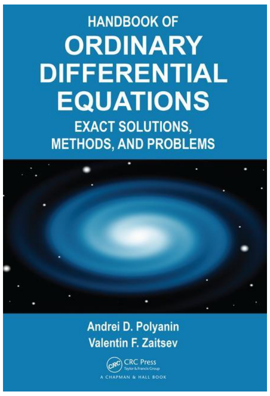

Handbook of Ordinary Differential Equations
Exact Solutions, Methods, and Problems

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Chapter 18: Some Systems of Ordinary Differential Equations

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Chapter 18

Some Systems of Ordinary Differential Equations

18.1 Linear Systems of Two Equations

18.1.1 Systems of First-Order Equations

1. $x'_t = ax + by, \quad y'_t = cx + dy.$

System of two constant-coefficient first-order linear homogeneous differential equations.

Let us write out the characteristic equation

$$\lambda^2 - (a+d)\lambda + ad - bc = 0 \tag{1}$$

and find its discriminant

$$D = (a-d)^2 + 4bc. \tag{2}$$

1°. Case $ad - bc \neq 0$. The origin of coordinates $x = y = 0$ is the only one stationary point; it is

$$\begin{aligned} &\text{a node if } D = 0;\\ &\text{a node if } D > 0 \text{ and } ad - bc > 0;\\ &\text{a saddle if } D > 0 \text{ and } ad - bc < 0;\\ &\text{a focus if } D < 0 \text{ and } a + d \neq 0;\\ &\text{a center if } D < 0 \text{ and } a + d = 0. \end{aligned}$$

1.1. Suppose $D > 0$. The characteristic equation (1) has two distinct real roots, λ_1 and λ_2. The general solution of the original system of differential equations is expressed as

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t},$$
$$y = C_1(\lambda_1 - a)e^{\lambda_1 t} + C_2(\lambda_2 - a)e^{\lambda_2 t},$$

where C_1 and C_2 are arbitrary constants.

1.2. Suppose $D < 0$. The characteristic equation (1) has two complex conjugate roots, $\lambda_{1,2} = \sigma \pm i\beta$. The general solution of the original system of differential equations is given by

$$x = be^{\sigma t}\big[C_1 \sin(\beta t) + C_2 \cos(\beta t)\big],$$
$$y = e^{\sigma t}\big\{[(\sigma - a)C_1 - \beta C_2]\sin(\beta t) + [\beta C_1 + (\sigma - a)C_2]\cos(\beta t)\big\},$$

where C_1 and C_2 are arbitrary constants.

1.3. Suppose $D=0$ and $a\neq d$. The characteristic equation (1) has two equal real roots, $\lambda_1=\lambda_2$. The general solution of the original system of differential equations is

$$x=2b\left(C_1+\frac{C_2}{a-d}+C_2t\right)\exp\left(\frac{a+d}{2}t\right),$$
$$y=[(d-a)C_1+C_2+(d-a)C_2t]\exp\left(\frac{a+d}{2}t\right),$$

where C_1 and C_2 are arbitrary constants.

1.4. Suppose $a=d\neq 0$ and $b=0$. Solution:

$$x=C_1e^{at},\qquad y=(cC_1t+C_2)e^{at}.$$

1.5. Suppose $a=d\neq 0$ and $c=0$. Solution:

$$x=(bC_1t+C_2)e^{at},\qquad y=C_1e^{at}.$$

2°. Case $ad-bc=0$ and $a^2+b^2>0$. The whole of the line $ax+by=0$ consists of singular points. The system in question may be rewritten in the form

$$x'_t=ax+by,\qquad y'_t=k(ax+by).$$

2.1. Suppose $a+bk\neq 0$. Solution:

$$x=bC_1+C_2e^{(a+bk)t},\qquad y=-aC_1+kC_2e^{(a+bk)t}.$$

2.2. Suppose $a+bk=0$. Solution:

$$x=C_1(bkt-1)+bC_2t,\qquad y=k^2bC_1t+(bk^2t+1)C_2.$$

2. $\boldsymbol{x'_t=a_1x+b_1y+c_1,\quad y'_t=a_2x+b_2y+c_2.}$

The general solution of this system is given by the sum of any one of its particular solutions and the general solution of the corresponding homogeneous system (see system 18.1.1.1).

1°. Suppose $a_1b_2-a_2b_1\neq 0$. A particular solution:

$$x=x_0,\qquad y=y_0,$$

where the constants x_0 and y_0 are determined by solving the linear algebraic system of equations

$$a_1x_0+b_1y_0+c_1=0,\qquad a_2x_0+b_2y_0+c_2=0.$$

2°. Suppose $a_1b_2-a_2b_1=0$ and $a_1^2+b_1^2>0$. Then the original system can be rewritten as

$$x'_t=ax+by+c_1,\qquad y'_t=k(ax+by)+c_2.$$

2.1. If $\sigma=a+bk\neq 0$, the original system has a particular solution of the form

$$x=b\sigma^{-1}(c_1k-c_2)t-\sigma^{-2}(ac_1+bc_2),\qquad y=kx+(c_2-c_1k)t.$$

2.2. If $\sigma=a+bk=0$, the original system has a particular solution of the form

$$x=\tfrac{1}{2}b(c_2-c_1k)t^2+c_1t,\qquad y=kx+(c_2-c_1k)t.$$

3. $\boldsymbol{x'_t=f(t)x+g(t)y,\quad y'_t=g(t)x+f(t)y.}$

Solution:

$$x=e^F(C_1e^G+C_2e^{-G}),\qquad y=e^F(C_1e^G-C_2e^{-G}),$$

where C_1 and C_2 are arbitrary constants, and

$$F = \int f(t)\,dt, \qquad G = \int g(t)\,dt.$$

4. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = -g(t)x + f(t)y.}$

Solution:

$$x = F(C_1 \cos G + C_2 \sin G), \qquad y = F(-C_1 \sin G + C_2 \cos G),$$

where C_1 and C_2 are arbitrary constants, and

$$F = \exp\biggl[\int f(t)\,dt\biggr], \qquad G = \int g(t)\,dt.$$

5. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = ag(t)x + [f(t) + bg(t)]y.}$

The transformation

$$x = \exp\biggl[\int f(t)\,dt\biggr]u, \qquad y = \exp\biggl[\int f(t)\,dt\biggr]v, \qquad \tau = \int g(t)\,dt$$

leads to a system of constant coefficient linear differential equations of the form 18.1.1.1:

$$u'_\tau = v, \qquad v'_\tau = au + bv.$$

6. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = a[f(t) + ah(t)]x + a[g(t) - h(t)]y.}$

Let us multiply the first equation by $-a$ and add it to the second equation to obtain

$$y'_t - ax'_t = -ah(t)(y - ax).$$

By setting $U = y - ax$ and then integrating, one obtains

$$y - ax = C_1 \exp\biggl[-a \int h(t)\,dt\biggr], \tag{$*$}$$

where C_1 is an arbitrary constant. On solving ($*$) for y and on substituting the resulting expression into the first equation of the system, one arrives at a first-order linear differential equation for x.

7. $\boldsymbol{x'_t = f(t)x + g(t)y, \quad y'_t = h(t)x + p(t)y.}$

1°. Let us express y from the first equation and substitute into the second one to obtain a second-order linear equation:

$$gx''_{tt} - (fg + gp + g'_t)x'_t + (fgp - g^2h + fg'_t - f'_tg)x = 0. \tag{1}$$

This equation is easy to integrate if, for example, the following conditions are met:

1) $fgp - g^2h + fg'_t - f'_tg = 0;$
2) $fgp - g^2h + fg'_t - f'_tg = ag, \quad fg + gp + g'_t = bg.$

In the first case, equation (1) has a particular solution $u = C = \text{const}$. In the second case, it is a constant-coefficient equation.

A considerable number of other solvable cases of equation (1) can be found in Section 14.1.

