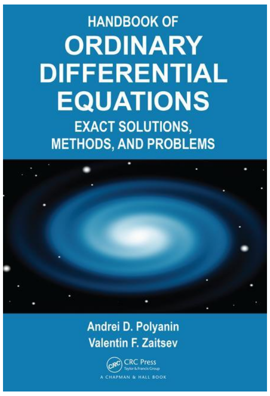

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

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Chapter 2: Methods for Second?Order Linear Differential Equations

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Chapter 2

Methods for Second-Order Linear Differential Equations

2.1 Homogeneous Linear Equations

2.1.1 Formulas for the General Solution. Wronskian Determinant

► **General form of a homogeneous linear equation.**

Consider a second-order homogeneous linear equation in the general form

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = 0. \tag{2.1.1.1}$$

The *trivial solution*, $y = 0$, is a particular solution of the homogeneous linear equation.

► **Two particular solutions are known.**

Let $y_1(x)$, $y_2(x)$ be a fundamental system of solutions (nontrivial linearly independent particular solutions) of equation (2.1.1.1). Then the general solution is given by

$$y = C_1 y_1(x) + C_2 y_2(x), \tag{2.1.1.2}$$

where C_1 and C_2 are arbitrary constants.

► **One particular solution is known.**

Let $y_1 = y_1(x)$ be any nontrivial particular solution of equation (2.1.1.1). Then its general solution can be represented as

$$y = y_1\left(C_1 + C_2\int \frac{e^{-F}}{y_1^2}\,dx\right), \quad \text{where} \quad F = \int \frac{f_1}{f_2}\,dx. \tag{2.1.1.3}$$

► **General solution of an equation of the canonical form.**

Consider the equation

$$y''_{xx} + f(x)y = 0,$$

which is written in the canonical form; see Section 2.1.2 for the reduction of equations to this form. Let $y_1 = y_1(x)$ be any nontrivial partial solution of this equation. The general solution can be constructed by formula (2.1.1.3) with $F = 0$ or formula (2.1.1.2) in which

$$y_2(x) = y_1 \int \frac{[f(x) - 1][y_1^2 - (y_1')^2]}{[y_1^2 + (y_1')^2]^2}\,dx + \frac{y_1'}{y_1^2 + (y_1')^2}.$$

Here the prime denotes differentiation with respect to x. The last formula is suitable where y_1 vanishes at some points.

► Special properties of some solutions.

1°. Suppose $y = C_1 f(x)[g(x)]^a + C_2 f(x)[g(x)]^b$ is the general solution of the homogeneous linear equation with $a \neq b$, where a and b are free parameters. Then the function $y = C_1 f(x)[g(x)]^a + C_2 f(x)[g(x)]^a \ln g(x)$ will be the general solution of this equation with $a = b$.

2°. Suppose a particular solution of a homogeneous linear equation is obtained in the closed form $y = [f(x)]^a$, with this formula valid for $f(x) \geq 0$. If the equation makes sense in a range of x where $f(x) < 0$, then the function $y = |f(x)|^a$ will be a particular solution of the equation in that range.

► Constant-coefficient linear equation.

The second-order constant-coefficient linear equation

$$y''_{xx} + ay'_x + by = 0 \tag{2.1.1.4}$$

has the following fundamental system of solutions:

$$\begin{aligned}
&y_1(x) = \exp\left(-\tfrac{1}{2}ax\right)\sinh\left(\tfrac{1}{2}x\sqrt{a^2-4b}\right), && y_2(x) = \exp\left(-\tfrac{1}{2}ax\right)\cosh\left(\tfrac{1}{2}x\sqrt{a^2-4b}\right) && \text{if } a^2 > 4b;\\
&y_1(x) = \exp\left(-\tfrac{1}{2}ax\right)\sin\left(\tfrac{1}{2}x\sqrt{4b-a^2}\right), && y_2(x) = \exp\left(-\tfrac{1}{2}ax\right)\cos\left(\tfrac{1}{2}x\sqrt{4b-a^2}\right) && \text{if } a^2 < 4b;\\
&y_1(x) = \exp\left(-\tfrac{1}{2}ax\right), && y_2(x) = x\exp\left(-\tfrac{1}{2}ax\right) && \text{if } a^2 = 4b.
\end{aligned}$$

► Euler equation.

The *Euler equation*

$$x^2 y''_{xx} + axy'_x + by = 0$$

is reduced by the change of variable $x = ke^t$ $(k \neq 0)$ to the second-order constant-coefficient linear equation $y''_{tt} + (a-1)y'_t + by = 0$, see Eq. (2.1.1.4).

◆ *Solutions to some other second-order linear equations can be found in Section 14.1.*

► Wronskian determinant and Liouville's formula.

The *Wronskian determinant* (or *Wronskian*) is defined by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv y_1(y_2)'_x - y_2(y_1)'_x,$$

where $y_1(x)$, $y_2(x)$ is a fundamental system of solutions of equation (2.1.1.1).

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Liouville's formula:

$$W(x) = W(x_0)\exp\biggl[-\int_{x_0}^{x}\frac{f_1(t)}{f_2(t)}\,dt\biggr].$$

2.1.2 Factorization and Some Transformations

► Factorization.

