = \sum_{\nu=0}^{n-1} C_\nu x^\nu + \int_{x_0}^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)\,dt$, where x_0 is an arbitrary number.

2. $y_x^{(n)} = f(x)y.$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of similar form: $w_t^{(n)} = (-1)^n t^{-2n} f(1/t)w$.

3. $y_x^{(n)} = (cx+d)^{-2n} f\left(\frac{ax+b}{cx+d}\right)y.$

The transformation $\xi = \frac{ax+b}{cx+d}$, $w = \frac{y}{(cx+d)^{n-1}}$ leads to a simpler equation: $w_\xi^{(n)} = \Delta^{-n} f(\xi)w$, where $\Delta = ad - bc$.

4. $fy_x^{(n)} - f_x^{(n)}y = 0, \qquad f = f(x).$

Particular solution: $y_0 = f(x)$.

5. $fy_x^{(2n+1)} + f_x^{(2n+1)}y = g(x), \qquad f = f(x).$

First integral: $\sum_{k=0}^{2n} (-1)^k f_x^{(2n-k)} y_x^{(k)} = \int g(x)\,dx + C$.

6. $y_x^{(n)} + (ax+b)f(x)y'_x - af(x)y = 0.$

Particular solution: $y_0 = ax + b$.

7. $y_x^{(n)} + (ax+b)f(x)y'_x - 2af(x)y = 0.$

Particular solution: $y_0 = (ax+b)^2$.

8. $y_x^{(n)} + (ax+b)f(x)y'_x - 3af(x)y = 0.$

Particular solution: $y_0 = (ax+b)^3$.

9. $y_x^{(n)} + (ax+b)f(x)y'_x - (n-1)af(x)y = 0.$

Particular solution: $y_0 = (ax+b)^{n-1}$. The substitution $z = (ax+b)y'_x - a(n-1)y$ leads to an $(n-1)$st-order linear equation: $z_x^{(n-1)} + (ax+b)f(x)z = 0$.

10. $y_x^{(n)} + xf(x)y'_x - mf(x)y = 0, \qquad m = 1,\ 2,\ \dots,\ n-1.$

Particular solution: $y_0 = x^m$. The substitution $z = xy'_x - my$ leads to an $(n-1)$st-order equation:

$$D^{n-m-1}\Big(\frac{z_x^{(m)}}{x}\Big) + f(x)z = 0, \quad \text{where} \quad D = \frac{d}{dx}.$$

11. $y_x^{(n)} + f(x)y'_x + g(x)y + h(x) = 0.$

The transformation $x = t^{-1}$, $y = wt^{1-n}$ leads to an equation of similar form:

$$w_t^{(n)} + (-1)^n t^{-2n}\Big\{-t^2 f(1/t)w'_t + \big[(n-1)tf(1/t) + g(1/t)\big]w + t^{n-1}h(1/t)\Big\} = 0.$$

12. $y_x^{(n)} + f(x)y'_x + f'_x(x)y = g(x).$

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} + f(x)y = \int g(x)\,dx + C.$

13. $y_x^{(2n)} = y + f(x)(y'_x \cosh x - y \sinh x).$

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a $(2n-1)$st-order linear equation.

14. $y_x^{(2n)} = y + f(x)(y'_x \sinh x - y \cosh x).$

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a $(2n-1)$st-order linear equation.

15. $y_x^{(2n)} = (-1)^n y + f(x)(y'_x \sin x - y \cos x).$

The substitution $w = y'_x \sin x - y \cos x$ leads to a $(2n-1)$st-order linear equation.

16. $y_x^{(2n)} = (-1)^n y + f(x)(y'_x \cos x + y \sin x).$

The substitution $w = y'_x \cos x + y \sin x$ leads to a $(2n-1)$st-order linear equation.

17. $y_x^{(n)} = \frac{\varphi_x^{(n)}}{\varphi}y + f(x)\Big(y'_x - \frac{\varphi'_x}{\varphi}y\Big), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi}y$ leads to an $(n-1)$st-order linear equation.

► **Other equations.**

18. $y_x^{(2n)} = a^n y + f(x)(y''_{xx} - ay).$

The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order linear equation:

$$w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = f(x)w.$$

19. $y_x^{(n)} + f(x)(x^2 y''_{xx} - 2xy'_x + 2y) = 0.$

Particular solutions: $y_1 = x$, $y_2 = x^2$. The substitution $z = x^2 y''_{xx} - 2xy'_x + 2y$ leads to a linear equation of the $(n-2)$nd-order.

20. $y_x^{(n+2)} + f(x)[x^2 y''_{xx} - 2nxy'_x + n(n+1)y] = 0.$

The substitution $w(x) = x^2 y''_{xx} - 2nxy'_x + n(n+1)y$ leads to an nth-order linear equation: $w_x^{(n)} + x^2 f(x)w = 0.$

21. $\boldsymbol{y_x^{(2n)} = a^2 y + f(x)[y_x^{(n)} + ay].}$

The substitution $w = y_x^{(n)} + ay$ leads to an nth-order linear equation: $w_x^{(n)} = [f(x)+a]w$.

22. $\boldsymbol{y_x^{(n)} + f(x) y_x^{(m)} - [a^n + a^m f(x)]y = 0.}$

Particular solution: $y_0 = e^{ax}$.

23. $\boldsymbol{y_x^{(n)} + (f - a^{n-m}) y_x^{(m)} - a^m f y = 0, \qquad f = f(x).}$

Particular solution: $y_0 = e^{ax}$.

24. $\boldsymbol{y_x^{(n)} + a y_x^{(n-1)} + f y_x' + a f y = 0, \qquad f = f(x).}$

Particular solution: $y_0 = e^{-ax}$.

25. $\boldsymbol{y_x^{(n)} + f(x) y_x^{(n-1)} + g(x) y_x^{(n-2)} + h(x) = 0.}$

The substitution $w(x) = y_x^{(n-2)}$ leads to a second-order linear equation: $w''_{xx} + f(x)w'_x + g(x)w + h(x) = 0$.

26. $\boldsymbol{y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \cdots + a_1 y_x' + a_0 y = f(x).}$

Constant coefficient nonhomogeneous linear equation. The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation (see 5.1.2.25) and any particular solution of the nonhomogeneous equation.

If the roots $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the characteristic equation

$$\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$$

are all different, the original equation has the general solution:

$$y = \sum_{\nu=1}^{n} C_\nu e^{\lambda_\nu x} + \sum_{\nu=1}^{n} \frac{e^{\lambda_\nu x}}{P'_\lambda(\lambda_\nu)} \int f(x) e^{-\lambda_\nu x}\, dx$$

(with complex roots, the real part should be taken).

Section 4.1.2 lists the forms of particular solutions corresponding to some special forms of the right-hand side function of the nonhomogeneous linear equation.

27. $\boldsymbol{y_x^{(n)} + f(x) \sum_{k=0}^{n-1} (-1)^k k!\, C_{n-1}^k x^{n-k-1} y_x^{(n-k-1)} = 0.}$

Here, $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

Particular solutions: $y_m = x^m$, where $m = 1, 2, \ldots, n-1$.

The substitution $z = \sum_{k=0}^{n-1} (-1)^k k!\, C_{n-1}^k x^{n-k-1} y_x^{(n-k-1)}$ leads to a first-order linear equation: $z'_x + x^{n-1} f(x) z = 0$. Having solved this equation, we obtain an $(n-1)$st-order linear equation of the form 17.1.6.34 for $y(x)$.

28. $\boldsymbol{y_x^{(n)} = \sum_{k=0}^{n-1} (a_{k+1} f - a_k) y_x^{(k)}.}$

Here, $f = f(x)$; $a_n = 1$, $a_0 = 0$; a_k are arbitrary numbers ($k = 1, 2, \ldots, n-1$).

Particular solutions: $y_k = e^{\lambda_k x}$ ($k = 1, 2, \ldots, n-1$), where the λ_k are roots of the polynomial equation $\sum_{k=0}^{n-1} a_{k+1}\lambda^k = 0$.

29. $xy_x^{(n)} + (a+n-1)y_x^{(n-1)} = f(x)(xy_x' + ay).$

The substitution $w = xy_x' + ay$ leads to an $(n-1)$st-order linear equation: $w_x^{(n-1)} = f(x)w$.

30. $xy_x^{(n)} + xfy_x^{(m)} - [(x+m)f + x + n]y = 0, \qquad f = f(x).$

Particular solution: $y_0 = xe^x$.

31. $xy_x^{(n)} + ny_x^{(n-1)} = x^{1-2n}f(1/x)y + x^{-n-1}g(1/x).$

The transformation $t = x^{-1}$, $w = yx^{2-n}$ leads to an nth-order linear equation: $w_t^{(n)} = (-1)^n[f(t)w + g(t)]$.

32. $x^2y_x^{(n+2)} + \alpha xy_x^{(n+1)} + \beta y_x^{(n)} + f(x)[x^2y_{xx}'' + (\alpha-2n)xy_x' + (\beta - \alpha n + n^2 + n)y] = 0.$

The substitution $w(x) = x^2y_{xx}'' + (\alpha-2n)xy_x' + (\beta - \alpha n + n^2 + n)y$ leads to an nth-order linear equation: $w_x^{(n)} + f(x)w = 0$.

33. $x(x+m)y_x^{(n)} + x(f - x - n)y_x^{(m)} - (x+m)fy = 0, \qquad f = f(x).$

Particular solution: $y_0 = xe^x$.

34. $x^ny_x^{(n)} + b_{n-1}x^{n-1}y_x^{(n-1)} + \cdots + b_1xy_x' + b_0y = f(x).$

Nonhomogeneous Euler equation. The substitution $x = ae^t$ $(a \neq 0)$ leads to a constant coefficient nonhomogeneous linear equation of the form 17.1.6.26.

35. $x^ny_x^{(n)} + (n-m-1)x^{n-1}y_x^{(n-1)} + xfy_x' - mfy = 0, \qquad f = f(x).$

Particular solution: $y_0 = x^m$.

36. $x^ny_x^{(n)} + x^mfy_x^{(m)} - (n!\,C_a^n + m!\,C_a^m f)y = 0.$

Here, $f = f(x)$, $C_a^n = \dfrac{\Gamma(a+1)}{n!\,\Gamma(a-n+1)}$ are binomial coefficients, and $\Gamma(a)$ is the gamma function.

Particular solution: $y_0 = x^a$.

37. $x^my_x^{(n)} = \sum\limits_{k=0}^{n-1}[x^m(a_{k+1}f - a_k) + a_{k+1}]y_x^{(k)}.$

Here, $f = f(x)$; $a_n = 1$, $a_0 = 0$; m and a_k are arbitrary numbers ($k = 1, 2, \dots, n-1$).

Particular solutions: $y_k = e^{\lambda_k x}$ $(k = 1,\ 2,\ \dots,\ n-1)$, where the λ_k are roots of the polynomial equation $\sum\limits_{k=0}^{n-1} a_{k+1}\lambda^k = 0$.

38. $\sin x\, y_x^{(n)} + \sin x\, f(x)y_x^{(m)} - \left[\sin\left(x + \tfrac{1}{2}\pi n\right) + f(x)\sin\left(x + \tfrac{1}{2}\pi m\right)\right]y = 0.$

Particular solution: $y_0 = \sin x$.

39. $\cos x\, y_x^{(n)} + \cos x\, f(x)y_x^{(m)} - \left[\cos\left(x + \tfrac{1}{2}\pi n\right) + f(x)\cos\left(x + \tfrac{1}{2}\pi m\right)\right]y = 0.$

Particular solution: $y_0 = \cos x$.

40. $\sum\limits_{k=2}^{n} f_k(x)y_x^{(k)} = g(x)(xy_x' - y).$

Particular solution: $y_0 = x$. The substitution $w(x) = xy_x' - y$ leads to an $(n-1)$st-order linear equation.

41. $\displaystyle\sum_{k=m+1}^{n} f_k(x)y_x^{(k)} = g(x)(xy'_x - my), \qquad m = 1,\ 2,\ \ldots,\ n-1.$

Particular solution: $y_0 = x^m$. The substitution $w(x) = xy'_x - my$ leads to an $(n-1)$st-order linear equation.

42. $\displaystyle\sum_{k=3}^{n} f_k(x)y_x^{(k)} = g(x)(x^2y''_{xx} - 2xy'_x + 2y).$

Particular solutions: $y_1 = x$, $y_2 = x^2$. The substitution $w(x) = x^2y''_{xx} - 2xy'_x + 2y$ leads to an $(n-2)$nd-order linear equation.