2°. Suppose a particular solution of the system in question is known,

$$x = x_0(t), \quad y = y_0(t).$$

Then the general solution can be written out in the form

$$x(t) = C_1 x_0(t) + C_2 x_0(t) \int \frac{g(t)F(t)P(t)}{x_0^2(t)}\,dt,$$
$$y(t) = C_1 y_0(t) + C_2 \left[\frac{F(t)P(t)}{x_0(t)} + y_0(t) \int \frac{g(t)F(t)P(t)}{x_0^2(t)}\,dt\right],$$

where C_1 and C_2 are arbitrary constants, and

$$F(t) = \exp\left[\int f(t)\,dt\right], \quad P(t) = \exp\left[\int p(t)\,dt\right].$$

18.1.2 Systems of Second-Order Equations

1. $x''_{tt} = ax + by, \quad y''_{tt} = cx + dy.$

System of two constant-coefficient second-order linear homogeneous differential equations.

The characteristic equation has the form

$$\lambda^4 - (a+d)\lambda^2 + ad - bc = 0.$$

1°. Case $ad - bc \neq 0$.

1.1. Suppose $(a-d)^2 + 4bc \neq 0$. The characteristic equation has four distinct roots $\lambda_1, \dots, \lambda_4$. The general solution of the system in question is written as

$$x = C_1 b e^{\lambda_1 t} + C_2 b e^{\lambda_2 t} + C_3 b e^{\lambda_3 t} + C_4 b e^{\lambda_4 t},$$
$$y = C_1(\lambda_1^2 - a)e^{\lambda_1 t} + C_2(\lambda_2^2 - a)e^{\lambda_2 t} + C_3(\lambda_3^2 - a)e^{\lambda_3 t} + C_4(\lambda_4^2 - a)e^{\lambda_4 t},$$

where $C_1, \dots, C_4$ are arbitrary constants.

1.2. Solution with $(a-d)^2 + 4bc = 0$ and $a \neq d$:

$$x = 2C_1\left(bt + \frac{2bk}{a-d}\right)e^{kt/2} + 2C_2\left(bt - \frac{2bk}{a-d}\right)e^{-kt/2} + 2bC_3 t e^{kt/2} + 2bC_4 t e^{-kt/2},$$
$$y = C_1(d-a)te^{kt/2} + C_2(d-a)te^{-kt/2} + C_3[(d-a)t + 2k]e^{kt/2} + C_4[(d-a)t - 2k]e^{-kt/2},$$

where $C_1, \dots, C_4$ are arbitrary constants and $k = \sqrt{2(a+d)}$.

1.3. Solution with $a = d \neq 0$ and $b = 0$:

$$x = 2\sqrt{a}\,C_1 e^{\sqrt{a}\,t} + 2\sqrt{a}\,C_2 e^{-\sqrt{a}\,t},$$
$$y = cC_1 t e^{\sqrt{a}\,t} - cC_2 t e^{-\sqrt{a}\,t} + C_3 e^{\sqrt{a}\,t} + C_4 e^{-\sqrt{a}\,t}.$$

1.4. Solution with $a = d \neq 0$ and $c = 0$:

$$x = bC_1 t e^{\sqrt{a}\,t} - bC_2 t e^{-\sqrt{a}\,t} + C_3 e^{\sqrt{a}\,t} + C_4 e^{-\sqrt{a}\,t},$$
$$y = 2\sqrt{a}\,C_1 e^{\sqrt{a}\,t} + 2\sqrt{a}\,C_2 e^{-\sqrt{a}\,t}.$$

2°. Case $ad - bc = 0$ and $a^2 + b^2 > 0$. The original system can be rewritten in the form

$$x''_{tt} = ax + by, \quad y''_{tt} = k(ax + by).$$

2.1. Solution with $a + bk \neq 0$:

$$x = C_1 \exp\bigl(t\sqrt{a+bk}\,\bigr) + C_2 \exp\bigl(-t\sqrt{a+bk}\,\bigr) + C_3 bt + C_4 b,$$
$$y = C_1 k \exp\bigl(t\sqrt{a+bk}\,\bigr) + C_2 k \exp\bigl(-t\sqrt{a+bk}\,\bigr) - C_3 at - C_4 a.$$

2.2. Solution with $a + bk = 0$:

$$x = C_1 bt^3 + C_2 bt^2 + C_3 t + C_4,$$
$$y = kx + 6C_1 t + 2C_2.$$

2. $\boldsymbol{x''_{tt} = a_1 x + b_1 y + c_1, \quad y''_{tt} = a_2 x + b_2 y + c_2.}$

The general solution of this system is expressed as the sum of any one of its particular solutions and the general solution of the corresponding homogeneous system (see system 18.1.2.1).

1°. Suppose $a_1 b_2 - a_2 b_1 \neq 0$. A particular solution:

$$x = x_0, \qquad y = y_0,$$

where the constants x_0 and y_0 are determined by solving the linear algebraic system of equations

$$a_1 x_0 + b_1 y_0 + c_1 = 0, \qquad a_2 x_0 + b_2 y_0 + c_2 = 0.$$

2°. Suppose $a_1 b_2 - a_2 b_1 = 0$ and $a_1^2 + b_1^2 > 0$. Then the system can be rewritten as

$$x''_{tt} = ax + by + c_1, \qquad y''_{tt} = k(ax + by) + c_2.$$

2.1. If $\sigma = a + bk \neq 0$, the original system has a particular solution

$$x = \tfrac{1}{2} b\sigma^{-1}(c_1 k - c_2)t^2 - \sigma^{-2}(ac_1 + bc_2), \qquad y = kx + \tfrac{1}{2}(c_2 - c_1 k)t^2.$$

2.2. If $\sigma = a + bk = 0$, the system has a particular solution

$$x = \tfrac{1}{24} b(c_2 - c_1 k)t^4 + \tfrac{1}{2} c_1 t^2, \qquad y = kx + \tfrac{1}{2}(c_2 - c_1 k)t^2.$$

3. $\boldsymbol{x''_{tt} - ay'_t + bx = 0, \quad y''_{tt} + ax'_t + by = 0.}$

This system is used to describe the horizontal motion of a pendulum taking into account the rotation of the earth.

Solution with $a^2 + 4b > 0$:

$$x = C_1 \cos(\alpha t) + C_2 \sin(\alpha t) + C_3 \cos(\beta t) + C_4 \sin(\beta t),$$
$$y = -C_1 \sin(\alpha t) + C_2 \cos(\alpha t) - C_3 \sin(\beta t) + C_4 \cos(\beta t),$$

where $C_1, \ldots, C_4$ are arbitrary constants and

$$\alpha = \tfrac{1}{2} a + \tfrac{1}{2}\sqrt{a^2 + 4b}, \qquad \beta = \tfrac{1}{2} a - \tfrac{1}{2}\sqrt{a^2 + 4b}.$$

4. $\boldsymbol{x''_{tt} + a_1 x'_t + b_1 y'_t + c_1 x + d_1 y = k_1 e^{i\omega t},}$
$\boldsymbol{y''_{tt} + a_2 x'_t + b_2 y'_t + c_2 x + d_2 y = k_2 e^{i\omega t}.}$

Systems of this type often arise in oscillation theory (e.g., oscillations of a ship and a ship gyroscope). The general solution of this constant-coefficient linear nonhomogeneous system of differential equations is expressed as the sum of any one of its particular solutions and the general solution of the corresponding homogeneous system (with $k_1 = k_2 = 0$).

1°. A particular solution is sought by the method of undetermined coefficients in the form

$$x = A_* e^{i\omega t}, \quad y = B_* e^{i\omega t}.$$

On substituting these expressions into the system of differential equations in question, one arrives at a linear nonhomogeneous system of algebraic equations for the coefficients A_* and B_*.