1°. Let $y_1 = y_1(x)$ be any nontrivial particular solution of equation (2.1.1.1). Then the equation can be factored as

$$L_2L_1y = 0, \tag{2.1.2.1}$$

where

$$L_1 = y_1\frac{d}{dx} - y_1', \qquad L_2 = \frac{1}{y_1}\biggl(\frac{d}{dx} + \frac{f_1}{f_2}\biggr).$$

2°. Equation (2.1.1.1) also admits a more general factorization in the form (2.1.2.1) with

$$L_1 = \frac{1}{\psi}\biggl(\frac{d}{dx} - \frac{y_1'}{y_1}\biggr), \qquad L_2 = \psi\biggl(\frac{d}{dx} + \frac{y_1'}{y_1} + \frac{\psi'}{\psi} + \frac{f_1}{f_2}\biggr),$$

where $y_1 = y_1(x)$ is any nontrivial particular solution of the equation and $\psi = \psi(x)$ is an arbitrary function (the special case $\psi = 1/y_1$ coincides with Item 1°).

Remark 2.1. The factorization (2.1.2.1) of equation (2.1.1.1), with L_1 and L_2 being some first-order differential operators, is equivalent in complexity to seeking a nontrivial particular solution of the equation.

► Reduction to the canonical form.

1°. The substitution

$$y = u(x)\exp\biggl(-\frac{1}{2}\int\frac{f_1}{f_2}\,dx\biggr) \tag{2.1.2.2}$$

brings equation (2.1.1.1) to the canonical (or normal) form

$$u''_{xx} + f(x)u = 0, \qquad \text{where} \quad f = \frac{f_0}{f_2} - \frac{1}{4}\biggl(\frac{f_1}{f_2}\biggr)^2 - \frac{1}{2}\biggl(\frac{f_1}{f_2}\biggr)'_x. \tag{2.1.2.3}$$

2°. The substitution (2.1.2.2) is a special case of the more general transformation (φ is an arbitrary function)

$$x = \varphi(\xi), \quad y = u(\xi)\sqrt{|\varphi'_\xi(\xi)|}\,\exp\biggl(-\frac{1}{2}\int\frac{f_1(\varphi)}{f_2(\varphi)}\,d\varphi\biggr),$$

which also brings the original equation to the canonical form.

► Reduction to the Riccati equation.

The substitution $u = y'_x/y$ brings the second-order homogeneous linear equation (2.1.1.1) to the Riccati equation:

$$f_2(x)u'_x + f_2(x)u^2 + f_1(x)u + f_0(x) = 0,$$

which is discussed in Section 1.4.

▶ **Reduction to a constant-coefficient equation (a special case).**

Let $f_2 = 1$, $f_0 \neq 0$, and the condition

$$\frac{1}{|f_0|}\frac{d}{dx}\sqrt{|f_0|} + \frac{f_1}{\sqrt{|f_0|}} = a = \text{const}$$

be satisfied. Then the substitution $\xi = \int \sqrt{|f_0|}\,dx$ leads to a constant-coefficient linear equation,

$$y''_{\xi\xi} + ay'_{\xi} + y \operatorname{sign} f_0 = 0.$$

⊙ *Literature for Section 2.1*: G. M. Murphy (1960), E. Kamke (1977), D. Zwillinger (1997), S. Yu. Dobrokhotov (1998), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

2.2 Nonhomogeneous Linear Equations

2.2.1 Existence Theorem. Kummer–Liouville Transformation

▶ **Existence and uniqueness theorem.**

A second-order nonhomogeneous linear equation has the form

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x). \tag{2.2.1.1}$$

EXISTENCE AND UNIQUENESS THEOREM. *On an open interval $a < x < b$, let the functions f_2, f_1, f_0, and g be continuous and $f_2 \neq 0$. Also let*

$$y(x_0) = A, \quad y'_x(x_0) = B$$

be arbitrary initial conditions, where x_0 is any point such that $a < x_0 < b$, and A and B are arbitrary prescribed numbers. Then a solution of equation (2.2.1.1) exists and is unique. This solutions is defined for all $x \in (a, b)$.

▶ **Kummer–Liouville transformation.**

The transformation

$$x = \alpha(t), \quad y = \beta(t)z + \gamma(t), \tag{2.2.1.2}$$

where $\alpha(t)$, $\beta(t)$, and $\gamma(t)$ are arbitrary sufficiently smooth functions ($\beta \not\equiv 0$), takes any linear differential equation for $y(x)$ to a linear equation for $z = z(t)$. In the special case $\gamma \equiv 0$, a homogeneous equation is transformed to a homogeneous one.

Special cases of transformation (2.2.1.2) are widely used to simplify second- and higher-order linear differential equations.

2.2.2 Formulas for the General Solution

▶ **Representation of the general solution as the sum of two solutions.**

The general solution of the nonhomogeneous linear equation (2.2.1.1) is the sum of the general solution of the corresponding homogeneous equation (2.1.1.1) and any particular solution of the nonhomogeneous equation (2.2.1.1).

► **Two particular solutions are known.**

Let $y_1 = y_1(x)$, $y_2 = y_2(x)$ be a fundamental system of solutions of the corresponding homogeneous equation, with $g \equiv 0$. Then the general solution of equation (2.2.1.1) can be represented as

$$y = C_1 y_1 + C_2 y_2 + y_2 \int y_1 \frac{g}{f_2} \frac{dx}{W} - y_1 \int y_2 \frac{g}{f_2} \frac{dx}{W}, \qquad (2.2.2.1)$$

where $W = y_1 (y_2)'_x - y_2 (y_1)'_x$ is the Wronskian determinant.