43. $\displaystyle\sum_{k=4}^{n} f_k(x)y_x^{(k)} = g(x)(x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y).$

Particular solutions: $y_1 = x$, $y_2 = x^2$, $y_3 = x^3$. The substitution $w(x) = x^3y'''_{xxx} - 3x^2y''_{xx} + 6xy'_x - 6y$ leads to an $(n-3)$rd-order linear equation.

44. $\displaystyle\sum_{k=m+1}^{n} f_k(x)y_x^{(k)} + g(x)\sum_{k=0}^{m}(-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)} = 0.$

Here, $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

Particular solutions: $y_s = x^s$, where $s = 1,\ 2,\ \ldots,\ m$.

The substitution $z = \displaystyle\sum_{k=0}^{m}(-1)^k k!\, C_m^k x^{m-k} y_x^{(m-k)}$ leads to an $(n-m)$th-order linear equation: $\displaystyle\sum_{k=m+1}^{n} f_k(x) D^{k-m-1}\bigl(x^{-m}z'_x\bigr) + g(x)z = 0$, where $D = d/dx$.

45. $\displaystyle\sum_{k=0}^{n}(f_k - af_{k+1})y_x^{(k)} = 0.$

Here, $f_k = f_k(x)$ $(k = 1,\ 2,\ \ldots,\ n)$; $f_{n+1} \equiv f_0 \equiv 0$.

Particular solution: $y_0 = e^{ax}$.

46. $\displaystyle\sum_{k=0}^{n} x^k[f_k + (k-m)f_{k+1}]y_x^{(k)} = 0.$

Here, $f_k = f_k(x)$ $(k = 1,\ 2,\ \ldots,\ n)$; $f_{n+1} \equiv f_0 \equiv 0$.

Particular solution: $y_0 = x^m$.

17.2 Nonlinear Equations

17.2.1 Equations Containing Power Functions

► **Fifth- and sixth-order equations.**

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + bx + c.$

This is a special case of equation 17.2.6.1 with $f(x) = ax + b$.

2. $yy_x^{(5)} = ax + b.$

This is a special case of equation 17.2.6.17 with $n = 2$ and $f(x) = ax + b$.

3. $yy^{(5)}_x = ay'_x y''''_{xxxx}$.

1°. For $a \neq -1$, integrating the equation two times, we arrive at a third-order autonomous equation: $y'_x y'''_{xxx} - \frac{1}{2}(y''_{xx})^2 = C_1 y^{a+1} + C_2$. The substitution $w(y) = \frac{1}{2}(y'_x)^2$ leads to a second-order equation:

$$ww''_{yy} - \tfrac{1}{4}(w'_y)^2 = \tfrac{1}{2}C_1 y^{a+1} + \tfrac{1}{2}C_2.$$

For $a = 1$, this is a solvable equation of the form 2.8.1.53.

2°. For $a = -1$, integrating the equation two times, we arrive at a third-order autonomous equation: $y'_x y'''_{xxx} - \frac{1}{2}(y''_{xx})^2 = C_1 \ln|y| + C_2$.

3°. Particular solution: $y = C_1x^3 + C_2x^2 + C_3x + C_4$.

4. $3yy^{(5)}_x + 5y'_x y''''_{xxxx} = 0$.

This is a special case of equation 17.2.1.3 with $a = -\frac{5}{3}$. Integrating the equation three times, we arrive at a second-order equation: $3yy''_{xx} - 2(y'_x)^2 = C_1x^2 + C_2x + C_3$. The substitution $y = w^3$ leads to a solvable equation of the form 14.8.1.5: $w''_{xx} = \frac{1}{9}(C_1x^2 + C_2x + C_3)w^{-5}$.

5. $2yy^{(5)}_x + 5y'_x y''''_{xxxx} + 5y''_{xx} y'''_{xxx} = 0$.

This is a special case of equation 17.2.6.4 with $a = \frac{5}{2}$ and $f(x) = 0$. Integrating the equation three times, we arrive at a second-order equation of the form 14.8.1.53: $yy''_{xx} - \frac{1}{4}(y'_x)^2 = C_1x^2 + C_2x + C_3$.

6. $yy^{(5)}_x + ay'_x y''''_{xxxx} + (3a-5)y''_{xx} y'''_{xxx} = 0$.

Integrating the equation three times, we obtain a second-order nonlinear equation: $yy''_{xx} + \frac{1}{2}(a-3)(y'_x)^2 = C_1x^2 + C_2x + C_3$.

7. $yy^{(5)}_x + ay'_x y''''_{xxxx} + by''_{xx} y'''_{xxx} = 0$.

Integrating yields a fourth-order equation: $yy''''_{xxxx} + (a-1)y'_x y'''_{xxx} + \frac{1}{2}(1-a+b)(y''_{xx})^2 = C$.

8. $yy^{(5)}_x + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = ax^n$.

This is a special case of equation 17.2.6.2 with $f(x) = ax^n$.

9. $yy^{(5)}_x + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = ay^n$.

This is a special case of equation 17.2.6.3 with $f(y) = ay^n$.

10. $y^{(6)}_x = Ay^{-7/5}$.

This is a special case of equation 17.2.1.12 with $n = 3$.

Multiplying by $y^{7/5}$ and differentiating with respect to x, we obtain $5yy^{(7)}_x + 7y'_x y^{(6)}_x = 0$. Having integrated this equation three times, we arrive at the chain of equations:

$$5yy^{(6)}_x + 2y'_x y^{(5)}_x - 2y''_{xx} y''''_{xxxx} + (y'''_{xxx})^2 = 2C_2, \quad (1)$$

$$5yy^{(5)}_x - 3y'_x y''''_{xxxx} + y''_{xx} y'''_{xxx} = 2C_2x + C_1, \quad (2)$$

$$5yy''''_{xxxx} - 8y'_x y'''_{xxx} + \tfrac{9}{2}(y''_{xx})^2 = C_2x^2 + C_1x + C_0, \quad (3)$$

where C_0, C_1, and C_2 are arbitrary constants. Eliminating the highest derivatives from (1)–(3), with the aid of the original equation, we obtain a third-order equation that can be reduced to a second-order equation.

11. $yy^{(6)}_x + 6y'_x y^{(5)}_x + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = ax^n$.

This is a special case of equation 17.2.6.6 with $f(x) = ax^n$.

▶ **Equations of the form $y_x^{(n)} = f(x, y)$.**

12. $y_x^{(2n)} = Ay^{\frac{1+2n}{1-2n}}$.

Multiply both sides by $y^{\frac{2n+1}{2n-1}}$ and differentiate with respect to x. As a result, we obtain

$$(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = 0.$$

Three integrals containing arbitrary constants C_0, C_1, and C_2 are presented in 5.2.6.62, where one should let $f \equiv 0$. Eliminating the highest derivatives from those integrals and the original equation, one can always obtain a $(2n-3)$rd-order equation. With the aid of the transformation

$$t = \int \frac{dx}{P}, \quad w = yP^{\frac{1-2n}{2}}, \qquad \text{where} \quad P = C_2x^2 + C_1x + C_0,$$

this equation can be reduced to the autonomous form 17.2.6.77. Therefore, the substitution $z(w) = w'_t$ finally leads to a $(2n-4)$th-order equation with respect to $z = z(w)$.

13. $y_x^{(2n)} = ay^k + b, \qquad k \neq -1.$

This is a special case of equation 17.2.6.8. Integrating yields a $(2n-1)$st-order equation:

$$\sum_{m=1}^{n-1} (-1)^m y_x^{(m)} y_x^{(2n-m)} + \frac{1}{2}(-1)^n \big[y_x^{(n)}\big]^2 = -\frac{a}{k+1}y^{k+1} - by + C,$$

where C is an arbitrary constant. Furthermore, the order of the obtained autonomous equation can be reduced by one by the substitution $w(y) = y'_x$.

14. $y_x^{(n)} = ax^{-n}y^m$.

This is a special case of equation 17.2.6.11 with $f(y) = ay^m$.

15. $y_x^{(n)} = ax^k y^m$.

1°. The transformation $x = t^{-1}$, $y = t^{1-n}w(t)$ leads to an equation of the same form: $w_t^{(n)} = (-1)^n At^{-k-(n-1)m-n-1}w^m$.

2°. The transformation $\xi = x^{n+k}y^{m-1}$, $z = xy'_x/y$ leads to an $(n-1)$st-order equation.

16. $yy_x^{(2n+1)} = ax^k + b$.

This is a special case of equation 17.2.6.17 with $f(x) = ax^k + b$.

17. $y_x^{(n)} = x^{m-nm-n-1}(ay + bx^{n-1})^m$.

This is a special case of equation 17.2.6.13 with $f(w) = (aw + b)^m$.

18. $y_x^{(2n)} = x^{\frac{m-2nm-2n-1}{2}}\Big(ay + bx^{\frac{2n-1}{2}}\Big)^m$.

This is a special case of equation 17.2.6.14 with $f(w) = (aw + b)^m$.

19. $y_x^{(n)} = (ay + bx^k)^m; \qquad k = 1,\ 2,\ \ldots,\ n-1.$

The substitution $aw = ay + bx^k$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = a^m w^m$ (see also 5.2.1.12 and 5.2.1.13 with $b = 0$).

20. $y_x^{(n)} = (ax^2 + bx + c)^{\frac{m-nm-n-1}{2}} y^m.$

This is a special case of equation 17.2.6.22 with $f(w) = w^m$.

21. $y_x^{(n)} = (ax + b)^{-n}(cx + d)^{m-nm-1} y^m.$

This is a special case of equation 17.2.6.21 with $f(w) = w^m$.

► Equations of the form $y_x^{(n)} = f(x, y, y'_x, y''_{xx})$.

22. $y_x^{(n)} = ay^k y'_x + bx^m.$

This is a special case of equation 17.2.6.34 with $f(y) = ay^k$ and $g(x) = bx^m$. Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = \dfrac{a}{k+1} y^{k+1} + \dfrac{b}{m+1} x^{m+1} + C$.

23. $y_x^{(n)} = a^n y + b(y'_x - ay)^k.$

This is a special case of equation 17.2.6.38 with $f(x, w) = bw^k$. The substitution $w = y'_x - ay$ leads to an $(n-1)$st-order autonomous equation:

$$w_x^{(n-1)} + aw_x^{(n-2)} + \cdots + a^{n-1} w = bw^k.$$

24. $y_x^{(n)} = ax^{m-n} y^{1-m} (y'_x)^m.$

This is a special case of equation 17.2.6.37 with $f(w) = aw^m$.

25. $y_x^{(n)} = ax^m (xy'_x - y)^k.$

This is a special case of equation 17.2.6.39 with $f(x, w) = ax^m w^k$. The substitution $w = xy'_x - y$ leads to an $(n-1)$st-order equation.

26. $y_x^{(n)} = ax^k (xy'_x - my)^l.$

Here, m is a positive integer and $n \geq m + 1$. The substitution $w = xy'_x - my$ leads to an $(n-1)$st-order equation: $\zeta_x^{(n-m-1)} = ax^k w^l$, where $\zeta = w_x^{(m)}/x$.

27. $y_x^{(2n)} + ay''_{xx} + by = cyy''_{xx} - c(y'_x)^2 + k.$

1°. Particular solution:

$$y = C_1 \sinh(C_4 x) + C_2 \cosh(C_4 x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^{2n} + (a - cC_3)C_4^2 + b = 0,$$
$$c(C_2^2 - C_1^2)C_4^2 - bC_3 + k = 0.$$

2°. Particular solution:

$$y = C_1 \sin(C_4 x) + C_2 \cos(C_4 x) + C_3,$$

where the constants C_1, C_2, C_3, and C_4 are related by two constraints

$$C_4^{2n} - (a - cC_3)C_4^2 + b = 0,$$
$$c(C_1^2 + C_2^2)C_4^2 + bC_3 - k = 0.$$

28. $y_x^{(n)} + ayy''_{xx} - a(y'_x)^2 + by''_{xx} + cy'_x = 0.$

Particular solution: $y = C_1 \exp(C_2 x) - \dfrac{C_2^{n-1} + bC_2 + c}{aC_2}$.

29. $y_x^{(2n)} = a^n y + b(y''_{xx} - ay)^k.$

This is a special case of equation 17.2.6.50 with $f(x,w) = bw^k$. The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order autonomous equation: $w_x^{(2n-2)} + aw_x^{(2n-4)} + \cdots + a^{n-1}w = bw^k$.