2°. The general solution of a homogeneous system of differential equations is determined by a linear combination of its linearly independent particular solutions, which are sought using the method of undetermined coefficients in the form of exponential functions,

$$x = Ae^{\lambda t}, \quad y = Be^{\lambda t}.$$

On substituting these expressions into the system and on collecting the coefficients of the unknowns A and B, one obtains

$$(\lambda^2 + a_1\lambda + c_1)A + (b_1\lambda + d_1)B = 0,$$
$$(a_2\lambda + c_2)A + (\lambda^2 + b_2\lambda + d_2)B = 0.$$

For a nontrivial solution to exist, the determinant of this system must vanish. This requirement results in the characteristic equation

$$(\lambda^2 + a_1\lambda + c_1)(\lambda^2 + b_2\lambda + d_2) - (b_1\lambda + d_1)(a_2\lambda + c_2) = 0,$$

which is used to determine λ. If the roots of this equation, $k_1, \dots, k_4$, are all distinct, then the general solution of the original system of differential equations has the form

$$x = -C_1(b_1\lambda_1 + d_1)e^{\lambda_1 t} - C_2(b_1\lambda_2 + d_1)e^{\lambda_2 t} - C_3(b_1\lambda_1 + d_1)e^{\lambda_3 t} - C_4(b_1\lambda_4 + d_1)e^{\lambda_4 t},$$
$$y = C_1(\lambda_1^2 + a_1\lambda_1 + c_1)e^{\lambda_1 t} + C_2(\lambda_2^2 + a_1\lambda_2 + c_1)e^{\lambda_2 t}$$
$$+ C_3(\lambda_3^2 + a_1\lambda_3 + c_1)e^{\lambda_3 t} + C_4(\lambda_4^2 + a_1\lambda_4 + c_1)e^{\lambda_4 t},$$

where $C_1, \dots, C_4$ are arbitrary constants.

5. $x''_{tt} = a(ty'_t - y), \quad y''_{tt} = b(tx'_t - x).$

The transformation

$$u = tx_t - x, \quad v = ty'_t - y \tag{1}$$

leads to a first-order system:

$$u'_t = atv, \quad v'_t = btu.$$

The general solution of this system is expressed as

$$\text{with } ab > 0: \quad \begin{cases} u(t) = C_1 a \exp\bigl(\tfrac{1}{2}\sqrt{ab}\,t^2\bigr) + C_2 a \exp\bigl(-\tfrac{1}{2}\sqrt{ab}\,t^2\bigr), \\ v(t) = C_1\sqrt{ab}\,\exp\bigl(\tfrac{1}{2}\sqrt{ab}\,t^2\bigr) - C_2\sqrt{ab}\,\exp\bigl(-\tfrac{1}{2}\sqrt{ab}\,t^2\bigr); \end{cases}$$

$$\text{with } ab < 0: \quad \begin{cases} u(t) = C_1 a \cos\bigl(\tfrac{1}{2}\sqrt{|ab|}\,t^2\bigr) + C_2 a \sin\bigl(\tfrac{1}{2}\sqrt{|ab|}\,t^2\bigr), \\ v(t) = -C_1\sqrt{|ab|}\,\sin\bigl(\tfrac{1}{2}\sqrt{|ab|}\,t^2\bigr) + C_2\sqrt{|ab|}\,\cos\bigl(\tfrac{1}{2}\sqrt{|ab|}\,t^2\bigr), \end{cases} \tag{2}$$

where C_1 and C_2 are arbitrary constants. On substituting (2) into (1) and integrating, one arrives at the general solution of the original system in the form

$$x = C_3 t + t\int \frac{u(t)}{t^2}\,dt, \quad y = C_4 t + t\int \frac{v(t)}{t^2}\,dt,$$

where C_3 and C_4 are arbitrary constants.

6. $x''_{tt} = f(t)(a_1x + b_1y), \quad y''_{tt} = f(t)(a_2x + b_2y).$

Let k_1 and k_2 be roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0.$$

Then, on multiplying the equations of the system by appropriate constants and on adding them together, one can rewrite the system in the form of two independent equations:

$$z''_1 = k_1f(t)z_1, \quad z_1 = a_2x + (k_1 - a_1)y;$$
$$z''_2 = k_2f(t)z_2, \quad z_2 = a_2x + (k_2 - a_1)y.$$

Here, a prime stands for a derivative with respect to t.

7. $x''_{tt} = f(t)(a_1x'_t + b_1y'_t), \quad y''_{tt} = f(t)(a_2x'_t + b_2y'_t).$

Let k_1 and k_2 be roots of the quadratic equation

$$k^2 - (a_1 + b_2)k + a_1b_2 - a_2b_1 = 0.$$

Then, on multiplying the equations of the system by appropriate constants and on adding them together, one can reduce the system to two independent equations:

$$z''_1 = k_1f(t)z'_1, \quad z_1 = a_2x + (k_1 - a_1)y;$$
$$z''_2 = k_2f(t)z'_2, \quad z_2 = a_2x + (k_2 - a_1)y.$$

Integrating these equations and returning to the original variables, one arrives at a linear algebraic system for the unknowns x and y:

$$a_2x + (k_1 - a_1)y = C_1 \int \exp\bigl[k_1F(t)\bigr]\,dt + C_2,$$
$$a_2x + (k_2 - a_1)y = C_3 \int \exp\bigl[k_2F(t)\bigr]\,dt + C_4,$$

where $C_1, \dots, C_4$ are arbitrary constants and $F(t) = \int f(t)\,dt$.

8. $x''_{tt} = af(t)(ty'_t - y), \quad y''_{tt} = bf(t)(tx'_t - x).$

The transformation

$$u = tx_t - x, \qquad v = ty'_t - y \tag{1}$$

leads to a system of first-order equations:

$$u'_t = atf(t)v, \qquad v'_t = btf(t)u.$$

The general solution of this system is expressed as

$$\begin{aligned}
&\text{if } ab>0, \quad \begin{cases} u(t) = C_1 a \exp\Bigl(\sqrt{ab}\int tf(t)\,dt\Bigr) + C_2 a \exp\Bigl(-\sqrt{ab}\int tf(t)\,dt\Bigr),\\ v(t) = C_1\sqrt{ab}\,\exp\Bigl(\sqrt{ab}\int tf(t)\,dt\Bigr) - C_2\sqrt{ab}\,\exp\Bigl(-\sqrt{ab}\int tf(t)\,dt\Bigr);\end{cases}\\
&\text{if } ab<0, \quad \begin{cases} u(t) = C_1 a \cos\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr) + C_2 a \sin\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr),\\ v(t) = -C_1\sqrt{|ab|}\,\sin\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr) + C_2\sqrt{|ab|}\,\cos\Bigl(\sqrt{|ab|}\int tf(t)\,dt\Bigr),\end{cases}
\end{aligned} \tag{2}$$

where C_1 and C_2 are arbitrary constants. On substituting (2) into (1) and integrating, one obtains the general solution of the original system

$$x = C_3t + t\int \frac{u(t)}{t^2}\,dt, \qquad y = C_4t + t\int \frac{v(t)}{t^2}\,dt,$$

where C_3 and C_4 are arbitrary constants.

9. $t^2x''_{tt}+a_1tx'_t+b_1ty'_t+c_1x+d_1y=0, \quad t^2y''_{tt}+a_2tx'_t+b_2ty'_t+c_2x+d_2y=0.$

Linear system homogeneous in the independent variable (an Euler-type system).