► **One particular solution is known.**

Given a nontrivial particular solution $y_1 = y_1(x)$ of the homogeneous equation (with $g \equiv 0$), a second particular solution $y_2 = y_2(x)$ can be calculated from the formula

$$y_2 = y_1 \int \frac{e^{-F}}{y_1^2} \, dx, \qquad \text{where} \quad F = \int \frac{f_1}{f_2} \, dx, \quad W = e^{-F}. \qquad (2.2.2.2)$$

Then the general solution of equation (2.2.1.1) can be constructed by (2.2.2.1).

► **A property of nonhomogeneous linear ODEs.**

Let $\bar{y}_1$ and $\bar{y}_2$ be respective solutions of the nonhomogeneous linear differential equations $\mathrm{L}[\bar{y}_1] = g_1(x)$ and $\mathrm{L}[\bar{y}_2] = g_2(x)$, which have the same left-hand side but different right-hand sides, where $\mathrm{L}[y]$ is the left-hand side of equation (2.2.1.1). Then the function $\bar{y} = \bar{y}_1 + \bar{y}_2$ is a solution of the equation $\mathrm{L}[\bar{y}] = g_1(x) + g_2(x)$.

⊙ *Literature for Section 2.2*: G. M. Murphy (1960), E. Kamke (1977), D. Zwillinger (1997), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

2.3 Representation of Solutions as a Series in the Independent Variable

2.3.1 Equation Coefficients are Representable in the Ordinary Power Series Form

Let us consider a homogeneous linear differential equation of the general form

$$y''_{xx} + f(x) y'_x + g(x) y = 0. \qquad (2.3.1.1)$$

Assume that the functions $f(x)$ and $g(x)$ are representable, in the vicinity of a point $x = x_0$, in the power series form,

$$f(x) = \sum_{n=0}^{\infty} A_n (x - x_0)^n, \qquad g(x) = \sum_{n=0}^{\infty} B_n (x - x_0)^n, \qquad (2.3.1.2)$$

on the interval $|x - x_0| < R$, where R stands for the minimum radius of convergence of the two series in (2.3.1.2). In this case, the point $x = x_0$ is referred to as an *ordinary point*, and equation (2.3.1.1) possesses two linearly independent solutions of the form

$$y_1(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \qquad y_2(x) = \sum_{n=0}^{\infty} b_n (x - x_0)^n. \tag{2.3.1.3}$$

The coefficients a_n and b_n are determined by substituting the series (2.3.1.2) into equation (2.3.1.1) followed by extracting the coefficients of like powers of $(x - x_0)$.*

2.3.2 Equation Coefficients Have Poles at Some Point

Assume that the functions $f(x)$ and $g(x)$ are representable, in the vicinity of a point $x = x_0$, in the form

$$f(x) = \sum_{n=-1}^{\infty} A_n (x - x_0)^n, \qquad g(x) = \sum_{n=-2}^{\infty} B_n (x - x_0)^n, \tag{2.3.2.1}$$

on the interval $|x - x_0| < R$. In this case, the point $x = x_0$ is referred to as a *regular singular point*.

Let λ_1 and λ_2 be roots of the quadratic equation

$$\lambda^2 + (A_{-1} - 1)\lambda + B_{-2} = 0, \tag{2.3.2.2}$$

where A_{-1} and B_{-2} are the leading terms in formulas (2.3.2.1) at $x \to x_0$. There are three cases, depending on the values of the exponents of the singularity.

1°. *Case $\lambda_1 \neq \lambda_2$ and $\lambda_1 - \lambda_2$ is not an integer.*

Equation (2.3.1.1) has two linearly independent solutions of the form

$$\begin{aligned} y_1(x) &= |x - x_0|^{\lambda_1} \left[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\right], \\ y_2(x) &= |x - x_0|^{\lambda_2} \left[1 + \sum_{n=1}^{\infty} b_n (x - x_0)^n\right]. \end{aligned} \tag{2.3.2.3}$$

2°. *Case $\lambda_1 = \lambda_2 = \lambda$.*

Equation (2.3.1.1) possesses two linearly independent solutions:

$$\begin{aligned} y_1(x) &= |x - x_0|^{\lambda} \left[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\right], \\ y_2(x) &= y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda} \sum_{n=0}^{\infty} b_n (x - x_0)^n. \end{aligned} \tag{2.3.2.4}$$

3°. *Case $\lambda_1 = \lambda_2 + N$, where N is a positive integer.*

*Prior to that, the terms containing the same powers $(x - x_0)^k$, $k = 0, 1, \ldots$, should be collected.

Equation (2.3.1.1) has two linearly independent solutions of the form

$$y_1(x) = |x - x_0|^{\lambda_1}\Big[1 + \sum_{n=1}^{\infty} a_n (x - x_0)^n\Big],$$
$$y_2(x) = k y_1(x) \ln|x - x_0| + |x - x_0|^{\lambda_2} \sum_{n=0}^{\infty} b_n (x - x_0)^n, \tag{2.3.2.5}$$

where k is a constant to be determined (it may be equal to zero). If $k \neq 0$, then we can set $k = 1$ without loss of generality.