30. $y_x^{(n)} = ax^m(xy'_x - y)^k (y''_{xx})^l.$

The substitution $w(x) = xy'_x - y$ leads to an $(n-1)$st-order equation:

$$\frac{d^{n-2}}{dx^{n-2}}\left(\frac{w'_x}{x}\right) = ax^{m-l}w^k(w'_x)^l.$$

31. $y_x^{(2n)} = ay\big(yy''_{xx} - y'^2_x\big)^k.$

This is a special case of equation 17.2.6.52 with $f(w) = 0$ and $g(w) = aw^k$.

► Other equations.

32. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2.$

1°. Integrating the equation two times, we obtain an $(n-2)$nd-order equation:

$$y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1 x + C_2.$$

2°. Particular solutions:

$$\begin{aligned} &y = C_1\exp(C_3x) + C_2\exp(-C_3x) + a^{-1}C_3^{n-4} && \text{if } n \text{ is an even number,}\\ &y = C_1\sin(C_3x) + C_2\cos(C_3x) + (-1)^{n/2}a^{-1}C_3^{n-4} && \text{if } n \text{ is an even number,}\\ &y = C_1\exp(C_2x) + a^{-1}C_2^{n-4} && \text{if } n \text{ is an odd number,}\\ &y = C_1x + C_2 && \text{if } n \ge 2 \text{ is any number.} \end{aligned}$$

33. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + bx + k.$

This is a special case of equation 17.2.6.54 with $f(x) = bx + k$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + \frac{1}{6}bx^3 + \frac{1}{2}kx^2 + C_1x + C_2$.

34. $y_x^{(2n)} = a^2 y + b[y_x^{(n)} + ay]^k.$

The substitution $w = y_x^{(n)} + ay$ leads to an nth-order autonomous equation: $w_x^{(n)} = aw + bw^k$.

35. $y_x^{(n)} + axy_x^{(n-1)} + 2byy'_x + abxy^2 + cx = 0.$

The functions that solve the $(n-1)$st-order autonomous equation $y_x^{(n-1)} = -by^2 - c/a$ are solutions of the original equation.

36. $y_x^{(n)} + ayy_x^{(n-1)} + by'_x + aby^2 + cy = 0.$

The functions that solve the $(n-1)$st-order constant coefficient nonhomogeneous linear equation $y_x^{(n-1)} + by = -c/a$ are solutions of the original equation.

37. $xy_x^{(n)} + ny_x^{(n-1)} = ax^m y^m.$

This is a special case of equation 17.2.6.59 with $f(w) = aw^m$.

38. $xy_x^{(n)} + (a+n-1)y_x^{(n-1)} = b(xy'_x + ay)^k.$

This is a special case of equation 17.2.6.60 with $f(x,w) = bw^k$. The substitution $w = xy'_x + ay$ leads to an $(n-1)$st-order autonomous equation: $w_x^{(n-1)} = bw^k$.

39. $x^2 y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = ax^{2m}y^m.$

This is a special case of equation 17.2.6.61 with $f(w) = aw^m$.

40. $yy_x^{(2n+1)} = ay'_x y_x^{(2n)}.$

The equation admits two different (with $a \neq -1$) first integrals:

$$y_x^{(2n)} = \widetilde{C}_1 y^a,$$

$$yy_x^{(2n)} + (a+1)\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + \tfrac{1}{2}(-1)^n(a+1)\big[y_x^{(n)}\big]^2 = \widetilde{C}_2,$$

where $\widetilde{C}_1$ and $\widetilde{C}_2$ are arbitrary constants. Eliminating the highest derivative from the first integrals, we arrive at a $(2n-1)$st-order autonomous equation:

$$\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + \tfrac{1}{2}(-1)^n\big[y_x^{(n)}\big]^2 = C_1 y^{a+1} + C_2,$$

where $C_1 = -\dfrac{\widetilde{C}_1}{a+1}$, $C_2 = \dfrac{\widetilde{C}_2}{a+1}$. The order of the obtained equation next can be lowered by the standard substitution $w(y) = y'_x$.

41. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = ax^m.$

This is a special case of equation 17.2.6.62 with $f(x) = ax^m$.

42. $yy_x^{(n)} - y'_x y_x^{(n-1)} = ay^2.$

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = (ax + C)y$. The transformation $z = x + C/a$ brings it to an equation of the form 17.1.2.3 with $b = 0$.

43. $yy_x^{(n)} = y'_x y_x^{(n-1)} + ay'_x.$

Integrating yields an $(n-1)$st-order constant coefficient nonhomogeneous linear equation: $y_x^{(n-1)} = Cy - a$.

44. $\displaystyle\sum_{k=0}^{n} a_k y_x^{(k)} = byy''_{xx} - b(y'_x)^2 + k.$

Particular solutions: $y = Ce^{\lambda x} + ka_0^{-1}$, where C is an arbitrary constant and λ are roots of the algebraic equation $a_0 \sum_{k=0}^{n} a_k \lambda^n = bk\lambda^2$.

45. $xyy_x^{(n)} = (xy'_x + ay)y_x^{(n-1)}.$

Integrating yields an $(n-1)$st-order linear equation of the form 17.1.2.4: $y_x^{(n-1)} = Cx^a y$.

46. $\boldsymbol{(y+ax^{m-1})y_x^{(n)}-y_x^{(m)}y_x^{(n-m)}+bx^{m-1}y_x^{(m)}=0,\qquad n>m.}$

The functions that solve the $(n-m)$th-order linear equation

$$y_x^{(n-m)}=Cy+(aC+b)x^{m-1}$$

are solutions of the original equation.

47. $\boldsymbol{y_x^{(n-2)}y_x^{(n)}=a\big[y_x^{(n-1)}\big]^2.}$

Solution: $y=\begin{cases}C_0+C_1x+\cdots+C_{n-3}x^{n-3}+(C_{n-2}+C_{n-1}x)^{n-2+\frac{1}{1-a}} & \text{if } a\neq 1,\\ C_0+C_1x+\cdots+C_{n-3}x^{n-3}+C_{n-2}\exp(C_{n-1}x) & \text{if } a=1.\end{cases}$

48. $\boldsymbol{y_x^{(n)}=ay^ky'_x\big[y_x^{(n-1)}\big]^m.}$

This is a special case of equation 17.2.6.73 with $f(y)=y^k,\ g(w)=aw^m$.

49. $\boldsymbol{y_x^{(n)}=ax^{m_1}y^{m_2}(y'_x)^{m_3}\cdots(y_x^{(n-1)})^{m_{n+1}}.}$

Generalized homogeneous equation. The transformation $\xi=x^\lambda y^\mu,\ w=xy'_x/y,$ where

$$\lambda=n+m_1-m_3-2m_4-\cdots-(n-1)m_{n+1},\quad \mu=m_2+m_3+\cdots+m_{n+1}-1,$$

leads to an $(n-1)$st-order equation.

50. $\boldsymbol{\Big(\sqrt{y}\,\frac{d}{dx}\Big)^{n-1}(y'_x)=ax+b.}$

The transformation $x=x(t),\ y=(x'_t)^2$ leads to a constant coefficient linear equation: $2x_t^{(n+1)}=ax+b$.

51. $\boldsymbol{2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)}y_x^{(2n-m)}+(-1)^n\big[y_x^{(n)}\big]^2+\lambda(y'_x)^2=ay^2+by+c.}$

Differentiating both sides with respect to x and dividing by y'_x, we arrive at a constant coefficient linear equation: $2y_x^{(2n)}-2\lambda y''_{xx}+2ay+b=0$.

52. $\boldsymbol{2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)}y_x^{(2n-m)}+(-1)^n\big[y_x^{(n)}\big]^2=\alpha(xy'_x-y)+\beta y'_x+\gamma.}$

Differentiating both sides of the equation with respect to x, we have

$$y''_{xx}\big[2y_x^{(2n-1)}-\alpha x-\beta\big]=0. \qquad (1)$$

Equate the second factor to zero to obtain:

$$y=\frac{\alpha x^{2n}}{2(2n)!}+\frac{\beta x^{2n-1}}{2(2n-1)!}+\sum_{k=0}^{2n-2}C_kx^k.$$

The integration constants C_k and parameters α, β, and γ are related by

$$2\sum_{m=2}^{n-1}(-1)^m m!\,(2n-m)!\,C_mC_{2n-m}+(-1)^n(n!)^2C_n^2=\beta C_1-\alpha C_0+\gamma,$$

which is obtained as a result of substituting the above solution into the original equation.

In addition, there is a solution corresponding to equating the first factor in (1) to zero: $y=\widetilde{C}_1x+\widetilde{C}_0$, where $\beta\widetilde{C}_1-\alpha\widetilde{C}_0+\gamma=0$.

53. $$2\sum_{m=1}^{n-1}(-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n\big[y_x^{(n)}\big]^2 + s(y''_{xx})^2 = \alpha(xy'_x - y) + \beta y'_x + \gamma, \qquad n \geq 3.$$

For the case $s = 0$, see equation 17.2.1.52. Let now $s \neq 0$. Differentiating the equation with respect to x, we have

$$y''_{xx}\big[2y_x^{(2n-1)} + 2sy'''_{xxx} - \alpha x - \beta\big] = 0.$$

Equate the second factor to zero and integrate to obtain:

$$y = \frac{\alpha x^4}{48s} + \frac{\beta x^3}{12s} + C_2x^2 + C_1x + C_0 + \iiint w\,dx\,dx\,dx,$$

where $w = w(x)$ is the general solution of a linear constant coefficient linear equation of the form 17.1.2.2: $w_x^{(2n-4)} + sw = 0$. The constants of integration are related by the constraint that results from substituting the obtained solution into the original equation.

In addition, there is the solution $y = \widetilde{C}_1x + \widetilde{C}_0$, where the constants of integration are related by $\beta\widetilde{C}_1 - \alpha\widetilde{C}_0 + \gamma = 0$.

54. $$\sum_{m=1}^{n} a_m\Big\{2\sum_{\nu=1}^{m-1}(-1)^\nu y_x^{(\nu)} y_x^{(2m-\nu)} + (-1)^m\big[y_x^{(m)}\big]^2\Big\} = \alpha y^2 + 2\beta y + \gamma.$$

Differentiating with respect to x, we arrive at a constant coefficient linear equation:

$$\sum_{m=1}^{n} a_m y_x^{(2m)} + \alpha y + \beta = 0.$$

17.2.2 Equations Containing Exponential Functions

▶ Fifth- and sixth-order equations.

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + be^{\lambda x}$.

1°. This is a special case of equation 17.2.6.1 with $f(x) = be^{\lambda x}$. Integrating the equation two times, we obtain a third-order equation: $y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}e^{\lambda x}$.

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{C\lambda^5 - b}{aC\lambda^4},$$
$$y = \frac{b}{2\lambda^5}\exp(\lambda x) + C\exp(-\lambda x) - \frac{\lambda}{a}.$$

2. $y_x^{(5)} = ae^{\lambda y}y'_x y''''_{xxxx}$.

Integrating yields a fourth-order autonomous equation of the form 17.2.6.8 with $n = 4$: $y''''_{xxxx} = C\exp\Big(\dfrac{a}{\lambda}e^{\lambda y}\Big)$.

3. $yy_x^{(5)} = ae^{\lambda x} + b$.

This is a special case of equation 17.2.6.17 with $n = 2$ and $f(x) = ae^{\lambda x} + b$.

4. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = ae^{\lambda x}$.

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-5}e^{\lambda x}$.

5. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = ae^{\lambda y}$.

This is a special case of equation 17.2.6.3 with $f(y) = ae^{\lambda y}$.

6. $y_x^{(6)} = ae^{\lambda y} + b$.

This is a special case of equation 17.2.6.8 with $n = 6$ and $f(y) = ae^{\lambda y} + b$.

7. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = ae^{\lambda x}$.

This is a special case of equation 17.2.6.6 with $f(x) = ae^{\lambda x}$.

► **Equations of the form $y_x^{(n)} = f(x, y)$.**

8. $y_x^{(n)} = ae^{\lambda y} + b$.

This is a special case of equation 17.2.6.8 with $f(y) = ae^{\lambda y} + b$.

9. $y_x^{(n)} = ae^{\lambda y+\beta x} + b$.

This is a special case of equation 17.2.6.9 with $m = 1$. The substitution $w = y + (\beta/\lambda)x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = ae^{\lambda w} + b$.

10. $y_x^{(n)} = ax^{-n}e^{\lambda y}$.

This is a special case of equation 17.2.6.11 with $f(y) = ae^{\lambda y}$.