1°. The general solution is determined by a linear combination of linearly independent particular solutions that are sought by the method of undetermined coefficients in the form of power-law functions

$$x = A|t|^k, \quad y = B|t|^k.$$

On substituting these expressions into the system and on collecting the coefficients of the unknowns A and B, one obtains

$$A + (b_1k + d_1)B = 0,$$
$$(a_2k + c_2)A + [k^2 + (b_2 - 1)k + d_2]B = 0.$$

For a nontrivial solution to exist, the determinant of this system must vanish. This requirement results in the characteristic equation

$$[k^2 + (a_1 - 1)k + c_1][k^2 + (b_2 - 1)k + d_2] - (b_1k + d_1)(a_2k + c_2) = 0,$$

which is used to determine k. If the roots of this equation, $k_1, \dots, k_4$, are all distinct, then the general solution of the system of differential equations in question has the form

$$x = -C_1(b_1k_1+d_1)|t|^{k_1} - C_2(b_1k_2+d_1)|t|^{k_2} - C_3(b_1k_1+d_1)|t|^{k_3} - C_4(b_1k_4+d_1)|t|^{k_4},$$
$$y = C_1[k_1^2+(a_1-1)k_1+c_1]|t|^{k_1} + C_2[k_2^2+(a_1-1)k_2+c_1]|t|^{k_2}$$
$$+ C_3[k_3^2+(a_1-1)k_3+c_1]|t|^{k_3} + C_4[k_4^2+(a_1-1)k_4+c_1]|t|^{k_4},$$

where $C_1, \dots, C_4$ are arbitrary constants.

2°. The substitution $t = \sigma e^\tau$ ($\sigma \neq 0$) leads to a system of constant-coefficient linear differential equations:

$$x''_{\tau\tau} + (a_1 - 1)x'_\tau + b_1y'_\tau + c_1x + d_1y = 0,$$
$$y''_{\tau\tau} + a_2x'_\tau + (b_2 - 1)y'_\tau + c_2x + d_2y = 0.$$

10. $(\alpha t^2 + \beta t + \gamma)^2 x''_{tt} = ax + by, \quad (\alpha t^2 + \beta t + \gamma)^2 y''_{tt} = cx + dy.$

The transformation

$$\tau = \int \frac{dt}{\alpha t^2 + \beta t + \gamma}, \quad u = \frac{x}{\sqrt{|\alpha t^2 + \beta t + \gamma|}}, \quad v = \frac{y}{\sqrt{|\alpha t^2 + \beta t + \gamma|}}$$

leads to a constant-coefficient linear system of equations of the form 18.1.2.1:

$$u''_{\tau\tau} = (a - \alpha\gamma + \tfrac{1}{4}\beta^2)u + bv,$$
$$v''_{\tau\tau} = cu + (d - \alpha\gamma + \tfrac{1}{4}\beta^2)v.$$

11. $x''_{tt} = f(t)(tx'_t - x) + g(t)(ty'_t - y), \quad y''_{tt} = h(t)(tx'_t - x) + p(t)(ty'_t - y).$

The transformation

$$u = tx_t - x, \quad v = ty'_t - y \qquad (1)$$

leads to a linear system of first-order equations

$$u'_t = tf(t)u + tg(t)v, \quad v'_t = th(t)u + tp(t)v. \qquad (2)$$

In order to find the general solution of this system, it suffices to know any one of its particular solutions (see system 18.1.1.7).

For solutions of some systems of the form (2), see systems 18.1.1.3–18.1.1.6.

If all functions in (2) are proportional, that is,

$$f(t)=a\varphi(t),\quad g(t)=b\varphi(t),\quad h(t)=c\varphi(t),\quad p(t)=d\varphi(t),$$

then the introduction of the new independent variable $\tau=\int t\varphi(t)\,dt$ leads to a constant-coefficient system of the form 18.1.1.1.

2°. Suppose a solution of system (2) has been found in the form

$$u=u(t,C_1,C_2),\qquad v=v(t,C_1,C_2), \tag{3}$$

where C_1 and C_2 are arbitrary constants. Then, on substituting (3) into (1) and integrating, one obtains a solution of the original system:

$$x=C_3t+t\int\frac{u(t,C_1,C_2)}{t^2}\,dt,\qquad y=C_4t+t\int\frac{v(t,C_1,C_2)}{t^2}\,dt,$$

where C_3 and C_4 are arbitrary constants.

18.2 Linear Systems of Three and More Equations

1. $x'_t=ax,\quad y'_t=bx+cy,\quad z'_t=dx+ky+pz.$

Solution:

$$\begin{aligned}x&=C_1e^{at},\\ y&=\frac{bC_1}{a-c}e^{at}+C_2e^{ct},\\ z&=\frac{C_1}{a-p}\Big(d+\frac{bk}{a-c}\Big)e^{at}+\frac{kC_2}{c-p}e^{ct}+C_3e^{pt},\end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants.

2. $x'_t=cy-bz,\quad y'_t=az-cx,\quad z'_t=bx-ay.$

1°. First integrals:

$$ax+by+cz=A, \tag{1}$$
$$x^2+y^2+z^2=B^2, \tag{2}$$

where A and B are arbitrary constants. It follows that the integral curves are circles formed by the intersection of planes (1) and spheres (2).

2°. Solution:

$$\begin{aligned}x&=aC_0+kC_1\cos(kt)+(cC_2-bC_3)\sin(kt),\\ y&=bC_0+kC_2\cos(kt)+(aC_3-cC_1)\sin(kt),\\ z&=cC_0+kC_3\cos(kt)+(bC_1-aC_2)\sin(kt),\end{aligned}$$

where $k=\sqrt{a^2+b^2+c^2}$ and the three of four constants of integration $C_0,\dots,C_3$ are related by the constraint

$$aC_1+bC_2+cC_3=0.$$

3. $ax'_t=bc(y-z),\quad by'_t=ac(z-x),\quad cz'_t=ab(x-y).$

1°. First integral:

$$a^2x+b^2y+c^2z=A,$$

where A is an arbitrary constant. It follows that the integral curves are plane curves.

2°. Solution:

$$x = C_0 + kC_1\cos(kt) + a^{-1}bc(C_2 - C_3)\sin(kt),$$
$$y = C_0 + kC_2\cos(kt) + ab^{-1}c(C_3 - C_1)\sin(kt),$$
$$z = C_0 + kC_3\cos(kt) + abc^{-1}(C_1 - C_2)\sin(kt),$$

where $k = \sqrt{a^2 + b^2 + c^2}$ and three of the four constants of integration $C_0, \dots, C_3$ are related by the constraint

$$a^2C_1 + b^2C_2 + c^2C_3 = 0.$$

4. $x'_t = (a_1f + g)x + a_2fy + a_3fz,$
$y'_t = b_1fx + (b_2f + g)y + b_3fz,\ \ z'_t = c_1fx + c_2fy + (c_3f + g)z.$

Here, $f = f(t)$ and $g = g(t)$.

The transformation

$$x = \exp\left[\int g(t)\,dt\right]u,\quad y = \exp\left[\int g(t)\,dt\right]v,\quad z = \exp\left[\int g(t)\,dt\right]w,\quad \tau = \int f(t)\,dt$$

leads to the system of constant coefficient linear differential equations

$$u'_\tau = a_1u + a_2v + a_3w,\quad v'_\tau = b_1u + b_2v + b_3w,\quad w'_\tau = c_1u + c_2v + c_3w.$$

5. $x'_t = h(t)y - g(t)z,\ \ y'_t = f(t)z - h(t)x,\ \ z'_t = g(t)x - f(t)y.$

1°. First integral:

$$x^2 + y^2 + z^2 = C^2,$$

where C is an arbitrary constant.

2°. The system concerned can be reduced to a Riccati equation (see Kamke, 1977).

6. $x'_k = a_{k1}x_1 + a_{k2}x_2 + \dots + a_{kn}x_n;\qquad k = 1, 2, \dots, n.$

System of n constant-coefficient first-order linear homogeneous differential equations.

The general solution of a homogeneous system of differential equations is determined by a linear combination of linearly independent particular solutions, which are sought by the method of undetermined coefficients in the form of exponential functions,

$$x_k = A_ke^{\lambda t};\qquad k = 1, 2, \dots, n.$$

On substituting these expressions into the system and on collecting the coefficients of the unknowns A_k, one obtains a linear homogeneous system of algebraic equations:

$$a_{k1}A_1 + a_{k2}A_2 + \dots + (a_{kk} - \lambda)A_k + \dots + a_{kn}A_n = 0;\qquad k = 1, 2, \dots, n.$$

For a nontrivial solution to exist, the determinant of this system must vanish. This requirement results in a characteristic equation that serves to determine λ.