To construct the solution in each of the three cases, the following procedure should be performed: substitute the above expressions of y_1 and y_2 into the original equation (2.3.1.1) and equate the coefficients of $(x - x_0)^n$ and $(x - x_0)^n \ln|x - x_0|$ for different values of n to obtain recurrence relations for the unknown coefficients. From these recurrence relations the solution sought can be found.

Example 2.1. The Bessel equation

$$x^2 y''_{xx} + x y'_x + (x^2 - \nu^2) y = 0 \tag{2.3.2.6}$$

is a special case of equation (2.3.1.1) with the functions of the form (2.3.2.1), where

$$f(x) = \frac{1}{x}, \quad g(x) = -\frac{\nu^2}{x^2} + 1, \quad x_0 = 0.$$

Therefore $A_{-1} = 1$ and $B_{-2} = -\nu^2$, and the quadratic equation (2.3.2.2) has the form

$$\lambda^2 - \nu^2 = 0. \tag{2.3.2.7}$$

The roots of the equation are $\lambda_1 = \nu$ and $\lambda_2 = -\nu$.

1°. If $\lambda_1 - \lambda_2 = 2\nu$ is not an arbitrary integer, then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.3).

2°. If $\nu = \lambda_1 = \lambda_2 = 0$, then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.4).

3°. If $\lambda_1 - \lambda_2 = 2\nu$ is an arbitrary integer, then there are two cases, depending on the values ν.

3.1. Case ν is an arbitrary integer (i.e., $\lambda_1 - \lambda_2$ is an even number). Then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.5) with $k = 1$.

3.2. Case $\nu = n + \frac{1}{2}$, where $n = 0, 1, 2, \ldots$ (i.e., $\lambda_1 - \lambda_2$ is an odd number). Then equation (2.3.2.6) has two linearly independent solutions of the form (2.3.2.5) with $k = 0$.

For more detailed information on solutions to the Bessel equation (2.3.2.6), see also Section 14.1.2 (Eq. 126) and Section S4.6.

⊙ *Literature for Section 2.3*: G. M. Murphy (1960), E. Kamke (1977), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003).

2.4 Asymptotic Solutions

This section presents asymptotic solutions, as $\varepsilon \to 0$ ($\varepsilon > 0$), of some second-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being real.

2.4.1 Equations Not Containing y'_x

► **Leading asymptotic terms.**

Consider the equation

$$\varepsilon^2 y''_{xx} - f(x)y = 0 \tag{2.4.1.1}$$

on a closed interval $a \le x \le b$.

Case 1. With the condition $f \neq 0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to 0$, are given by the formulas

$$y_1 = f^{-1/4}\exp\Bigl(-\frac{1}{\varepsilon}\int\sqrt{f}\,dx\Bigr),\qquad y_2 = f^{-1/4}\exp\Bigl(\frac{1}{\varepsilon}\int\sqrt{f}\,dx\Bigr)\qquad \text{if } f>0,$$

$$y_1 = (-f)^{-1/4}\cos\Bigl(\frac{1}{\varepsilon}\int\sqrt{-f}\,dx\Bigr),\quad y_2 = (-f)^{-1/4}\sin\Bigl(\frac{1}{\varepsilon}\int\sqrt{-f}\,dx\Bigr)\quad \text{if } f<0.$$

Case 2. Discuss the asymptotic solution of equation (2.4.1.1) in the vicinity of the point $x = x_0$, where function $f(x)$ vanishes, $f(x_0) = 0$ (such a point is referred to as a *transition point*). We assume that the function f can be presented in the form

$$f(x) = (x_0 - x)\psi(x), \qquad \text{where} \quad \psi(x) > 0.$$

In this case, the fundamental solutions, as $\varepsilon \to 0$, are described by three different formulas:

$$y_1 = \begin{cases} \dfrac{1}{|f(x)|^{1/4}}\sin\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_{x_0}^{x}\sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Bigr] & \text{if } x - x_0 \ge \delta,\\ \dfrac{\sqrt{\pi}}{[\varepsilon\psi(x_0)]^{1/6}}\,\mathrm{Ai}(z) & \text{if } |x - x_0| \le \delta,\\ \dfrac{1}{2[f(x)]^{1/4}}\exp\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_{x}^{x_0}\sqrt{f(x)}\,dx\Bigr] & \text{if } x_0 - x \ge \delta, \end{cases}$$

$$y_2 = \begin{cases} \dfrac{1}{|f(x)|^{1/4}}\cos\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_{x_0}^{x}\sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Bigr] & \text{if } x - x_0 \ge \delta,\\ \dfrac{\sqrt{\pi}}{[\varepsilon\psi(x_0)]^{1/6}}\,\mathrm{Bi}(z) & \text{if } |x - x_0| \le \delta,\\ \dfrac{1}{[f(x)]^{1/4}}\exp\Bigl[\dfrac{1}{\varepsilon}\displaystyle\int_{x}^{x_0}\sqrt{f(x)}\,dx\Bigr] & \text{if } x_0 - x \ge \delta, \end{cases}$$

where $\mathrm{Ai}(z)$ and $\mathrm{Bi}(z)$ are the Airy functions of the first and second kind, respectively (see Section S4.8), $z = \varepsilon^{-2/3}[\psi(x_0)]^{1/3}(x_0 - x)$, and $\delta = O(\varepsilon^{2/3})$.