11. $y_x^{(n)} = ax^k e^{\alpha y}$.

This is a special case of equation 17.2.6.26 with $f(w) = aw$ and $m = k + n$.

12. $y_x^{(n)} = Ae^{\alpha x}y^m$.

This is a special case of equation 17.2.2.23 with $m = m_1$ and $m_2 = m_3 = \cdots = m_n = 0$.

13. $yy_x^{(2n+1)} = ae^{\lambda x} + b$.

This is a special case of equation 17.2.6.17 with $f(x) = ae^{\lambda x} + b$.

14. $y_x^{(n)} = a\exp(\lambda y + \beta x^m) + b, \qquad m = 1,\ 2,\ \ldots,\ n-1$.

The substitution $w = y + (\beta/\lambda)x^m$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = ae^{\lambda w} + b$.

► **Other equations.**

15. $y_x^{(n)} = ae^{\lambda y}y'_x + be^{\beta x}$.

Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = \dfrac{a}{\lambda}e^{\lambda y} + \dfrac{b}{\beta}e^{\beta x} + C$.

16. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + be^{\lambda x}$.

1°. This is a special case of equation 17.2.6.54 with $f(x) = be^{\lambda x}$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1 x + C_2 + b\lambda^{-2}e^{\lambda x}$.

2°. Particular solutions:

$$y = C\exp(\lambda x) + \frac{C\lambda^n - b}{aC\lambda^4} \qquad (n \text{ is any number}),$$

$$y = \frac{b}{2\lambda^n}\exp(\lambda x) + C\exp(-\lambda x) - \frac{\lambda^{n-4}}{a} \qquad (n \text{ is an odd number}).$$

17. $y_x^{(n)} = a^n y + be^{\lambda x}(y'_x - ay)^k$.

This is a special case of equation 17.2.6.38 with $f(x, w) = be^{\lambda x}w^k$.

18. $y_x^{(n)} = be^{\lambda x}(xy'_x - y)^k$.

This is a special case of equation 17.2.6.39 with $f(x, w) = be^{\lambda x}w^k$.

19. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = ae^{\lambda x}$.

This is a special case of equation 17.2.6.62 with $f(x) = ae^{\lambda x}$.

20. $y_x^{(n)} = ae^{\lambda y}y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = ae^{\lambda y}$.

21. $y_x^{(n)} = (ae^{\lambda y}y'_x + be^{\beta x})y_x^{(n-1)}$.

This is a special case of equation 17.2.6.58 with $f(y) = ae^{\lambda y}$ and $g(x) = be^{\beta x}$.

22. $y_x^{(n)} = ae^{\lambda y}y'_x\big[y_x^{(n-1)}\big]^m$.

This is a special case of equation 17.2.6.73 with $f(y) = ae^{\lambda y}$ and $g(w) = w^m$.

23. $y_x^{(n)} = Ae^{\alpha x}y^{m_1}(y'_x)^{m_2}\dots\big(y_x^{(n-1)}\big)^{m_n}$.

The substitution $w(x) = ye^{\beta x}$, where $\beta = \dfrac{\alpha}{m_1 + m_2 + \dots + m_n - 1}$, leads to an autonomous equation of the form 17.2.6.77.

24. $y_x^{(n)} = Ae^{\alpha y}x^{m_1}(y'_x)^{m_2}(y''_{xx})^{m_3}\dots\big(y_x^{(n-1)}\big)^{m_n}$.

The transformation $z = x^\sigma e^{\alpha y}$, $w = xy'_x$, where $\sigma = n + m_1 - m_2 - 2m_3 - 3m_4 - \dots - (n-1)m_n$, leads to an $(n-1)$st-order equation.

17.2.3 Equations Containing Hyperbolic Functions

► **Equations with hyperbolic sine.**

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sinh(\lambda x)$.

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\sinh(\lambda x)$. Integrating the equation two times, we obtain a third-order equation: $y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1 x + C_2 + b\lambda^{-2}\sinh(\lambda x)$.

2°. Particular solution: $y = \dfrac{b}{\lambda^4(\lambda^2 - a^2C^2)}\big[aC\sinh(\lambda x) + \lambda\cosh(\lambda x)\big] + C$.

2. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\sinh(\lambda x).$

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-5}\cosh(\lambda x)$.

3. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\sinh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\sinh^m(\lambda x) + b$.

4. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\sinh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\sinh^m(\lambda y) + b$.

5. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = a\sinh^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\sinh^m(\lambda x)$.

6. $y_x^{(n)} = a\sinh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\sinh^m(\lambda y) + b$.

7. $y_x^{(n)} = ax^{-n}\sinh^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\sinh^m(\lambda y)$.

8. $yy_x^{(2n+1)} = a\sinh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\sinh^m(\lambda x) + b$.

9. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sinh(\lambda x).$

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\sinh(\lambda x)$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}\sinh(\lambda x)$.

2°. Particular solutions:

$$y = C\sinh(\lambda x) + \frac{C\lambda^n - b}{aC\lambda^4} \quad \text{if } n \text{ is an even number,}$$
$$y = \frac{b}{\lambda^{2n-4} - a^2C^2\lambda^4}\left[aC\sinh(\lambda x) + \lambda^{n-4}\cosh(\lambda x)\right] + C \quad \text{if } n \text{ is an odd number.}$$

10. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\sinh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\sinh^m(\lambda x) + b$.

11. $y_x^{(n)} = a\sinh^k(\lambda y)y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\sinh^k(\lambda y)$.

12. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\sinh(\lambda x)y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a\sinh(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \left[\frac{a}{\lambda}\cosh(\lambda x) + C\right]y$.

► **Equations with hyperbolic cosine.**

13. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cosh(\lambda x).$

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\cosh(\lambda x)$. Integrating the equation twice yields the third-order equation

$$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}\cosh(\lambda x).$$

2°. Particular solution: $y = \dfrac{b}{\lambda^4(\lambda^2 - a^2C^2)}\big[aC\cosh(\lambda x) + \lambda\sinh(\lambda x)\big] + C.$

14. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cosh(\lambda x).$

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-5}\sinh(\lambda x).$

15. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cosh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\cosh^m(\lambda x) + b$.

16. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cosh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\cosh^m(\lambda y) + b$.

17. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\cosh^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\cosh^m(\lambda x)$.

18. $y_x^{(n)} = a\cosh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\cosh^m(\lambda y) + b$.

19. $y_x^{(n)} = ax^{-n}\cosh^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\cosh^m(\lambda y)$.

20. $yy_x^{(2n+1)} = a\cosh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\cosh^m(\lambda x) + b$.

21. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cosh(\lambda x).$

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\cosh(\lambda x)$. Integrating the equation twice yields the $(n-2)$nd-order equation

$$y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + b\lambda^{-2}\cosh(\lambda x).$$

2°. Particular solutions:

$$y = C\cosh(\lambda x) + \frac{C\lambda^n - b}{aC\lambda^4} \quad \text{if } n \text{ is an even number,}$$

$$y = \frac{b}{\lambda^{2n-4} - a^2C^2\lambda^4}\big[aC\cosh(\lambda x) + \lambda^{n-4}\sinh(\lambda x)\big] + C \quad \text{if } n \text{ is an odd number.}$$

22. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\cosh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\cosh^m(\lambda x) + b$.

23. $y_x^{(n)} = a\cosh^k(\lambda y)y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\cosh^k(\lambda y)$.

24. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\cosh(\lambda x)y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a\cosh(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Big[\dfrac{a}{\lambda}\sinh(\lambda x) + C\Big]y.$

► **Equations with hyperbolic tangent.**

25. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\tanh(\lambda x) + c.$

This is a special case of equation 17.2.6.1 with $f(x) = b\tanh(\lambda x) + c$.

26. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\tanh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\tanh^m(\lambda x) + b$.

27. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\tanh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\tanh^m(\lambda y) + b$.

28. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx} y''''_{xxxx} + 10(y'''_{xxx})^2 = a\tanh^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\tanh^m(\lambda x)$.

29. $y_x^{(n)} = a\tanh^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\tanh^m(\lambda y) + b$.

30. $y_x^{(n)} = ax^{-n}\tanh^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\tanh^m(\lambda y)$.

31. $yy_x^{(2n+1)} = a\tanh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\tanh^m(\lambda x) + b$.

32. $y_x^{(2n)} = y + a(y'_x - y\tanh x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cosh x$.

33. $y_x^{(2n+1)} = y\tanh x + a(y'_x - y\tanh x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cosh x$.

34. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\tanh^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\tanh^m(\lambda x) + b$.

35. $y_x^{(n)} = a\tanh^k(\lambda y) y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\tanh^k(\lambda y)$.

36. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\tanh(\lambda x) y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a\tanh(\lambda x)$.

► **Equations with hyperbolic cotangent.**

37. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\coth(\lambda x) + c.$

This is a special case of equation 17.2.6.1 with $f(x) = b\coth(\lambda x) + c$.

38. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\coth^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\coth^m(\lambda x) + b$.

39. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx} y'''_{xxx} = a\coth^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\coth^m(\lambda y) + b$.

40. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\coth^m(\lambda x)$.

This is a special case of equation 17.2.6.6 with $f(x) = a\coth^m(\lambda x)$.

41. $y_x^{(n)} = a\coth^m(\lambda y) + b$.

This is a special case of equation 17.2.6.8 with $f(y) = a\coth^m(\lambda y) + b$.

42. $y_x^{(n)} = ax^{-n}\coth^m(\lambda y)$.

This is a special case of equation 17.2.6.11 with $f(y) = a\coth^m(\lambda y)$.

43. $yy_x^{(2n+1)} = a\coth^m(\lambda x) + b$.

This is a special case of equation 17.2.6.17 with $f(x) = a\coth^m(\lambda x) + b$.

44. $y_x^{(2n)} = y + a(y'_x - y\coth x)^k$.

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \sinh x$.

45. $y_x^{(2n+1)} = y\coth x + a(y'_x - y\coth x)^k$.

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \sinh x$.

46. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\coth^m(\lambda x) + b$.

This is a special case of equation 17.2.6.62 with $f(x) = a\coth^m(\lambda x) + b$.

47. $y_x^{(n)} = a\coth^k(\lambda y)y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\coth^k(\lambda y)$.

48. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\coth(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\coth(\lambda x)$.

17.2.4 Equations Containing Logarithmic Functions

► **Equations of the form** $y_x^{(n)} = f(x,y)$.

1. $y_x^{(n)} = a\ln^m(by) + c$.

This is a special case of equation 17.2.6.8 with $f(y) = a\ln^m(by) + c$.

2. $yy_x^{(2n+1)} = a\ln^m(bx)$.

This is a special case of equation 17.2.6.17 with $f(x) = a\ln^m(bx)$.

3. $y_x^{(n)} = y(\alpha x + m\ln y + b)$.

This is a special case of equation 17.2.6.25 with $f(w) = \ln w + b$.

4. $y_x^{(n)} = x^{-n}(\alpha y + m\ln x + b)$.

This is a special case of equation 17.2.6.26 with $f(w) = \ln w + b$.

5. $y_x^{(n)} = ax^{-n}\ln^m(by)$.

This is a special case of equation 17.2.6.11 with $f(y) = a\ln^m(by)$.

6. $y_x^{(n)} = ax^{-n-1}[\ln y + (1-n)\ln x]$.

This is a special case of equation 17.2.6.13 with $f(w) = a\ln w$.

7. $y_x^{(n)} = ax^{-n-k}(\ln y + k \ln x).$

This is a special case of equation 17.2.6.15 with $f(w) = a \ln w$.

8. $y_x^{(n)} = ayx^{-n}(m \ln y + k \ln x).$

This is a special case of equation 17.2.6.16 with $f(w) = a \ln w$.

9. $y_x^{(2n)} = ax^{-\frac{2n+1}{2}}[2 \ln y + (1 - 2n) \ln x].$

This is a special case of equation 17.2.6.14 with $f(w) = 2a \ln w$.

10. $y_x^{(n)} = (ax^2 + c)^{-\frac{n+1}{2}}[2 \ln y + (1 - n) \ln(ax^2 + c)].$

This is a special case of equation 17.2.6.22 with $b = 0$ and $f(w) = 2 \ln w$.

11. $y_x^{(n)} = be^{\alpha x}(\ln y - \alpha x).$

This is a special case of equation 17.2.6.24 with $f(w) = b \ln w$.

► **Other equations.**

12. $y_x^{(n)} = ay'_x \ln y + b \ln x.$

This is a special case of equation 17.2.6.34 with $f(y) = a \ln y$ and $g(x) = b \ln x$.