18.3 Nonlinear Systems of Two Equations

18.3.1 Systems of First-Order Equations

1. $x'_t = x^nF(x, y),\quad y'_t = g(y)F(x, y).$

Solution:

$$x = \varphi(y),\quad \int\frac{dy}{g(y)F(\varphi(y), y)} = t + C_2,$$

where

$$\varphi(y) = \begin{cases} \Big[C_1 + (1-n)\displaystyle\int \frac{dy}{g(y)}\Big]^{\frac{1}{1-n}} & \text{if } n \neq 1, \\ C_1 \exp\Big[\displaystyle\int \frac{dy}{g(y)}\Big] & \text{if } n = 1, \end{cases}$$

C_1 and C_2 are arbitrary constants.

2. $\boldsymbol{x'_t = e^{\lambda x} F(x,y), \quad y'_t = g(y) F(x,y).}$

Solution:

$$x = \varphi(y), \quad \int \frac{dy}{g(y)F(\varphi(y),y)} = t + C_2,$$

where

$$\varphi(y) = \begin{cases} -\dfrac{1}{\lambda} \ln\Big[C_1 - \lambda \displaystyle\int \frac{dy}{g(y)}\Big] & \text{if } \lambda \neq 0, \\ C_1 + \displaystyle\int \frac{dy}{g(y)} & \text{if } \lambda = 0, \end{cases}$$

C_1 and C_2 are arbitrary constants.

3. $\boldsymbol{x'_t = F(x,y), \quad y'_t = G(x,y).}$

Autonomous system of general form.

Suppose

$$y = y(x, C_1),$$

where C_1 is an arbitrary constant, is the general solution of the first-order equation

$$F(x,y) y'_x = G(x,y).$$

Then the general solution of the system in question results in the following dependence for the variable x:

$$\int \frac{dx}{F(x, y(x, C_1))} = t + C_2.$$

4. $\boldsymbol{x'_t = f_1(x) g_1(y) \Phi(x,y,t), \quad y'_t = f_2(x) g_2(y) \Phi(x,y,t).}$

First integral:

$$\int \frac{f_2(x)}{f_1(x)}\, dx - \int \frac{g_1(y)}{g_2(y)}\, dy = C, \qquad (*)$$

where C is an arbitrary constant.

On solving $(*)$ for x (or y) and on substituting the resulting expression into one of the equations of the system concerned, one arrives at a first-order equation for y (or x).

5. $\boldsymbol{x = t x'_t + F(x'_t, y'_t), \quad y = t y'_t + G(x'_t, y'_t).}$

Clairaut system.

The following are solutions of the system:

(i) straight lines

$$x = C_1 t + F(C_1, C_2), \quad y = C_2 t + G(C_1, C_2),$$

where C_1 and C_2 are arbitrary constants;

(ii) envelopes of these lines;

(iii) continuously differentiable curves that are formed by segments of curves (i) and (ii).

18.3.2 Systems of Second-Order Equations

1. $x''_{tt} = xf(ax-by) + g(ax-by), \quad y''_{tt} = yf(ax-by) + h(ax-by).$

Let us multiply the first equation by a and the second one by $-b$ and add them together to obtain the autonomous equation

$$z''_{tt} = zf(z) + ag(z) - bh(z), \qquad z = ax - by. \tag{1}$$

We will consider this equation in conjunction with the first equation of the system,

$$x''_{tt} = xf(z) + g(z). \tag{2}$$

Autonomous equation (1) can be treated separately; its general solution can be written out in implicit form (see Eq. 14.9.1.1). The function $x = x(t)$ can be determined by solving the linear equation (2), and the function $y = y(t)$ is found as $y = (ax - z)/b$.

2. $x''_{tt} = xf(y/x), \quad y''_{tt} = yg(y/x).$

A periodic particular solution:

$$x = C_1 \sin(kt) + C_2 \cos(kt), \qquad k = \sqrt{-f(\lambda)},$$
$$y = \lambda[C_1 \sin(kt) + C_2 \cos(kt)],$$

where C_1 and C_2 are arbitrary constants and λ is a root of the transcendental (algebraic) equation

$$f(\lambda) = g(\lambda). \tag{1}$$

2°. Particular solution:

$$x = C_1 \exp(kt) + C_2 \exp(-kt), \qquad k = \sqrt{f(\lambda)},$$
$$y = \lambda[C_1 \exp(kt) + C_2 \exp(-kt)],$$

where C_1 and C_2 are arbitrary constants and λ is a root of the transcendental (algebraic) equation (1).

3. $x''_{tt} = kxr^{-3}, \quad y''_{tt} = kyr^{-3}, \quad$ **where** $\quad r = \sqrt{x^2 + y^2}.$

Equation of motion of a point mass in the xy-plane under gravity.

Passing to polar coordinates by the formulas

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad r = r(t), \quad \varphi = \varphi(t),$$

one may obtain the first integrals

$$r^2\varphi'_t = C_1, \quad (r'_t)^2 + r^2(\varphi'_t)^2 = -2kr^{-1} + C_2, \tag{1}$$

where C_1 and C_2 are arbitrary constants. Assuming that $C_1 \neq 0$ and integrating further, one finds that

$$r[C\cos(\varphi - \varphi_0) - k] = C_1^2, \qquad C^2 = C_1^2 C_2 + k^2.$$

This is an equation of a conic section. The dependence $\varphi(t)$ may be found from the first equation in (1).

4. $x''_{tt} = xf(r), \quad y''_{tt} = yf(r), \quad$ **where** $\quad r = \sqrt{x^2 + y^2}.$

Equation of motion of a point mass in the xy-plane under a central force.

Passing to polar coordinates by the formulas

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad r = r(t), \quad \varphi = \varphi(t),$$

one may obtain the first integrals

$$r^2\varphi'_t = C_1, \qquad (r'_t)^2 + r^2(\varphi'_t)^2 = 2\int rf(r)\,dr + C_2,$$

where C_1 and C_2 are arbitrary constants. Integrating further, one finds that

$$t + C_3 = \pm\int \frac{r\,dr}{\sqrt{2r^2F(r) + r^2C_2 - C_1^2}}, \qquad \varphi = C_1\int \frac{dt}{r} + C_4, \qquad (*)$$

where C_3 and C_4 are arbitrary constants and

$$F(r) = \int rf(r)\,dr.$$

It is assumed in the second relation in $(*)$ that the dependence $r = r(t)$ is obtained by solving the first equation in $(*)$ for $r(t)$.

5. $\boldsymbol{x''_{tt} + a(t)x = x^{-3}f(y/x), \quad y''_{tt} + a(t)y = y^{-3}g(y/x).}$

Generalized Ermakov system.

1°. First integral:

$$\frac{1}{2}(xy'_t - yx'_t)^2 + \int^{y/x}\big[uf(u) - u^{-3}g(u)\big]\,du = C,$$

where C is an arbitrary constant.