► **Two-term asymptotic expansions.**

The two-term asymptotic expansions of the solution of equation (2.4.1.1) with $f > 0$, as $\varepsilon \to 0$, on a closed interval $a \le x \le b$, has the form

$$y_1 = f^{-1/4}\exp\Bigl(-\frac{1}{\varepsilon}\int_{x_0}^{x}\sqrt{f}\,dx\Bigr)\Bigl\{1 - \varepsilon I(x) + O(\varepsilon^2)\Bigr\},$$

$$y_2 = f^{-1/4}\exp\Bigl(\frac{1}{\varepsilon}\int_{x_0}^{x}\sqrt{f}\,dx\Bigr)\Bigl\{1 + \varepsilon I(x) + O(\varepsilon^2)\Bigr\}, \tag{2.4.1.2}$$

$$I(x) = \int_{x_0}^{x}\Bigl[\frac{1}{8}\frac{f''_{xx}}{f^{3/2}} - \frac{5}{32}\frac{(f'_x)^2}{f^{5/2}}\Bigr]dx,$$

where x_0 is an arbitrary number satisfying the inequality $a \le x_0 \le b$.

The asymptotic expansions of the fundamental system of solutions of equation (2.4.1.1) with $f < 0$ are derived by separating the real and imaginary parts in either formula (2.4.1.2).

► **Equations of the special form.**

Consider the equation

$$\varepsilon^2 y''_{xx} - x^{m-2} f(x) y = 0 \tag{2.4.1.3}$$

on a closed interval $a \le x \le b$, where $a < 0$ and $b > 0$, under the conditions that m is a positive integer and $f(x) \ne 0$. In this case, the leading term of the asymptotic solution, as $\varepsilon \to 0$, in the vicinity of the point $x = 0$ is expressed in terms of a simpler model equation, which results from substituting the function $f(x)$ in equation (2.4.1.3) by the constant $f(0)$ (the solution of the model equation is expressed in terms of the Bessel functions of order $1/m$).

We specify below formulas by which the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2.4.1.3) with $a < x < 0$ and $0 < x < b$ are related (excluding a small vicinity of the point $x = 0$). Three different cases can be extracted.

$1°$. Let m be an even integer and $f(x) > 0$. Then,

$$y_1 = \begin{cases} [f(x)]^{-1/4} \exp\Big[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Big] & \text{if } x < 0, \\ k^{-1}[f(x)]^{-1/4} \exp\Big[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Big] & \text{if } x > 0, \end{cases}$$

$$y_2 = \begin{cases} [f(x)]^{-1/4} \exp\Big[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Big] & \text{if } x < 0, \\ k[f(x)]^{-1/4} \exp\Big[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Big] & \text{if } x > 0, \end{cases}$$

where $f = f(x)$, $k = \sin\Big(\dfrac{\pi}{m}\Big)$.

$2°$. Let m be an even integer and $f(x) < 0$. Then,

$$y_1 = \begin{cases} |f(x)|^{-1/4} \cos\Big[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Big] & \text{if } x < 0, \\ k^{-1}|f(x)|^{-1/4} \cos\Big[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\Big] & \text{if } x > 0, \end{cases}$$

$$y_2 = \begin{cases} |f(x)|^{-1/4} \cos\Big[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\Big] & \text{if } x < 0, \\ k|f(x)|^{-1/4} \cos\Big[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Big] & \text{if } x > 0, \end{cases}$$

where $f = f(x)$, $k = \tan\Big(\dfrac{\pi}{2m}\Big)$.

$3°$. Let m be an odd integer. Then,

$$y_1 = \begin{cases} |f(x)|^{-1/4} \cos\Big[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx + \dfrac{\pi}{4}\Big] & \text{if } x < 0, \\ \dfrac{1}{2}k^{-1}[f(x)]^{-1/4} \exp\Big[\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Big] & \text{if } x > 0, \end{cases}$$

$$y_2 = \begin{cases} |f(x)|^{-1/4}\cos\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{|f(x)|}\,dx - \dfrac{\pi}{4}\Bigr] & \text{if } x < 0, \\ k[f(x)]^{-1/4}\exp\Bigl[-\dfrac{1}{\varepsilon}\displaystyle\int_0^x \sqrt{f(x)}\,dx\Bigr] & \text{if } x > 0, \end{cases}$$

where $f = f(x)$, $k = \sin\Bigl(\dfrac{\pi}{2m}\Bigr)$.

► **Equation coefficients are dependent on ε.**

Consider an equation of the form

$$\varepsilon^2 y''_{xx} - f(x,\varepsilon)y = 0 \tag{2.4.1.4}$$

on a closed interval $a \le x \le b$ under the condition that $f \ne 0$. Assume that the following asymptotic relation holds:

$$f(x,\varepsilon) = \sum_{k=0}^{\infty} f_k(x)\varepsilon^k, \qquad \varepsilon \to 0.$$

Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2.4.1.4) are given by the formulas

$$y_1 = f_0^{-1/4}(x)\exp\Bigl[-\frac{1}{\varepsilon}\int\sqrt{f_0(x)}\,dx + \frac{1}{2}\int\frac{f_1(x)}{\sqrt{f_0(x)}}\,dx\Bigr]\bigl[1 + O(\varepsilon)\bigr],$$
$$y_2 = f_0^{-1/4}(x)\exp\Bigl[\frac{1}{\varepsilon}\int\sqrt{f_0(x)}\,dx + \frac{1}{2}\int\frac{f_1(x)}{\sqrt{f_0(x)}}\,dx\Bigr]\bigl[1 + O(\varepsilon)\bigr].$$