13. $y_x^{(n)} = a^n y + b \ln x\, (y'_x - ay)^k.$

This is a special case of equation 17.2.6.38 with $f(x, w) = bw^k \ln x$.

14. $y_x^{(n)} = a \ln x\, (xy'_x - y)^k.$

This is a special case of equation 17.2.6.39 with $f(x, w) = aw^k \ln x$.

15. $y_x^{(n)} = a \ln x\, (xy'_x - 2y)^k.$

This is a special case of equation 17.2.6.40 with $m = 2$ and $f(x, w) = aw^k \ln x$.

16. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b \ln x + c.$

This is a special case of equation 17.2.6.54 with $f(x) = b \ln x + c$.

17. $xy_x^{(n)} + ny_x^{(n-1)} = a \ln x + a \ln y.$

This is a special case of equation 17.2.6.59 with $f(w) = a \ln w$.

18. $x^2 y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = 2a \ln x + a \ln y.$

This is a special case of equation 17.2.6.61 with $f(w) = a \ln w$.

19. $yy_x^{(n)} - y'_x y_x^{(n-1)} = ay^2 \ln x.$

This is a special case of equation 17.2.6.64 with $f(w) = a \ln x$.

20. $(2n - 1)yy_x^{(2n+1)} + (2n + 1)y'_x y_x^{(2n)} = a \ln^m(bx) + c.$

This is a special case of equation 17.2.6.62 with $f(x) = a \ln^m(bx) + c$.

21. $y_x^{(n)} = a \ln^k(by)\, y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a \ln^k(by)$.

22. $y_x^{(n)} = ay^m y'_x \ln y_x^{(n-1)}.$

This is a special case of equation 17.2.6.73 with $f(y) = ay^m$ and $g(w) = \ln w$.

17.2.5 Equations Containing Trigonometric Functions

▶ Equations with sine.

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sin(\lambda x).$

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\sin(\lambda x)$. Integrating the equation twice, we obtain a third-order equation:

$$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 - b\lambda^{-2}\sin(\lambda x).$$

2°. Particular solution: $y = -\dfrac{b}{\lambda^4(a^2C^2+\lambda^2)}\big[aC\sin(\lambda x) + \lambda\cos(\lambda x)\big] + C.$

2. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\sin(\lambda x).$

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 - 2a\lambda^{-5}\cos(\lambda x).$

3. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\sin^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\sin^m(\lambda x) + b.$

4. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\sin^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\sin^m(\lambda y) + b.$

5. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\sin^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\sin^m(\lambda x).$

6. $y_x^{(n)} = a\sin^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\sin^m(\lambda y) + b.$

7. $y_x^{(n)} = ax^{-n}\sin^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\sin^m(\lambda y).$

8. $yy_x^{(2n+1)} = a\sin^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\sin^m(\lambda x) + b.$

9. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\sin(\lambda x).$

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\sin(\lambda x)$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 - b\lambda^{-2}\sin(\lambda x)$.

2°. Particular solutions:

$$y = C\sin(\lambda x) + \frac{(-1)^{n/2}C\lambda^n - b}{aC\lambda^4} \qquad \text{if } n \text{ is even,}$$
$$y = -\frac{b}{\lambda^{2n-4} + a^2C^2\lambda^4}\Big[aC\sin(\lambda x) + (-1)^{\frac{n-1}{2}}\lambda^{n-4}\cos(\lambda x)\Big] + C \qquad \text{if } n \text{ is odd.}$$

10. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\sin^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\sin^m(\lambda x) + b.$

11. $y_x^{(n)} = a\sin^k(\lambda y)y'_x y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\sin^k(\lambda y)$.

12. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\sin(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\sin(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Bigl[C - \dfrac{a}{\lambda}\cos(\lambda x)\Bigr]y$.

► **Equations with cosine.**

13. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cos(\lambda x)$.

1°. This is a special case of equation 17.2.6.1 with $f(x) = b\cos(\lambda x)$. Integrating the equation twice, we obtain a third-order equation:

$$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 - b\lambda^{-2}\cos(\lambda x).$$

2°. Particular solution: $y = -\dfrac{b}{\lambda^4(a^2C^2+\lambda^2)}\bigl[aC\cos(\lambda x) - \lambda\sin(\lambda x)\bigr] + C$.

14. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cos(\lambda x)$.

Solution: $y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + 2a\lambda^{-5}\sin(\lambda x)$.

15. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cos^m(\lambda x) + b$.

This is a special case of equation 17.2.6.2 with $f(x) = a\cos^m(\lambda x) + b$.

16. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\cos^m(\lambda y) + b$.

This is a special case of equation 17.2.6.3 with $f(y) = a\cos^m(\lambda y) + b$.

17. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\cos^m(\lambda x)$.

This is a special case of equation 17.2.6.6 with $f(x) = a\cos^m(\lambda x)$.

18. $y_x^{(n)} = a\cos^m(\lambda y) + b$.

This is a special case of equation 17.2.6.8 with $f(y) = a\cos^m(\lambda y) + b$.

19. $y_x^{(n)} = ax^{-n}\cos^m(\lambda y)$.

This is a special case of equation 17.2.6.11 with $f(y) = a\cos^m(\lambda y)$.

20. $yy_x^{(2n+1)} = a\cos^m(\lambda x) + b$.

This is a special case of equation 17.2.6.17 with $f(x) = a\cos^m(\lambda x) + b$.

21. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\cos(\lambda x)$.

1°. This is a special case of equation 17.2.6.54 with $f(x) = b\cos(\lambda x)$. Integrating the equation two times, we obtain an $(n-2)$nd-order equation: $y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 - b\lambda^{-2}\cos(\lambda x)$.

2°. Particular solutions:

$$y = C\cos(\lambda x) + \frac{(-1)^{n/2}C\lambda^n - b}{aC\lambda^4} \qquad \text{if } n \text{ is even,}$$

$$y = -\frac{b}{\lambda^{2n-4} + a^2C^2\lambda^4}\left[aC\cos(\lambda x) + (-1)^{\frac{n+1}{2}}\lambda^{n-4}\sin(\lambda x)\right] + C \qquad \text{if } n \text{ is odd.}$$

22. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\cos^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\cos^m(\lambda x) + b$.

23. $y_x^{(n)} = a\cos^k(\lambda y)y'_x y_x^{(n-1)}.$

This is a special case of equation 17.2.6.57 with $f(y) = a\cos^k(\lambda y)$.

24. $yy_x^{(n)} - y'_x y_x^{(n-1)} = a\cos(\lambda x)y^2.$

This is a special case of equation 17.2.6.64 with $f(x) = a\cos(\lambda x)$. Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \left[\frac{a}{\lambda}\sin(\lambda x) + C\right]y$.

► **Equations with tangent.**

25. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + b\tan(\lambda x) + c.$

This is a special case of equation 17.2.6.1 with $f(x) = b\tan(\lambda x) + c$.

26. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\tan^m(\lambda x) + b.$

This is a special case of equation 17.2.6.2 with $f(x) = a\tan^m(\lambda x) + b$.

27. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = a\tan^m(\lambda y) + b.$

This is a special case of equation 17.2.6.3 with $f(y) = a\tan^m(\lambda y) + b$.

28. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = a\tan^m(\lambda x).$

This is a special case of equation 17.2.6.6 with $f(x) = a\tan^m(\lambda x)$.

29. $y_x^{(n)} = a\tan^m(\lambda y) + b.$

This is a special case of equation 17.2.6.8 with $f(y) = a\tan^m(\lambda y) + b$.

30. $y_x^{(n)} = ax^{-n}\tan^m(\lambda y).$

This is a special case of equation 17.2.6.11 with $f(y) = a\tan^m(\lambda y)$.

31. $yy_x^{(2n+1)} = a\tan^m(\lambda x) + b.$

This is a special case of equation 17.2.6.17 with $f(x) = a\tan^m(\lambda x) + b$.

32. $y_x^{(2n)} = (-1)^n y + a(y'_x + y\tan x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cos x$.

33. $y_x^{(2n+1)} = (-1)^{n+1}y\tan x + a(y'_x + y\tan x)^k.$

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \cos x$.

34. $(2n-1)yy_x^{(2n+1)} + (2n+1)y'_x y_x^{(2n)} = a\tan^m(\lambda x) + b.$

This is a special case of equation 17.2.6.62 with $f(x) = a\tan^m(\lambda x) + b$.

35. $y_x^{(n)} = a\tan^k(\lambda y)y_x' y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\tan^k(\lambda y)$.

36. $yy_x^{(n)} - y_x' y_x^{(n-1)} = a\tan(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\tan(\lambda x)$.

► **Equations with cotangent.**

37. $y_x^{(5)} = ayy_{xxxx}'''' - a(y_{xx}'')^2 + b\cot(\lambda x) + c$.

This is a special case of equation 17.2.6.1 with $f(x) = b\cot(\lambda x) + c$.

38. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = a\cot^m(\lambda x) + b$.

This is a special case of equation 17.2.6.2 with $f(x) = a\cot^m(\lambda x) + b$.

39. $yy_x^{(5)} + 5y_x' y_{xxxx}'''' + 10y_{xx}'' y_{xxx}''' = a\cot^m(\lambda y) + b$.

This is a special case of equation 17.2.6.3 with $f(y) = a\cot^m(\lambda y) + b$.

40. $yy_x^{(6)} + 6y_x' y_x^{(5)} + 15y_{xx}'' y_{xxxx}'''' + 10(y_{xxx}''')^2 = a\cot^m(\lambda x)$.

This is a special case of equation 17.2.6.6 with $f(x) = a\cot^m(\lambda x)$.

41. $y_x^{(n)} = a\cot^m(\lambda y) + b$.

This is a special case of equation 17.2.6.8 with $f(y) = a\cot^m(\lambda y) + b$.

42. $y_x^{(n)} = ax^{-n}\cot^m(\lambda y)$.

This is a special case of equation 17.2.6.11 with $f(y) = a\cot^m(\lambda y)$.

43. $yy_x^{(2n+1)} = a\cot^m(\lambda x) + b$.

This is a special case of equation 17.2.6.17 with $f(x) = a\cot^m(\lambda x) + b$.

44. $y_x^{(2n)} = (-1)^n y + a(y_x' - y\cot x)^k$.

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \sin x$.

45. $y_x^{(2n+1)} = (-1)^n y\cot x + a(y_x' - y\cot x)^k$.

This is a special case of equation 17.2.6.47 with $f(x,u) = au^k$ and $\varphi(x) = \sin x$.

46. $(2n-1)yy_x^{(2n+1)} + (2n+1)y_x' y_x^{(2n)} = a\cot^m(\lambda x) + b$.

This is a special case of equation 17.2.6.62 with $f(x) = a\cot^m(\lambda x) + b$.

47. $y_x^{(n)} = a\cot^k(\lambda y)y_x' y_x^{(n-1)}$.

This is a special case of equation 17.2.6.57 with $f(y) = a\cot^k(\lambda y)$.

48. $yy_x^{(n)} - y_x' y_x^{(n-1)} = a\cot(\lambda x)y^2$.

This is a special case of equation 17.2.6.64 with $f(x) = a\cot(\lambda x)$.

17.2.6 Equations Containing Arbitrary Functions

► **Fifth- and sixth-order equations.**

1. $y_x^{(5)} = ayy''''_{xxxx} - a(y''_{xx})^2 + f(x)$.

Integrating the equation two times, we obtain a third-order equation:

$$y'''_{xxx} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + \int_{x_0}^{x} (x-t)f(t)\,dt, \quad \text{where } x_0 \text{ is an arbitrary number.}$$

2. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = f(x)$.

Solution:

$$y^2 = C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + \frac{1}{12}\int_{x_0}^{x} (x-t)^4 f(t)\,dt,$$

where x_0 is an arbitrary number.

3. $yy_x^{(5)} + 5y'_x y''''_{xxxx} + 10y''_{xx}y'''_{xxx} = f(y)$.

The substitution $w = y^2$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(5)} = 2f\left(\pm\sqrt{w}\right)$.

4. $yy_x^{(5)} + ay'_x y''''_{xxxx} + (3a-5)y''_{xx}y'''_{xxx} = f(x)$.

Integrating the equation three times, we obtain a second-order equation:

$$yy''_{xx} + \frac{a-3}{2}(y'_x)^2 = C_2x^2 + C_1x + C_0 + \frac{1}{2}\int_{x_0}^{x} (x-t)^2 f(t)\,dt,$$

where x_0 is an arbitrary number.