2°. Suppose $\varphi = \varphi(t)$ is a nontrivial solution of the second-order linear differential equation

$$\varphi''_{tt} + a(t)\varphi = 0. \qquad (1)$$

Then the transformation

$$\tau = \int \frac{dt}{\varphi^2(t)}, \quad u = \frac{x}{\varphi(t)}, \quad v = \frac{y}{\varphi(t)} \qquad (2)$$

leads to the autonomous system of equations

$$u''_{\tau\tau} = u^{-3}f(v/u), \qquad v''_{\tau\tau} = v^{-3}g(v/u). \qquad (3)$$

3°. Particular solution of system (3) is

$$u = A\sqrt{C_2\tau^2 + C_1\tau + C_0}, \qquad v = Ak\sqrt{C_2\tau^2 + C_1\tau + C_0}, \qquad A = \left[\frac{f(k)}{C_0C_2 - \frac{1}{4}C_1^2}\right]^{1/4},$$

where C_0, C_1, and C_2 are arbitrary constants, and k is a root of the algebraic (transcendental) equation

$$k^4f(k) = g(k).$$

6. $\boldsymbol{x''_{tt} = f(y'_t/x'_t), \quad y''_{tt} = g(y'_t/x'_t).}$

1°. The transformation

$$u = x'_t, \qquad w = y'_t \qquad (1)$$

leads to a system of the first-order equations

$$u'_t = f(w/u), \qquad w'_t = g(w/u). \qquad (2)$$

Eliminating t yields a homogeneous first-order equation, whose solution is given by

$$\int \frac{f(\xi)\,d\xi}{g(\xi) - \xi f(\xi)} = \ln|u| + C, \quad \xi = \frac{w}{u}, \qquad (3)$$

where C is an arbitrary constant. On solving (3) for w, one obtains $w = w(u, C)$. On substituting this expression into the first equation of (2), one can find $u = u(t)$ and then $w = w(t)$. Finally, one can determine $x = x(t)$ and $y = y(t)$ from (1) by simple integration.

2°. *The Suslov problem.* The problem of a point particle sliding down an inclined rough plane is described by the equations

$$x''_{tt} = 1 - \frac{kx'_t}{\sqrt{(x'_t)^2 + (y'_t)^2}}, \qquad y''_{tt} = -\frac{ky'_t}{\sqrt{(x'_t)^2 + (y'_t)^2}},$$

which correspond to a special case of the system in question with

$$f(z) = 1 - \frac{k}{\sqrt{1+z^2}}, \qquad g(z) = -\frac{kz}{\sqrt{1+z^2}}.$$

The solution of the corresponding Cauchy problem under the initial conditions

$$x(0) = y(0) = x'_t(0) = 0, \qquad y'_t(0) = 1$$

leads, for the case $k = 1$, to the following dependences $x(t)$ and $y(t)$ written in parametric form:

$$x = -\tfrac{1}{16} + \tfrac{1}{16}\xi^4 - \tfrac{1}{4}\ln\xi, \quad y = \tfrac{2}{3} - \tfrac{1}{2}\xi - \tfrac{1}{6}\xi^3, \quad t = \tfrac{1}{4} - \tfrac{1}{4}\xi^2 - \tfrac{1}{2}\ln\xi \qquad (0 \le \xi \le 1).$$

7. $\boldsymbol{x''_{tt} = x\Phi(x, y, t, x'_t, y'_t), \quad y''_{tt} = y\Phi(x, y, t, x'_t, y'_t).}$

1°. First integral:

$$xy'_t - yx'_t = C,$$

where C is an arbitrary constant.

Remark 18.1. The function Φ can also be dependent on the second and higher derivatives with respect to t.

2°. Particular solution: $y = C_1 x$, where C_1 is an arbitrary constant and the function $x = x(t)$ is determined by the ordinary differential equation

$$x''_{tt} = x\Phi(x, C_1x, t, x'_t, C_1x'_t).$$

8. $\boldsymbol{x''_{tt} + x^{-3}f(y/x) = x\Phi(x, y, t, x'_t, y'_t), \; y''_{tt} + y^{-3}g(y/x) = y\Phi(x, y, t, x'_t, y'_t).}$

First integral:

$$\frac{1}{2}(xy'_t - yx'_t)^2 + \int^{y/x} \bigl[u^{-3}g(u) - uf(u)\bigr]\, du = C,$$

where C is an arbitrary constant.

Remark 18.2. The function Φ can also be dependent on the second and higher derivatives with respect to t.

9. $\boldsymbol{x''_{tt} = F(t, tx'_t - x, ty'_t - y), \quad y''_{tt} = G(t, tx'_t - x, ty'_t - y).}$

1°. The transformation

$$u = tx_t - x, \qquad v = ty'_t - y \tag{1}$$

leads to a system of first-order equations

$$u'_t = tF(t, u, v), \qquad v'_t = tG(t, u, v). \tag{2}$$

2°. Suppose a solution of system (2) has been found in the form

$$u = u(t, C_1, C_2), \qquad v = v(t, C_1, C_2), \tag{3}$$

where C_1 and C_2 are arbitrary constants. Then, substituting (3) into (1) and integrating, one obtains a solution of the original system,

$$x = C_3 t + t\int \frac{u(t, C_1, C_2)}{t^2}\, dt, \qquad y = C_4 t + t\int \frac{v(t, C_1, C_2)}{t^2}\, dt.$$

3°. If the functions F and G are independent of t, then, on eliminating t from system (2), one arrives at a first-order equation

$$g(u, v)u'_v = F(u, v).$$

18.4 Nonlinear Systems of Three or More Equations

18.4.1 Systems of Three Equations

1. $\boldsymbol{ax'_t = (b-c)yz, \quad by'_t = (c-a)zx, \quad cz'_t = (a-b)xy.}$

First integrals:

$$ax^2 + by^2 + cz^2 = C_1,$$
$$a^2x^2 + b^2y^2 + c^2z^2 = C_2,$$

where C_1 and C_2 are arbitrary constants. On solving the first integrals for y and z and on substituting the resulting expressions into the first equation of the system, one arrives at a separable first-order equation.

2. $\boldsymbol{ax'_t = (b-c)yzF(x,y,z,t),}$
$\boldsymbol{by'_t = (c-a)zxF(x,y,z,t), \quad cz'_t = (a-b)xyF(x,y,z,t).}$

First integrals:

$$ax^2 + by^2 + cz^2 = C_1,$$
$$a^2x^2 + b^2y^2 + c^2z^2 = C_2,$$

where C_1 and C_2 are arbitrary constants. On solving the first integrals for y and z and on substituting the resulting expressions into the first equation of the system, one arrives at a separable first-order equation; if F is independent of t, this equation will be separable.

3. $\boldsymbol{x'_t = cF_2 - bF_3, \quad y'_t = aF_3 - cF_1, \quad z'_t = bF_1 - aF_2,}$
where $\boldsymbol{F_n = F_n(x,y,z).}$

First integral:

$$ax + by + cz = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one arrives at the first-order equation

$$\frac{dy}{dx} = \frac{aF_3(x,y,z) - cF_1(x,y,z)}{cF_2(x,y,z) - bF_3(x,y,z)}, \quad \text{where} \quad z = \frac{1}{c}(C_1 - ax - by).$$

4. $\boldsymbol{x'_t = czF_2 - byF_3, \quad y'_t = axF_3 - czF_1, \quad z'_t = byF_1 - axF_2.}$

Here, $F_n = F_n(x,y,z)$ are arbitrary functions ($n = 1, 2, 3$).

First integral:

$$ax^2 + by^2 + cz^2 = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one arrives at the first-order equation

$$\frac{dy}{dx} = \frac{axF_3(x,y,z) - czF_1(x,y,z)}{czF_2(x,y,z) - byF_3(x,y,z)}, \quad \text{where} \quad z = \pm\sqrt{\frac{1}{c}(C_1 - ax^2 - by^2)}.$$

5. $\boldsymbol{x'_t = x(cF_2 - bF_3), \quad y'_t = y(aF_3 - cF_1), \quad z'_t = z(bF_1 - aF_2).}$

Here, $F_n = F_n(x,y,z)$ are arbitrary functions ($n = 1, 2, 3$).

First integral:

$$|x|^a|y|^b|z|^c = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one may obtain a first-order equation.

6. $x'_t = h(z)F_2 - g(y)F_3, \quad y'_t = f(x)F_3 - h(z)F_1, \quad z'_t = g(y)F_1 - f(x)F_2.$

Here, $F_n = F_n(x, y, z)$ are arbitrary functions ($n = 1, 2, 3$).