2.4.2 Equations Containing y'_x

► **Equations of a special form.**

1°. Consider an equation of the form

$$\varepsilon y''_{xx} + g(x)y'_x + f(x)y = 0$$

on a closed interval $0 \le x \le 1$. With $g(x) > 0$, the asymptotic solution of this equation, satisfying the boundary conditions $y(0) = C_1$ and $y(1) = C_2$, can be represented in the form

$$y = (C_1 - kC_2)\exp\bigl[-\varepsilon^{-1}g(0)x\bigr] + C_2\exp\Bigl[\int_x^1 \frac{f(x)}{g(x)}\,dx\Bigr] + O(\varepsilon),$$

where $k = \exp\Bigl[\displaystyle\int_0^1 \frac{f(x)}{g(x)}\,dx\Bigr]$.

2°. Now let us take a look at an equation of the form

$$\varepsilon^2 y''_{xx} + \varepsilon g(x)y'_x + f(x)y = 0 \tag{2.4.2.1}$$

on a closed interval $a \le x \le b$. Assume

$$D(x) \equiv [g(x)]^2 - 4f(x) \ne 0.$$

Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (2.4.2.1), as $\varepsilon \to 0$, are expressed by

$$y_1 = |D(x)|^{-1/4} \exp\Big[-\frac{1}{2\varepsilon}\int \sqrt{D(x)}\,dx - \frac{1}{2}\int \frac{g'_x(x)}{\sqrt{D(x)}}\,dx\Big]\big[1+O(\varepsilon)\big],$$

$$y_2 = |D(x)|^{-1/4} \exp\Big[\frac{1}{2\varepsilon}\int \sqrt{D(x)}\,dx - \frac{1}{2}\int \frac{g'_x(x)}{\sqrt{D(x)}}\,dx\Big]\big[1+O(\varepsilon)\big].$$

► **Equations of the general form.**

The more general equation

$$\varepsilon^2 y''_{xx} + \varepsilon g(x,\varepsilon) y'_x + f(x,\varepsilon) y = 0$$

is reducible, with the aid of the substitution $y = w\exp\Big(-\frac{1}{2\varepsilon}\int g\,dx\Big)$, to an equation of the form (2.4.1.4),

$$\varepsilon^2 w''_{xx} + \big(f - \tfrac{1}{4}g^2 - \tfrac{1}{2}\varepsilon g'_x\big)w = 0,$$

which can be solved using the asymptotic formulas given above.

⊙ *Literature for Section 2.4*: W. Wasov (1965), F. W. J. Olver (1974), A. H. Nayfeh (1973, 1981), M. V. Fedoryuk (1993), A. D. Polyanin and V. F. Zaitsev (2003).

2.5 Boundary Value Problems. Green’s Function

2.5.1 First, Second, Third, and Some Other Boundary Value Problems

We consider the second-order nonhomogeneous linear differential equation

$$f_2(x)y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x) \qquad (2.5.1.1)$$

on a bounded interval $x_1 < x < x_2$. We assume that $f_2(x) \neq 0$.

► **First boundary value problem.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *first-type boundary conditions* (or *Dirichlet conditions*)

$$y = a_1 \quad \text{at} \quad x = x_1, \qquad y = a_2 \quad \text{at} \quad x = x_2. \qquad (2.5.1.2)$$

(The values of the unknown are prescribed at two distinct points x_1 and x_2.)

► **Second boundary value problem.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *second-type boundary conditions* (or *Neumann boundary conditions*)

$$y'_x = a_1 \quad \text{at} \quad x = x_1, \qquad y'_x = a_2 \quad \text{at} \quad x = x_2. \qquad (2.5.1.3)$$

(The values of the derivative of the unknown are prescribed at two distinct points x_1 and x_2.)

▶ **Third boundary value problem.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *third-type boundary conditions* (or *Robin boundary conditions*)

$$\begin{aligned} y'_x - k_1 y = a_1 \quad &\text{at} \quad x = x_1, \\ y'_x + k_2 y = a_2 \quad &\text{at} \quad x = x_2. \end{aligned} \tag{2.5.1.4}$$

▶ **Mixed boundary value problems.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying the *mixed-type boundary conditions*

$$y = a_1 \quad \text{at} \quad x = x_1, \qquad y'_x = a_2 \quad \text{at} \quad x = x_2. \tag{2.5.1.5}$$

(The unknown itself is prescribed at one point, and its derivative at another point.)

Other mixed boundary value problem: Find a solution of equation (2.5.1.1) satisfying the boundary conditions

$$y'_x = a_1 \quad \text{at} \quad x = x_1, \qquad y = a_2 \quad \text{at} \quad x = x_2. \tag{2.5.1.6}$$

Boundary conditions (2.5.1.2), (2.5.1.3), (2.5.1.4), (2.5.1.5) and (2.5.1.6) are called *homogeneous* if $a_1 = a_2 = 0$.

▶ **Problems with boundary conditions involving the values of the unknown (or/and its derivative) at both endpoints of the interval.**

Sometimes, one has to deal with problems whose boundary conditions involve the values of the unknown (or/and its derivative) at both ends of the interval.