5. $(a+y)y_x^{(5)} + by'_x y''''_{xxxx} + cy''_{xx}y'''_{xxx} = f(x)$.

Integrating yields a fourth-order equation:

$$(a+y)y''''_{xxxx} + (b-1)y'_x y'''_{xxx} + \frac{1}{2}(1-b+c)(y''_{xx})^2 = \int f(x)\,dx + C.$$

6. $yy_x^{(6)} + 6y'_x y_x^{(5)} + 15y''_{xx}y''''_{xxxx} + 10(y'''_{xxx})^2 = f(x)$.

Solution: $y^2 = C_5x^5 + C_4x^4 + C_3x^3 + C_2x^2 + C_1x + C_0 + \dfrac{1}{60}\displaystyle\int_{x_0}^{x} (x-t)^5 f(t)\,dt.$

7. $y_x^{(6)} = (ax^2+bx+c)^{-7/2} f\big(y(ax^2+bx+c)^{-5/2}\big)$.

This is a special case of equation 17.2.6.22 with $n = 6$.

► **Equations of the form $y_x^{(n)} = f(x, y)$.**

8. $y_x^{(n)} = f(y)$.

Autonomous equation. This is a special case of equation 17.2.6.77.

1°. The substitution $w(y) = y'_x$ leads to an $(n-1)$st-order equation.

2°. For even $n = 2m$, the first integral of the equation is:

$$\sum_{k=1}^{m-1} (-1)^k y_x^{(k)} y_x^{(2m-k)} + \tfrac{1}{2}(-1)^m \left[y_x^{(m)}\right]^2 + \int f(y)\,dy = C.$$

Furthermore, the order of the obtained equation can be reduced by one by the substitution $w(y) = y'_x$.

9. $\boldsymbol{y_x^{(n)} = f(y + ax^m), \qquad m = 0,\ 1,\ \dots,\ n-1.}$

The substitution $w = y + ax^m$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

10. $\boldsymbol{y_x^{(n)} = f\big(y + a_n x^n + a_{n-1}x^{n-1} + \dots + a_0\big).}$

The substitution $w = y + a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = a_n n! + f(w)$.

11. $\boldsymbol{y_x^{(n)} = x^{-n} f(y).}$

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 17.2.6.77.

12. $\boldsymbol{y_x^{(n)} = x^{1-n} f(y/x).}$

Homogeneous equation. This is a special case of equation 17.2.6.83. The transformation $t = \ln x$, $w = y/x$ leads to an autonomous equation of the form 17.2.6.77.

13. $\boldsymbol{y_x^{(n)} = x^{-n-1} f(x^{1-n} y).}$

The transformation $x = t^{-1}$, $y = t^{1-n} w$ leads to an autonomous equation of the form 17.2.6.8: $w_t^{(n)} = (-1)^n f(w)$.

14. $\boldsymbol{y_x^{(2n)} = x^{-\frac{2n+1}{2}} f\Big(x^{\frac{1-2n}{2}} y\Big).}$

The transformation $x = e^t$, $y = x^{\frac{2n-1}{2}} w(t)$ leads to an autonomous equation of the form 17.2.6.68, whose order can be reduced by two.

15. $\boldsymbol{y_x^{(n)} = x^{-n-k} f(yx^k).}$

This is a special case of equation 17.2.6.86.

1°. The transformation $t = \ln x$, $z = yx^k$ leads to an autonomous equation of the form 17.2.6.77.

2°. The transformation $z = yx^k$, $w = xy'_x/y$ leads to an $(n-1)$st-order equation.

16. $\boldsymbol{y_x^{(n)} = yx^{-n} f(x^k y^m).}$

This is a special case of equation 17.2.6.89. The transformation $t = x^k y^m$, $w = xy'_x/y$ leads to an $(n-1)$st-order equation.

17. $\boldsymbol{yy_x^{(2n+1)} = f(x).}$

Integrating yields a $2n$th-order equation:

$$2\sum_{m=0}^{n-1} (-1)^m y_x^{(m)} y_x^{(2n-m)} + (-1)^n \big[y_x^{(n)}\big]^2 = 2\int f(x)\,dx + C,$$

where the notation $y_x^{(0)} \equiv y$ is used.

18. $\boldsymbol{y_x^{(n)} = f(x, y).}$

The transformation $x = z^{-1}$, $y = z^{1-n} w(z)$ leads to an equation of the same form: $w_z^{(n)} = (-1)^n z^{-n-1} f(z^{-1},\ z^{1-n} w)$.

19. $\boldsymbol{y_x^{(n)} = (ax+by+c)^{1-n} f\Big(\frac{ax+by+c}{\alpha x+\beta y+\gamma}\Big).}$

1°. For $a\beta - b\alpha = 0$, the substitution $bw = ax + by + c$ leads to an autonomous equation of the form 17.2.6.8.

2°. For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants which are determined by solving the linear algebraic system

$$ax_0 + by_0 + c = 0,$$
$$\alpha x_0 + \beta y_0 + \gamma = 0,$$

leads to a homogeneous equation of the form 17.2.6.12:

$$w_z^{(n)} = z^{1-n} F\Big(\frac{w}{z}\Big), \qquad \text{where} \quad F(\xi) = (a+b\xi)^{1-n} f\Big(\frac{a+b\xi}{\alpha+\beta\xi}\Big).$$

20. $\boldsymbol{y_x^{(n)} = (a_1x+b_1y+c_1)^{1-n} f\bigg(\frac{a_2x+b_2y+c_2}{a_3x+b_3y+c_3}\bigg).}$

Suppose the following condition holds: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

For $a_2b_3 - a_3b_2 \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system

$$a_2x_0 + b_2y_0 + c_2 = 0,$$
$$a_3x_0 + b_3y_0 + c_3 = 0,$$

leads to a homogeneous equation of the form 17.2.6.12:

$$w_z^{(n)} = z^{1-n} F\Big(\frac{w}{z}\Big), \qquad \text{where} \quad F(\xi) = (a_1+b_1\xi)^{1-n} f\Big(\frac{a_2+b_2\xi}{a_3+b_3\xi}\Big).$$

21. $\boldsymbol{(ax+b)^n(cx+d)y_x^{(n)} = f\bigg(\frac{y}{(cx+d)^{n-1}}\bigg).}$

The transformation $\xi = \ln\Big|\frac{ax+b}{cx+d}\Big|$, $w = \frac{y}{(cx+d)^{n-1}}$ leads to an autonomous equation of the form 17.2.6.77.

22. $\boldsymbol{y_x^{(n)} = (ax^2+bx+c)^{-\frac{1+n}{2}} f\Big(y(ax^2+bx+c)^{\frac{1-n}{2}}\Big).}$

1°. The transformation

$$t = \int \frac{dx}{ax^2+bx+c}, \qquad w = y(ax^2+bx+c)^{\frac{1-n}{2}} \tag{1}$$

leads to an autonomous equation with respect to $w = w(t)$, which admits reduction of order by the substitution $z(w) = w'_t$.

2°. Let $n = 2m$ be an even integer ($m = 1, 2, 3, \ldots$). In this case, transformation (1) yields an equation of the form 17.2.6.68, whose order can be reduced by two.

Setting $P = ax^2 + bx + c$, $y = wP^{\frac{2m-1}{2}}$ and multiplying both sides of the original equation by $w'_x = P^{-\frac{1+2m}{2}}\Big(Py'_x + \dfrac{1-2m}{2}P'_x y\Big)$, we obtain

$$\Big(Py'_x + \frac{1-2m}{2}P'_x y\Big)y^{(2m)}_x = f(w)w'_x.$$

Integrating both sides of this equality with respect to x (the left-hand side is integrated by parts), we have

$$\sum_{k=0}^{m-2}(-1)^k\psi^{(k)}_x y^{(2m-1-k)}_x + (-1)^{m-1}\int \psi^{(m-1)}_x y^{(m+1)}_x\,dx = \int f(w)\,dw + C, \qquad (2)$$

where

$$\psi^{(k)}_x = \frac{d^k}{dx^k}\Big(Py'_x + \frac{1-2m}{2}P'_x y\Big) = Py^{(k+1)}_x + \Big(k-m+\frac{1}{2}\Big)P'_x y^{(k)}_x + ak(k-2m)y^{(k-1)}_x$$

(remember that $n = 2m$). It can be shown that the integrand on the left-hand side of (2) is a total differential. Finally, we arrive at the first integral

$$\sum_{k=0}^{m-2}(-1)^k\big[Py^{(k+1)}_x + \big(k-m+\tfrac{1}{2}\big)P'_x y^{(k)}_x + ak(k-2m)y^{(k-1)}_x\big]y^{(2m-1-k)}_x$$
$$+(-1)^{m-1}\Big\{\tfrac{1}{2}P\big[y^{(m)}_x\big]^2 - \tfrac{1}{2}P'_x y^{(m-1)}_x y^{(m)}_x + a(1-m^2)y^{(m-2)}_x y^{(m)}_x + \tfrac{1}{2}am^2\big[y^{(m-1)}_x\big]^2\Big\}$$
$$= \int f(w)\,dw + C.$$

23. $\boldsymbol{y^{(n)}_x = y^{\frac{1+n}{1-n}} f\Big(y(ax^2+bx+c)^{\frac{1-n}{2}}\Big).}$

1°. Setting $f(u) = u^{\frac{n+1}{n-1}} f_1(u)$, we have equation 17.2.6.22 with the function f_1 (instead of f).

2°. The transformation $x = z^{-1}$, $y = z^{1-n}w(z)$ leads to an equation of similar form: $w^{(n)}_z = (-1)^n w^{\frac{1+n}{1-n}} f\Big(w(cz^2+bz+a)^{\frac{1-n}{2}}\Big)$.

24. $\boldsymbol{y^{(n)}_x = e^{\alpha x} f(ye^{-\alpha x}).}$

The substitution $w(x) = ye^{-\alpha x}$ leads to an autonomous equation of the form 17.2.6.77.

25. $\boldsymbol{y^{(n)}_x = yf(e^{\alpha x}y^m).}$

The transformation $z = e^{\alpha x}y^m$, $w(z) = y'_x/y$ leads to an $(n-1)$st-order equation.

26. $\boldsymbol{y^{(n)}_x = x^{-n} f(x^m e^{\alpha y}).}$

The transformation $z = x^m e^{\alpha y}$, $w(z) = xy'_x$ leads to an $(n-1)$st-order equation.

27. $\boldsymbol{y^{(n)}_x = f(y + ae^{\lambda x}) - a\lambda^n e^{\lambda x}.}$

The substitution $w(x) = y + ae^{\lambda x}$ leads to an autonomous equation of the form 17.2.6.8: $w^{(n)}_x = f(w)$.

28. $\boldsymbol{y_x^{(2n)} = f(y + a\cosh x) - a\cosh x.}$

The substitution $w(x) = y + a\cosh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n)} = f(w)$.

29. $\boldsymbol{y_x^{(2n)} = f(y + a\sinh x) - a\sinh x.}$

The substitution $w(x) = y + a\sinh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n)} = f(w)$.

30. $\boldsymbol{y_x^{(2n+1)} = f(y + a\cosh x) - a\sinh x.}$

The substitution $w(x) = y + a\cosh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n+1)} = f(w)$.

31. $\boldsymbol{y_x^{(2n+1)} = f(y + a\sinh x) - a\cosh x.}$

The substitution $w(x) = y + a\sinh x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(2n+1)} = f(w)$.

32. $\boldsymbol{y_x^{(n)} = f(y + a\cos x) - a\cos\left(x + \tfrac{1}{2}\pi n\right).}$

The substitution $w(x) = y + a\cos x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

33. $\boldsymbol{y_x^{(n)} = f(y + a\sin x) - a\sin\left(x + \tfrac{1}{2}\pi n\right).}$

The substitution $w(x) = y + a\sin x$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

► Equations of the form $y_x^{(n)} = f(x, y, y_x')$.

34. $\boldsymbol{y_x^{(n)} = f(y)y_x' + g(x).}$

Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = \int f(y)\,dy + \int g(x)\,dx + C$.

35. $\boldsymbol{y_x^{(n)} = f(x, y_x').}$

The substitution $w(x) = y_x'$ leads to an $(n-1)$st-order equation: $w_x^{(n-1)} = f(x, w)$.

36. $\boldsymbol{y_x^{(n)} = f(y, y_x').}$

Autonomous equation. This is a special case of equation 17.2.6.77.

The substitution $w(y) = y_x'$ leads to an $(n-1)$st-order equation.