First integral:

$$\int f(x)\,dx + \int g(y)\,dy + \int h(z)\,dz = C_1,$$

where C_1 is an arbitrary constant. On eliminating t and z from the first two equations of the system (using the above first integral), one may obtain a first-order equation.

7. $x''_{tt} = \dfrac{\partial F}{\partial x}, \quad y''_{tt} = \dfrac{\partial F}{\partial y}, \quad z''_{tt} = \dfrac{\partial F}{\partial z},$ **where** $F = F(r), \; r = \sqrt{x^2 + y^2 + z^2}.$

Equations of motion of a point particle under gravity.

The system can be rewritten as a single vector equation:

$$\mathbf{r}''_{tt} = \operatorname{grad} F \qquad \text{or} \qquad \mathbf{r}''_{tt} = \frac{F'(r)}{r}\mathbf{r},$$

where $\mathbf{r} = (x, y, z)$.

1°. First integrals:

$$\begin{aligned} &(\mathbf{r}'_t)^2 = 2F(r) + C_1 && \text{(law of conservation of energy)},\\ &[\mathbf{r} \times \mathbf{r}'_t] = \mathbf{C} && \text{(law of conservation of areas)},\\ &(\mathbf{r} \cdot \mathbf{C}) = 0 && \text{(all trajectories are plane curves)}. \end{aligned}$$

2°. Solution:

$$\mathbf{r} = \mathbf{a}\, r \cos\varphi + \mathbf{b}\, r \sin\varphi.$$

Here, the constant vectors **a** and **b** must satisfy the conditions

$$|\mathbf{a}| = |\mathbf{b}| = 1, \qquad (\mathbf{a} \cdot \mathbf{b}) = 0,$$

and the functions $r = r(t)$ and $\varphi = \varphi(t)$ are given by

$$t = \int \frac{r\,dr}{\sqrt{2r^2F(r) + C_1r^2 - C_3^2}} + C_2, \quad \varphi = C_3 \int \frac{dr}{r\sqrt{2r^2F(r) + C_1r^2 - C_3^2}}, \quad C_3 = |\mathbf{C}|.$$

8. $x''_{tt} = xF, \; y''_{tt} = yF, \; z''_{tt} = zF,$ **where** $F = F(x, y, z, t, x'_t, y'_t, z'_t).$

First integrals (laws of conservation of areas):

$$\begin{aligned} zy'_t - yz'_t &= C_1,\\ xz'_t - zx'_t &= C_2,\\ yx'_t - xy'_t &= C_3, \end{aligned}$$

where C_1, C_2, and C_3 are arbitrary constants.

Corollary of the conservation laws:

$$C_1x + C_2y + C_3z = 0.$$

This implies that all integral curves are plane curves.

Remark 18.3. The function F can also be dependent on the second and higher derivatives with respect to t.

9. $x''_{tt} = F_1, \quad y''_{tt} = F_2, \quad z''_{tt} = F_3,$ **where** $F_n = F_n(t, tx'_t - x, ty'_t - y, tz'_t - z).$

1°. The transformation

$$u = tx_t - x, \qquad v = ty'_t - y, \qquad w = tz'_t - z \tag{1}$$

leads to the system of first-order equations

$$u'_t = tF_1(t,u,v,w), \quad v'_t = tF_2(t,u,v,w), \quad w'_t = tF_3(t,u,v,w). \tag{2}$$

2°. Suppose a solution of system (2) has been found in the form

$$u(t) = u(t,C_1,C_2,C_3), \quad v(t) = v(t,C_1,C_2,C_3), \quad w(t) = w(t,C_1,C_2,C_3), \tag{3}$$

where C_1, C_2, and C_3 are arbitrary constants. Then, substituting (3) into (1) and integrating, one obtains a solution of the original system:

$$x = C_4 t + t\int \frac{u(t)}{t^2}\,dt, \quad y = C_5 t + t\int \frac{v(t)}{t^2}\,dt, \quad z = C_6 t + t\int \frac{w(t)}{t^2}\,dt,$$

where C_4, C_5, and C_6 are arbitrary constants.

18.4.2 Dynamics of a Rigid Body with a Fixed Point*

► **Kinematic and dynamic Euler equations.**

The motion (rotation) of a rigid about a fixed point under the action of external forces is governed by a system of six first-order coupled ODEs:

$$Ap'_t + (C-B)qr = M_1, \tag{1}$$
$$Bq'_t + (A-C)pr = M_2, \tag{2}$$
$$Cr'_t + (B-A)pq = M_3, \tag{3}$$
$$p = \psi'_t \sin\theta \sin\varphi + \theta'_t \cos\varphi, \tag{4}$$
$$q = \psi'_t \sin\theta \cos\varphi - \theta'_t \sin\varphi, \tag{5}$$
$$r = \psi'_t \cos\theta + \varphi'_t, \tag{6}$$

where p, q, and r are the components of the body's angular velocity in a moving orthonormal reference frame, $\xi\eta\zeta$, rigidly connected with the body and formed by the principal axes of inertia (the origin placed at the fixed point); xyz is a fixed orthonormal reference frame with origin at the same point; A, B, and C are the moments of inertia about the principal axes; and M_1, M_2, and M_3 are the components of the moment of external forces in the frame $\xi\eta\zeta$, which usually depend of the Euler angles ψ, θ, and φ defining the position of the moving frame relative to the fixed one. The entries of the rotation matrix, $[a_{ij}]$, are expressed in terms of the Euler angles as follows:

$$a_{11} = \cos\varphi\cos\psi - \sin\varphi\cos\theta\sin\psi, \tag{7}$$
$$a_{11} = -\sin\varphi\cos\psi - \cos\varphi\cos\theta\sin\psi, \tag{8}$$
$$a_{13} = -\sin\theta\sin\varphi, \tag{9}$$
$$a_{21} = \cos\varphi\sin\psi + \sin\varphi\cos\theta\cos\psi, \tag{10}$$
$$a_{22} = -\sin\varphi\sin\psi + \cos\varphi\cos\theta\cos\psi, \tag{11}$$
$$a_{23} = -\sin\theta\cos\varphi, \tag{12}$$
$$a_{31} = \sin\varphi\sin\theta, \tag{13}$$
$$a_{32} = \cos\varphi\sin\theta, \tag{14}$$
$$a_{33} = \cos\theta. \tag{15}$$

*This section was written by Alexander Fomichev.

It is required to determine p, q, and r as functions of ψ, θ, and φ and time t from system (1)–(6).

From now on, the following quantities will be used in this section: m is the mass of the body, $\mathbf{r}$ is the position vector of the center of mass, $\mathbf{K} = (K_1, K_2, K_3)^{\mathrm{T}} = (Ap, Bq, Cr)^{\mathrm{T}}$ is the angular momentum of the body (in the frame $\xi\eta\zeta$), $\boldsymbol{\gamma} = (\gamma_1, \gamma_2, \gamma_3)$ is a vertical unit vector ($\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$), which is introduced when the body is in a homogeneous gravitational field so that the direction of $\boldsymbol{\gamma}$ is opposite to the gravitational acceleration $\mathbf{g}$, with $g = |\mathbf{g}|$.

Equations (1)–(3) are known as *Euler's dynamic equations* and (4)–(6) as *Euler's kinematic equations*. In general, system (1)–(6) cannot be solved by quadrature. However, there are three special cases where the system is reduced to quadratures for any initial conditions; this is due to the availability of first integrals, which do not exist in the general case. The three solvable cases are discussed below.

► Euler's case.

Euler's case takes place when the body has an arbitrary shape and the external moments are all zero:

$$M_1 = M_2 = M_3 = 0. \tag{16}$$

With formulas (16), the dynamic equations (1)–(3) can be solved independently of the kinematic equations.