Example 2.2. Here are two examples of such boundary conditions:

$$y(x_1) = a_1, \qquad y(x_2) + k y'_x(x_1) = a_2$$

and

$$y(x_1) + k y(x_2) = a_1, \qquad y'_x(x_2) = a_2.$$

▶ **Problems with a nonlocal condition.**

Statement of the problem: Find a solution of equation (2.5.1.1) satisfying a boundary condition of the first kind at x_1 (see the first boundary condition in (2.5.1.2)) and the nonlocal condition

$$\int_{x_1}^{x_2} h(x) y(x)\, dx = b, \tag{2.5.1.7}$$

where $h(x)$ is a given function and b is a given number.

Condition (2.5.1.7) can be interpreted as a conservation law (with weight h) for the unknown. In particular, if

$$h(x) = \frac{1}{x_2 - x_1} = \text{const}$$

condition (2.5.1.7) defines the integral mean of the unknown.

The nonlocal condition (2.5.1.7) can be set together with a boundary condition of the second or third kind at one of the ends of the interval where the solution is sought.

► Boundary value problems with a degeneration at the boundary.

Let us look at the situation where the coefficient of the highest derivative in (2.5.1.1) becomes zero at the left endpoint:

$$f_2(x_1) = 0, \qquad f_1^2(x_1) + f_0^2(x_1) \neq 0.$$

In this case, one of the solutions to equation (2.5.1.1) may tend to infinity as $x \to x_1$; in order to establish this fact, one has to find the leading asymptotic terms of the fundamental system of solutions as $x \to x_1$). If one of the solutions is unbounded as $x \to x_1$, then for the problem to be well-posed, a boundedness condition for the solution needs to be set at the left endpoint:

$$|y| \neq \infty \quad \text{at} \quad x = x_1. \tag{2.5.1.8}$$

The boundary at the other end, $x = x_2$, can be any of the those listed above.

Example 2.3. Suppose the coefficients of equation (2.5.1.1) can be expanded in a Taylor series about $x = x_1$, so that

$$f_2(x) \simeq (x - x_1), \qquad f_1(x) \simeq b \neq 0, \qquad f_0(x) \simeq c \qquad (x \to x_1). \tag{2.5.1.9}$$

In view of (2.5.1.9), the leading asymptotic term of the (potentially) singular solution to equation (2.5.1.1) as $x \to x_1$ will be sought in the form

$$y \simeq (x - x_1)^\lambda. \tag{2.5.1.10}$$

(By virtue of the linearity of the equation, the solutions are determined up to a constant factor). Substituting (2.5.1.10) into ODE (2.5.1.1), taking into account (2.5.1.9), and dividing by $(x - x_1)^\lambda$, we obtain $\lambda(\lambda - 1 + b)(x - x_1)^{-1} + O(1) = 0$. For the left-hand side of this relation to be bounded as $x \to x_1$, we must set

$$\lambda(\lambda - 1 + b) = 0. \tag{2.5.1.11}$$

The zero root $\lambda = 0$ corresponds to a regular solution to equation (2.5.1.1), which does not have a singularity at $x = x_1$ and is expandable in a Taylor series in powers of $x - x_1$. The other root of the quadratic equation (2.5.1.11) is

$$\lambda = 1 - b. \tag{2.5.1.12}$$

If $b > 1$, then $\lambda < 0$; hence, the solution with the asymptotic behavior (2.5.1.10) is unbounded as $x \to x_1$. In this case, the boundedness condition (2.5.1.8) should be set at the left endpoint. If $b = 1$, equation (2.5.1.11) has a double root $\lambda = 0$, which determines a solution with a logarithmic singularity; in this case, a boundedness condition should also be set at the left endpoint of the interval $[x_1, x_2]$.

► Boundary value problems on an unbounded interval.

Consider equation (2.5.1.1) on the unbounded interval $x_1 < x < \infty$ (i.e., $x_2 = \infty$). Let one of the fundamental solutions of the equation tend to zero and be bounded as $x \to \infty$ and let the modulus of the other solution increase without bound. Then, a boundedness condition has to be set at the right end of the interval:

$$|y| \neq \infty \quad \text{as} \quad x \to \infty. \tag{2.5.1.13}$$

The boundary condition at the left endpoint, $x = x_1$, can be any of those listed previously.

Remark 2.2. If the bounded fundamental solution is monotonic for sufficiently large x, the equivalent condition

$$y'_x \to 0 \quad \text{as} \quad x \to \infty \tag{2.5.1.14}$$

can be used instead of (2.5.1.13).

Example 2.4. Let the coefficients of equation (2.5.1.1) be expandable in Taylor series as $x \to \infty$ and tend to constant quantities:

$$f_2(\infty) = a, \quad f_1(\infty) = b, \quad f_0(\infty) = c. \tag{2.5.1.15}$$

Then the qualitative behavior of the fundamental system of equations as $x \to \infty$ is determined by the roots of the characteristic equation

$$a\lambda^2 + b\lambda + c = 0,$$

which is obtained by substituting $y = e^{\lambda x}$ into equation (2.5.1.1), whose coefficients are replaced with their leading asymptotic terms (2.5.1.15).

Condition (2.5.1.14) (or (2.5.1.13)) must be set if either (i) $ac < 0$ or (ii) $c = 0$ and $ab > 0$.