37. $\boldsymbol{y_x^{(n)} = yx^{-n} f(xy_x'/y).}$

The transformation $z = xy_x'/y$, $w = x^2 y_{xx}''/y$ leads to an $(n-2)$nd-order equation.

38. $\boldsymbol{y_x^{(n)} = a^n y + f(x, y_x' - ay).}$

The substitution $w = y_x' - ay$ leads to an $(n-1)$st-order equation:

$$w_x^{(n-1)} + aw_x^{(n-2)} + \cdots + a^{n-1}w = f(x, w).$$

39. $\boldsymbol{y_x^{(n)} = f(x, xy_x' - y).}$

The substitution $w = xy_x' - y$ leads to an $(n-1)$st-order equation: $\dfrac{d^{n-2}}{dx^{n-2}}\left(\dfrac{w_x'}{x}\right) = f(x, w)$.

40. $y_x^{(n)} = f\big(x,\, xy'_x - my\big).$

Here, m is a positive integer and $n \geq m+1$. The substitution $w = xy'_x - my$ leads to an $(n-1)$st-order equation: $\zeta_x^{(n-m-1)} = f(x,w)$, where $\zeta = w_x^{(m)}/x$.

41. $x^n y_x^{(n)} = f(x, xy'_x + ay) - (a)_n y.$

Here, $(a)_n = a(a+1)\dots(a+n-1)$ is the Pochhammer symbol. The substitution $w = xy'_x + ay$ leads to an $(n-1)$st-order equation.

42. $y_x^{(n)} = f\big(x, P_m y'_x - P'_m y\big), \quad P_m = \sum_{k=0}^{m} a_k x^k,\ P'_m = \sum_{k=0}^{m} a_k k x^{k-1},\ n > m.$

The substitution $w = P_m y'_x - P'_m y$ leads to an $(n-1)$st-order equation.

43. $y_x^{(2n)} = y + f(x, y'_x \cosh x - y \sinh x).$

The substitution $w = y'_x \cosh x - y \sinh x$ leads to a $(2n-1)$st-order equation.

44. $y_x^{(2n)} = y + f(x, y'_x \sinh x - y \cosh x).$

The substitution $w = y'_x \sinh x - y \cosh x$ leads to a $(2n-1)$st-order equation.

45. $y_x^{(2n)} = (-1)^n y + f(x, y'_x \sin x - y \cos x).$

The substitution $w = y'_x \sin x - y \cos x$ leads to a $(2n-1)$st-order equation.

46. $y_x^{(2n)} = (-1)^n y + f(x, y'_x \cos x + y \sin x).$

The substitution $w = y'_x \cos x + y \sin x$ leads to a $(2n-1)$st-order equation.

47. $y_x^{(n)} = \dfrac{\varphi_x^{(n)}}{\varphi} y + f\left(x, y'_x - \dfrac{\varphi'_x}{\varphi} y\right), \qquad \varphi = \varphi(x).$

The substitution $w = y'_x - \dfrac{\varphi'_x}{\varphi} y$ leads to an $(n-1)$st-order equation.

► **Equations of the form** $y_x^{(n)} = f(x, y, y'_x, y''_{xx})$.

48. $y_x^{(n)} = f(x, xy'_x - y, y''_{xx}).$

This is a special case of equation 17.2.6.78. The substitution $w(x) = xy'_x - y$ leads to an $(n-1)$st-order equation.

49. $y_x^{(n)} = f(x, x^2 y''_{xx} - 2xy'_x + 2y).$

This is a special case of equation 17.2.6.81. The substitution $w(x) = x^2 y''_{xx} - 2xy'_x + 2y$ leads to an $(n-2)$nd-order equation.

50. $y_x^{(2n)} = a^n y + f(x, y''_{xx} - ay).$

The substitution $w = y''_{xx} - ay$ leads to a $(2n-2)$nd-order equation:

$$w_x^{(2n-2)} + a w_x^{(2n-4)} + \dots + a^{n-1} w = f(x, w).$$

51. $y_x^{(2n)} = y f\big(y y''_{xx} - y'^2_x\big).$

This is a special case of equation 17.2.6.52.

52. $y_x^{(2n)} = y''_{xx} f\big(yy''_{xx} - y'^2_x\big) + y g\big(yy''_{xx} - y'^2_x\big).$

1°. Particular solution:

$$y = C_1 \exp(C_3 x) + C_2 \exp(-C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$C_3^{2n} - C_3^2 f(4C_1C_2C_3^2) - g(4C_1C_2C_3^2) = 0.$$

2°. Particular solution:

$$y = C_1 \cos(C_3 x) + C_2 \sin(C_3 x),$$

where the constants C_1, C_2, and C_3 are related by the constraint

$$(-1)^n C_3^{2n} + C_3^2 f(-C_1^2C_3^2 - C_2^2C_3^2) - g(-C_1^2C_3^2 - C_2^2C_3^2) = 0.$$

53. $y_x^{(n)} = y'_x f\Big(\dfrac{y''_{xx}}{y'_x},\, y'_x - y\dfrac{y''_{xx}}{y'_x}\Big).$

Particular solution: $y = C_1 \exp(C_2 x) + C_3$, where C_1 is an arbitrary constant and the constants C_2 and C_2 are related by the constraint $C_2^{n-1} = f(C_2, -C_2C_3)$.

► **Equations of the form** $f(x,y)y_x^{(n)} + g(x,y,y'_x)y_x^{(n-1)} = h\big(x,y,y'_x,\dots,y_x^{(n-2)}\big).$

54. $y_x^{(n)} = ayy''''_{xxxx} - a(y''_{xx})^2 + f(x).$

Integrating the equation two times, we obtain an $(n-2)$nd-order equation:

$$y_x^{(n-2)} = ayy''_{xx} - a(y'_x)^2 + C_1x + C_2 + \int_{x_0}^x (x-t) f(t)\,dt, \quad \text{where } x_0 \text{ is an arbitrary number.}$$

55. $y_x^{(2n)} = a^2 y + f\big(x, y_x^{(n)} + ay\big).$

The substitution $w = y_x^{(n)} + ay$ leads to an nth-order equation: $w_x^{(n)} = aw + f(x,w)$.

56. $y_x^{(n)} = f\big(y_x^{(n-2)}\big).$

Having set $u(x) = y_x^{(n-2)}$, we obtain a second-order equation $u''_{xx} = f(u)$, whose solution has the form:

$$x = \int \frac{du}{\varphi(u)} + C_2, \qquad \text{where} \quad \varphi(u) = \pm\Big[C_1 + 2\int f(u)\,du\Big]^{1/2}.$$

Expressing u in terms of x and integrating the resulting relation $n-2$ times, we find y.

Solution in parametric form:

$$x = \int_{C_2}^u \frac{du}{\varphi(u)}, \qquad y = \int_{C_3}^u \frac{du_1}{\varphi(u_1)} \int_{C_4}^{u_1} \frac{du_2}{\varphi(u_2)} \cdots \int_{C_{n-1}}^{u_{n-4}} \frac{du_{n-3}}{\varphi(u_{n-3})} \int_{C_n}^{u_{n-3}} \frac{u_{n-2}\,du_{n-2}}{\varphi(u_{n-2})}.$$

57. $y_x^{(n)} = f(y)y'_x y_x^{(n-1)}.$

Integrating yields an $(n-1)$st-order autonomous equation of the form 17.2.6.8:

$$y_x^{(n-1)} = F(y), \qquad \text{where} \quad F(y) = C\exp\Big[\int f(y)\,dy\Big].$$

58. $y_x^{(n)} = \big[f(y)y'_x + g(x)\big]y_x^{(n-1)}.$

Integrating yields an $(n-1)$st-order equation: $y_x^{(n-1)} = C\exp\Big[\int f(y)\,dy + \int g(x)\,dx\Big].$

59. $\boldsymbol{xy_x^{(n)} + ny_x^{(n-1)} = f(xy).}$

The substitution $w(x) = xy$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

60. $\boldsymbol{xy_x^{(n)} + (a+n-1)y_x^{(n-1)} = f(x, xy'_x + ay).}$

The substitution $w = xy'_x + ay$ leads to an $(n-1)$st-order equation: $w_x^{(n-1)} = f(x, w)$.

61. $\boldsymbol{x^2y_x^{(n)} + 2nxy_x^{(n-1)} + n(n-1)y_x^{(n-2)} = f(x^2y).}$

The substitution $w(x) = x^2y$ leads to an autonomous equation of the form 17.2.6.8: $w_x^{(n)} = f(w)$.

62. $\boldsymbol{(2n-1)yy_x^{(2n+1)} + (2n+1)y'_xy_x^{(2n)} = f(x).}$

Having integrated the equation, we obtain

$$(2n-1)yy_x^{(2n)} + 2\sum_{k=1}^{n-1}(-1)^{k+1}y_x^{(k)}y_x^{(2n-k)} + (-1)^{n+1}\big[y_x^{(n)}\big]^2 = \int f(x)\,dx + 2C_2.$$

The second integration leads to a $(2n-1)$st-order equation:

$$\sum_{k=0}^{n-1}(2n-1-2k)(-1)^k y_x^{(k)}y_x^{(2n-1-k)} = 2C_2x + C_1 + \int_{x_0}^{x}(x-t)f(t)\,dt.$$

The third integration leads to a $(2n-2)$nd-order equation:

$$\sum_{k=0}^{n-2}(k+1)(2n-k-1)(-1)^k y_x^{(k)}y_x^{(2n-2-k)} + \frac{1}{2}(-1)^{n-1}n^2\big[y_x^{(n-1)}\big]^2 = C_2x^2 + C_1x + C_0 + \frac{1}{2}\int_{x_0}^{x}(x-t)^2f(t)\,dt.$$

63. $\boldsymbol{(2n-1)yy_x^{(2n+1)} + (2n+1)y'_xy_x^{(2n)} = f(y)y'_x + g(x).}$

Integrating yields an $(n-1)$st-order equation:

$$(2n-1)yy_x^{(2n)}+2\sum_{k=1}^{n-1}(-1)^{k+1}y_x^{(k)}y_x^{(2n-k)}+(-1)^{n+1}\big[y_x^{(n)}\big]^2=\int f(y)\,dy+\int g(x)\,dx+C.$$

64. $\boldsymbol{yy_x^{(n)} - y'_xy_x^{(n-1)} = f(x)y^2.}$

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = \Big[\int f(x)\,dx + C\Big]y$.

65. $\boldsymbol{yy_x^{(n)} = y'_xy_x^{(n-1)} + f(x)yy_x^{(n-1)}.}$

Integrating yields an $(n-1)$st-order linear equation: $y_x^{(n-1)} = C\exp\Big[\int f(x)\,dx\Big]y$.

66. $\boldsymbol{yy_x^{(n)} + (f-1)y'_xy_x^{(n-1)} + fgyy'_x + g'_xy^2 = 0, \qquad f = f(x), \quad g = g(x).}$

This equation is solved by the functions that are solutions of the $(n-1)$st-order linear equation $y_x^{(n-1)} + g(x)y = 0$.

67. $\boldsymbol{[y + f(x)]y_x^{(n)} = [y'_x + f'_x(x)]y_x^{(n-1)} + af(x)y'_x - af'_x(x)y.}$

Integrating yields an $(n-1)$st-order constant coefficient nonhomogeneous linear equation: $y_x^{(n-1)} - Cy = (C-a)f(x)$. There is also the trivial solution $y = 0$.

68. $\displaystyle\sum_{m=1}^{n} a_m y_x^{(2m)} = f(y).$

The first integral has the form:

$$\sum_{m=1}^{n} a_m \left\{ \sum_{\nu=1}^{m-1} (-1)^\nu y_x^{(\nu)} y_x^{(2m-\nu)} + \frac{1}{2}(-1)^m \bigl[y_x^{(m)}\bigr]^2 \right\} + \int f(y)\,dy = C,$$

where C is an arbitrary constant. Furthermore, the order of the obtained equation can be reduced by one by the substitution $w(y) = y'_x$.

69. $\displaystyle\sum_{m=1}^{n} a_m x^m y_x^{(m)} = f(y).$

The substitution $t = \ln|x|$ leads to an autonomous equation of the form 17.2.6.77.

70. $\displaystyle y \sum_{m=0}^{n} a_m y_x^{(2m+1)} = f(x).$

Integrating yields a $2n$th-order equation:

$$\sum_{m=0}^{n} a_m \left\{ 2 \sum_{\nu=0}^{m-1} (-1)^\nu y_x^{(\nu)} y_x^{(2m-\nu)} + (-1)^m \bigl[y_x^{(m)}\bigr]^2 \right\} = 2\int f(x)\,dx + C,$$

where $y_x^{(0)}$ stands for y.