To be specific, we assume that $A \ge B \ge C$ and $A > C$ (the case $A = B = C$ is trivial). System (1)–(3) with (16) has the following first integrals:

$$\begin{aligned} Ap^2 + Bq^2 + Cr^2 &= 2T \qquad \text{(conservation of energy)},\\ A^2p^2 + B^2q^2 + C^2r^2 &= K^2 \qquad \text{(conservation of angular momentum)}, \end{aligned}$$

where $T > 0$ and K are arbitrary constants. In Euler's case, the angular momentum $\mathbf{K}$ is constant in the fixed frame xyz.

For $A > C$, p and r can always be expressed via q:

$$p = \pm\sqrt{a - bq^2}, \qquad r = \pm\sqrt{c - dq^2}, \tag{17}$$

with the constants a, b, c, and d expressible in terms of the initial parameters of the problem. Substituting (17) into the equation for q yields

$$bq'_t \pm (A - C)\sqrt{(a - bq^2)(c - dq^2)} = 0.$$

Integrating gives the solution in implicit form

$$t - t_0 = \pm\frac{B}{A - C}\int_0^q \frac{dq}{\sqrt{(a - bq^2)(c - dq^2)}}.$$

Effectively, the problem is reduced to the inversion of an elliptic integral, resulting in expressions of $p(t)$, $q(t)$, and $r(t)$ in terms of elliptic functions of time.

To solve the kinematic equations, it is convenient to direct the z-axis of the fixed frame along the constant angular momentum **K**, in which case we obtain

$$\begin{aligned}
&K_1 = K\sin\theta\sin\varphi,\\
&K_2 = K\sin\theta\cos\varphi, \quad\Longrightarrow\quad \cos\theta(t) = \frac{Cr(t)}{K}, \quad\Longrightarrow\quad \cos\varphi(t) = \frac{Bq(t)}{K\sin\theta(t)},\\
&K_3 = K\cos\theta,\\
&\psi(t) = \psi_0 + \int_0^t \frac{p(t)\sin\varphi(t) + q(t)\cos\varphi(t)}{\sin\theta(t)}\,dt.
\end{aligned}$$

This solution is known to have geometric interpretations suggested by Poinsot and MacCullagh (e.g., see Zhuravlev (1996), Borisov and Mamaev (2001), and Teodorescu (2009)).

► Lagrange's case.

The body, which is in a homogeneous gravitational field, is dynamically symmetric and its center of mass lies on the dynamic symmetry axis (the ζ-azis). Then, in equations (1)–(3), one should set

$$A = B, \quad \mathbf{M} = (M_1, M_2, M_3)^T = mg(\mathbf{r}\times\boldsymbol{\gamma}). \tag{18}$$

The easiest way to integrate the equations is to use the Euler angles. System (1)–(6) with (18) admits the following three first integrals:

$K_3 = \text{const}$ (conservation of the angular momentum projection onto the ζ-axis);

$(\mathbf{K}\cdot\boldsymbol{\gamma}) = K_1\gamma_1 + K_2\gamma_2 + K_3\gamma_3 = C_1$ (conservation of the angular momentum projection onto the direction of $\boldsymbol{\gamma}$);

$$\frac{h}{2}(\theta'_t)^2 + \frac{K_3^2}{2C} + \frac{(C_1 - K_3\cos\theta)^2}{2A\sin^\theta} + mgl\cos\theta = h = \text{const} \quad \text{(energy integral)}.$$

The availability of these integrals reduces the problem to the equation

$$(\theta'_t)^2 = 2h - \frac{K_3^2}{C} - \frac{(C_1 - K_3\cos\theta)^2}{\sin^\theta} - 2\cos\theta,$$

which is obtained if one sets $A = mgl = 1$ (without loss of generality). With the change of variable $u = \cos\theta$, this equation can be reduced to the elliptic quadrature

$$u'_t = \sqrt{R(u)},$$
$$R(u) = 2(h_1 - u)(1 - u^2) - (C_1 - K_3u)^2, \quad h_1 = h - \frac{K_3^2}{2C}.$$

To determine the full motion of the system, one has to integrate the following two equations:

$$\psi'_t = \frac{C_1 - K_3u}{1 - u^2}, \quad \varphi'_t = \Big(\frac{1}{C} - 1\Big)K_3 + \frac{C_1 - K_3u}{1 - u^2}.$$

Depending on the initial data and specific parameters of the problem, the solution defines four types of motion, in one of which the axis of the top asymptotically tends to a vertical positions.

► **Sofia Kovalevskaya's case.**

The body is dynamically symmetric with $A = B$ and, in addition, the condition $A = 2C$ holds. The center of mass lies in the equatorial plane of the inertia ellipsoid (its center at the fixed point) and its position in the frame $\xi\eta\zeta$ is $\mathbf{r} = (L, 0, 0)^T$. The system is in a homogeneous gravitational field, so that $\mathbf{M} = mg(\mathbf{r} \times \boldsymbol{\gamma})$. For simplicity, we assume that $A = 1$, $mg = 1$, and $L = 1$.

This case is much more complex than the previous two, both in the way how the equations are integrated and from the viewpoint of the qualitative analysis of the motion. The Euler equations (1)–(6) admit the following three first integrals:

$$(\mathbf{K}\cdot\boldsymbol{\gamma}) = K_1\gamma_1 + K_2\gamma_2 + K_3\gamma_3 = c = \text{const} \quad \text{(conservation of the angular momentum projection onto the vertical)};$$

$$\tfrac{1}{2}(K_1^2 + K_2^2 + K_3^2) - L\gamma_1 = h = \text{const} \quad \text{(energy integral)};$$

$$\left(\frac{K_1^2 + K_2^2}{2} + \gamma_1 x\right)^2 + (K_1K_2 + \gamma_2 x)^2 = k = \text{const} \quad \text{(integral having no clear physical meaning)}.$$

The equations of motion are integrated using Kovalevskaya's variables (s_1, s_2), which are defined as follows:

$$s_1 = \frac{R - \sqrt{R_1R_2}}{2(z_1 - z_2)^2}, \quad s_2 = \frac{R + \sqrt{R_1R_2}}{2(z_1 - z_2)^2},$$
$$z_1 = K_\xi + iK_\eta, \quad z_2 = K_\xi - iK_\eta, \quad i^2 = -1,$$
$$R = R(z_1, z_2) = \tfrac{1}{4}z_1^2z_2^2 - \tfrac{1}{2}h(z_1^2 + z_2^2) + c(z_1 + z_2) + \tfrac{1}{4}k^2 - 1,$$
$$R_1 = R(z_1, z_1), \quad R_2 = R(z_2, z_2).$$

In these variables, the equations of motion become

$$\frac{ds_1}{dt} = \frac{\sqrt{P(s_1)}}{s_1 - s_2}, \quad \frac{ds_2}{dt} = \frac{\sqrt{P(s_2)}}{s_2 - s_1}, \tag{19}$$

where

$$P(s) = \left[(2s + \tfrac{1}{2}h)^2 - \tfrac{1}{16}k^2\right]\left[4s^3 + 2hs^2 + \tfrac{1}{16}(4h^2 - k^2 + 4)s + \tfrac{1}{16}c^2\right].$$

By eliminating t, system (19) can be reduced to a separable equation, which is easy to integrate. As a results, equations (19) also convert into separable equations.

⊙ *Literature for Section 18*: C. G. J. Jacobi (1884), S. Kowalewsky (1889, 1890), J. L. Lagrange (1889), F. Klein and A. Sommerfeld (1965), E. Kamke (1977), J. R. Ray and J. L. Reid (1979), V. F. Zhuravlev (1996), A. V. Borisov and I. S. Mamaev (2001), A. P. Markeev (2001), V. Ph. Zhuravlev (2001), F. R. Gantmakher (2002), D. M. Klimov and V. Ph. Zhuravlev (2002), A. D. Polyanin (2006), A. D. Polyanin and A. V. Manzhirov (2007), P. P. Teodorescu (2009).