71. $\displaystyle\sum_{m=0}^{n} a_m y_x^{(m)} y_x^{(2n+1-m)} = f(x).$

The first integral has the form:

$$2\sum_{m=0}^{n-1} A_m y_x^{(m)} y_x^{(2n-m)} + A_n \bigl[y_x^{(n)}\bigr]^2 = 2\int f(x)\,dx + C,$$

where

$$A_m = \sum_{k=0}^{m} (-1)^{m+k} a_k = a_m - a_{m-1} + a_{m-2} - \cdots.$$

If the condition

$$A_n = 2\sum_{m=0}^{n-1} (-1)^{n-1+m} A_m$$

is satisfied, the obtained equation can be integrated two times more (in particular, see equation 17.2.6.62).

► **Equations of the form** $y_x^{(n)} = f\bigl(x, y, y'_x, \ldots, y_x^{(n-1)}\bigr)$.

72. $y_x^{(n)} = f\bigl(y_x^{(n-1)}\bigr).$

Having set $u(x) = y_x^{(n-1)}$, we obtain a first-order equation $u'_x = f(u)$. Further, find u from the relation $x = \displaystyle\int \frac{du}{f(u)} + C_1$. Then the $(n-1)$-fold integration yields y.

Solution in parametric form:

$$x = \int_{C_1}^{u} \frac{du}{f(u)}, \qquad y = \int_{C_2}^{u} \frac{du_1}{f(u_1)} \int_{C_3}^{u_1} \frac{du_2}{f(u_2)} \cdots \int_{C_{n-1}}^{u_{n-3}} \frac{du_{n-2}}{f(u_{n-2})} \int_{C_n}^{u_{n-2}} \frac{u_{n-1}\,du_{n-1}}{f(u_{n-1})}.$$

73. $y_x^{(n)} = f(y)y_x' g\big(y_x^{(n-1)}\big).$

Integrating yields an $(n-1)$st-order equation:

$$\int \frac{dw}{g(w)} = \int f(y)\,dy + C, \qquad \text{where} \quad w = y_x^{(n-1)}.$$

Furthermore, the order of this equation can be reduced by one by the substitution $z(y) = y_x'$.

74. $y_x^{(n)} = \big[f(y)y_x' + g(x)\big]h(y_x^{(n-1)}).$

Integrating yields an $(n-1)$st-order equation:

$$\int \frac{dw}{h(w)} = \int f(y)\,dy + \int g(x)\,dx + C, \qquad w = y_x^{(n-1)}.$$

75. $y_x^{(n)} = f\big(x,\, y_x^{(n-2)},\, y_x^{(n-1)}\big).$

The substitution $w(x) = y_x^{(n-2)}$ leads to a second-order equation: $w_{xx}'' = f(x, w, w_x')$.

► Equations of the general form $F\big(x, y, y_x', \ldots, y_x^{(n)}\big) = 0.$

76. $F\big(x,\, y_x',\, y_{xx}'',\, \ldots,\, y_x^{(n)}\big) = 0.$

The equation does not depend on y explicitly. Hence, the substitution $w(x) = y_x'$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, w_x',\, \ldots,\, w_x^{(n-1)}\big) = 0.$$

77. $F\big(y,\, y_x',\, y_{xx}'',\, \ldots,\, y_x^{(n)}\big) = 0.$

Autonomous equation. It does not depend on x explicitly. The substitution $w(y) = y_x'$ leads to an $(n-1)$st-order equation. The derivatives of the original equation and the transformed one are related by

$$y_{xx}'' = ww_y', \quad y_{xxx}''' = w^2 w_{yy}'' + w(w_y')^2, \quad \ldots, \quad y_x^{(n)} = w\big(y_x^{(n-1)}\big)_y'.$$

78. $F\big(x,\, xy_x' - y,\, y_{xx}'',\, y_{xxx}''',\, \ldots,\, y_x^{(n)}\big) = 0.$

The substitution $w(x) = xy_x' - y$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta_x',\, \ldots,\, \zeta_x^{(n-2)}\big) = 0, \quad \text{where} \quad \zeta = w_x'/x.$$

79. $F\big(x,\, xy_x' - 2y,\, y_{xxx}''',\, y_{xxxx}'''',\, \ldots,\, y_x^{(n)}\big) = 0.$

The substitution $w = xy_x' - 2y$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta_x',\, \ldots,\, \zeta_x^{(n-3)}\big) = 0, \quad \text{where} \quad \zeta = w_{xx}''/x.$$

80. $F\big(x,\, xy_x' - my,\, y_x^{(m+1)},\, y_x^{(m+2)},\, \ldots,\, y_x^{(n)}\big) = 0,$
$\qquad n \geq m+1, \quad m = 1,\, \ldots,\, n-1.$

The substitution $w = xy_x' - my$ leads to an $(n-1)$st-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta_x',\, \ldots,\, \zeta_x^{(n-m-1)}\big) = 0, \quad \text{where} \quad \zeta = w_x^{(m)}/x.$$

81. $F\big(x,\, x^2 y_{xx}'' - 2xy_x' + 2y,\, y_{xxx}''',\, \ldots,\, y_x^{(n)}\big) = 0.$

The substitution $w(x) = x^2 y_{xx}'' - 2xy_x' + 2y$ leads to an $(n-2)$nd-order equation:

$$F\big(x,\, w,\, \zeta,\, \zeta_x',\, \ldots,\, \zeta_x^{(n-3)}\big) = 0, \quad \text{where} \quad \zeta = x^{-2} w_x'.$$

82. $\displaystyle\sum_{k=0}^{m}(-1)^k k!\,C_m^k x^{m-k} y_x^{(m-k)} = F(x,\, y_x^{(m+1)},\, \dots,\, y_x^{(n)}).$

Here, $C_m^k = \dfrac{m!}{k!\,(m-k)!}$ are binomial coefficients.

The substitution $w(x) = \sum_{k=0}^{m}(-1)^k k!\,C_m^k x^{m-k} y_x^{(m-k)}$ leads to an $(n-m)$th-order equation; the derivatives on the right-hand side are calculated in consecutive manner using the formula $y_x^{(m+1)} = x^{-m} w_x'$.

83. $F\Big(\dfrac{y}{x},\, y_x',\, xy_{xx}'',\, \dots,\, x^{n-1}y_x^{(n)}\Big) = 0.$

Homogeneous equation. The transformation $t = \ln x$, $w = y/x$ leads to an autonomous equation of the form 17.2.6.77.

84. $F\Big(\dfrac{ax+by+c}{\alpha x+\beta y+\gamma},\, y_x',\, \dots,\, (ax+by+c)^{n-1}y_x^{(n)}\Big) = 0.$

1°. For $a\beta - b\alpha = 0$, the substitution $bw = ax+by+c$ leads to an autonomous equation of the form 17.2.6.77.

2°. For $a\beta - b\alpha \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system

$$ax_0 + by_0 + c = 0, \qquad \alpha x_0 + \beta y_0 + \gamma = 0,$$

leads to a homogeneous equation of the form 17.2.6.83:

$$F\Big(\frac{a+bw/z}{\alpha+\beta w/z},\, w_z',\, \dots,\, (a+bw/z)^{n-1}z^{n-1}w_z^{(n)}\Big) = 0.$$

85. $F\Big(\dfrac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2},\, y_x',\, \dots,\, (a_3x+b_3y+c_3)^{n-1}y_x^{(n)}\Big) = 0.$

Let the following condition hold: $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$

For $a_1b_2 - a_2b_1 \neq 0$, the transformation

$$z = x - x_0, \qquad w = y - y_0,$$

where x_0 and y_0 are the constants determined by the linear algebraic system

$$a_1x_0 + b_1y_0 + c_1 = 0, \qquad a_2x_0 + b_2y_0 + c_2 = 0,$$

leads to a homogeneous equation of the form 17.2.6.83:

$$F\Big(\frac{a_1+b_1w/z}{a_2+b_2w/z},\, w_z',\, \dots,\, (a_3+b_3w/z)^{n-1}z^{n-1}w_z^{(n)}\Big) = 0.$$

86. $F\big(x^k y,\, x^{k+1}y_x',\, \dots,\, x^{k+n}y_x^{(n)}\big) = 0.$

Generalized homogeneous equation. The transformation $t = \ln x$, $w = x^k y$ leads to an autonomous equation of the form 17.2.6.77.

87. $F\left(\dfrac{xy'_x}{y}, \dfrac{x^2y''_{xx}}{y}, \dots, \dfrac{x^ny_x^{(n)}}{y}\right) = 0.$

Generalized homogeneous equation. The transformation $z = xy'_x/y$, $w = x^2y''_{xx}/y$ leads to an $(n-2)$nd-order equation.

88. $F\left(y'_x - y\dfrac{y''_{xx}}{y'_x}, \dfrac{y''_{xx}}{y'_x}, \dfrac{y'''_{xxx}}{y'_x}, \dots, \dfrac{y_x^{(n)}}{y'_x}\right) = 0.$

Autonomous equation. Particular solution: $y = C_1\exp(C_2x) + C_3$, where C_1 is an arbitrary constant and the constants C_2 and C_3 are related by $F(-C_2C_3, C_2, C_2^2, \dots, C_2^{n-1}) = 0$.

89. $F\left(x^ky^m, \dfrac{xy'_x}{y}, \dfrac{x^2y''_{xx}}{y}, \dots, \dfrac{x^ny_x^{(n)}}{y}\right) = 0.$

Generalized homogeneous equation. The transformation $t = x^ky^m$, $z = xy'_x/y$ leads to an $(n-1)$st-order equation.

90. $F\left(\dfrac{y_x^{(n)}}{y'_x}, y\dfrac{y_x^{(n)}}{y'_x} - y_x^{(n-1)}\right) = 0.$

A solution of this equation is any function that satisfies the $(n-1)$st-order constant coefficient linear equation $y_x^{(n-1)} = C_1y + C_2$, where the constants C_1 and C_2 are related by the constraint $F(C_1, -C_2) = 0$.

91. $F\left(\dfrac{y_x^{(n)}}{y_x^{(k)}}, x^{1-k}y\dfrac{y_x^{(n)}}{y_x^{(k)}} - x^{1-k}y_x^{(n-k)}\right) = 0, \qquad n > k.$

A solution of this equation is any function that satisfies the $(n-k)$th-order linear equation $y_x^{(n-k)} = C_1y + C_2x^{k-1}$, where the constants C_1 and C_2 are related by $F(C_1, -C_2) = 0$.

92. $F\big(x, y_x^{(n)} - y, y_x^{(m)} - y\big) = 0.$

The substitution $w = y'_x - y$ reduces the order of the equation by one.

93. $F\big(y_x^{(n)} - y, y_x^{(2n)} - y, y_x^{(2n)} - y_x^{(n)}\big) = 0.$

The substitution $u = y_x^{(n)} - y$ leads to an nth-order autonomous equation of the form $F\big(u, u_x^{(n)} + u, u_x^{(n)}\big) = 0$.

94. $F\big(x, y_x^{(n)} + ay, y_x^{(2n)} - a^2y, y_x^{(2n)} + ay_x^{(n)}\big) = 0.$

The substitution $u = y_x^{(n)} + ay$ leads to an nth-order equation $F\big(x, u, u_x^{(n)} - au, u_x^{(n)}\big) = 0$.

95. $F\big(e^{\alpha x}y, e^{\alpha x}y'_x, e^{\alpha x}y''_{xx}, \dots, e^{\alpha x}y_x^{(n)}\big) = 0.$

Equation invariant under "translation–dilatation" transformation. The substitution $u = e^{\alpha x}y$ leads to an autonomous equation of the form 17.2.6.77.

96. $F\left(e^{\alpha x}y^m, \dfrac{y'_x}{y}, \dfrac{y''_{xx}}{y}, \dots, \dfrac{y_x^{(n)}}{y}\right) = 0.$

Equation invariant under "translation–dilatation" transformation. The transformation $z = e^{\alpha x}y^m$, $w = y'_x/y$ leads to an $(n-1)$st-order equation. See also Section 5.2.4 (the first paragraph).

97. $F\big(x^me^{\alpha y}, xy'_x, x^2y''_{xx}, \dots, x^ny_x^{(n)}\big) = 0.$

Equation invariant under "dilatation–translation" transformation. The transformation $z = x^me^{\alpha y}$, $w = xy'_x$ leads to an $(n-1)$st-order equation. See also Section 5.2.4 (the second paragraph).