2.5.2 Simplification of Boundary Conditions. Self-Adjoint Form of Equations

► **Simplification of boundary conditions.**

Nonhomogeneous boundary conditions of the first-, second-, third-, and mixed kinds set at the endpoints of a bounded interval $[x_1, x_2]$ can be reduced to homogeneous ones by the change of variable

$$z = A_2x^2 + A_1x + A_0 + y,$$

with the constants A_2, A_1, and A_0 selected using the method of undetermined coefficients.

Table 2.1 gives examples of such transformations.

TABLE 2.1
Simple transformations of the form $z = A_2x^2 + A_1x + A_0 + y$ that lead to homogeneous boundary conditions ($x_1 \le x \le x_2$)

No	Problem	Boundary conditions	Transformation
1	First boundary value problem	$y = a_1$ at $x = x_1$ $y = a_2$ at $x = x_2$	$z = y - \dfrac{a_2 - a_1}{x_2 - x_1}(x - x_1) - a_1$
2	Second boundary value problem	$y'_x = a_1$ at $x = x_1$ $y'_x = a_2$ at $x = x_2$	$z = y + \dfrac{a_1 - a_2}{2(x_2 - x_1)}x^2 + \dfrac{a_2x_1 - a_1x_2}{x_2 - x_1}x$
3	Mixed boundary value problem	$y = a_1$ at $x = x_1$ $y'_x = a_2$ at $x = x_2$	$z = y - a_2x + a_2x_1 - a_1$
4	Mixed boundary value problem	$y'_x = a_1$ at $x = x_1$ $y = a_2$ at $x = x_2$	$z = y - a_1x + a_1x_2 - a_2$

► **Reduction of a bounded interval to a unit interval.**

The interval $x_1 \le x \le x_2$ on which a boundary problem is defined can be reduced with the change of variable $x = x_1 + (x_2 - x_1)\bar{x}$ to the unit interval $0 \le \bar{x} \le 1$. Homogeneous boundary conditions of the first-, second-, third-, and mixed kinds remain homogeneous under this transformation.

► **Self-adjoint form of equations.**

On multiplying by $p(x) = \exp\Big[\int \frac{f_1(x)}{f_2(x)}\,dx\Big]$, one reduces equation (2.5.1.1) to the self-adjoint form:

$$[p(x)y'_x]'_x + q(x)y = r(x), \tag{2.5.2.1}$$

where $q(x) = f_0(x)p(x)/f_2(x)$ and $r(x) = g(x)(x)p(x)/f_2(x)$.

Hence, without loss of generality, we can further deal with equation (2.5.2.1) instead of (2.5.1.1). We assume that the functions p, p'_x, q, and r are continuous on the interval $x_1 \le x \le x_2$, and p is positive.

2.5.3 Green's and Modified Green's Functions. Representation Solutions via Green's or Modified Green's Functions

► **Green's function. Linear problems for nonhomogeneous equations.**

A *Green's function* of the first boundary value problem for equation (2.5.2.1) with homogeneous boundary conditions (2.5.1.2) is a function of two variables $G(x,\xi)$ that satisfies the following conditions:

1°. $G(x,\xi)$ is continuous in x for fixed ξ, with $x_1 \le x \le x_2$ and $x_1 \le \xi \le x_2$.

2°. $G(x,\xi)$ is a solution of the homogeneous equation (2.5.2.1), with $r = 0$, for all $x_1 < x < x_2$ exclusive of the point $x = \xi$.

3°. $G(x,\xi)$ satisfies the homogeneous boundary conditions $G(x_1,\xi) = G(x_2,\xi) = 0$.

4°. The derivative $G'_x(x,\xi)$ has a jump of $1/p(\xi)$ at the point $x = \xi$, that is,

$$G'_x(x,\xi)\big|_{x\to\xi,\,x>\xi} - G'_x(x,\xi)\big|_{x\to\xi,\,x<\xi} = \frac{1}{p(\xi)}.$$

For the second, third, and mixed boundary value problems, the Green's function is defined likewise except that in 3° the homogeneous boundary conditions (2.5.1.3), (2.5.1.4), and (2.5.1.5), with $a_1 = a_2 = 0$, are adopted, respectively.

The solution of the nonhomogeneous equation (2.5.2.1) subject to appropriate homogeneous boundary conditions is expressed in terms of the Green's function as follows:*

$$y(x) = \int_{x_1}^{x_2} G(x,\xi)r(\xi)\,d\xi. \tag{2.5.3.1}$$

*The homogeneous boundary value problem—with $r(x) = 0$ and $a_1 = a_2 = 0$—is assumed to have only the trivial solution.

▶ **Representation of the Green's function in terms of particular solutions.**

1°. We consider the first boundary value problem. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be linearly independent particular solutions of the homogeneous equation (2.5.2.1), with $r=0$, that satisfy the conditions

$$y_1(x_1) = 0, \quad y_2(x_2) = 0. \tag{2.5.3.2}$$

(Each of the solutions satisfies one of the homogeneous boundary conditions.)

The Green's function is expressed in terms of solutions of the homogeneous equation as follows:

$$G(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{p(\xi)W(\xi)} & \text{for } x_1 \le x \le \xi, \\ \dfrac{y_1(\xi)y_2(x)}{p(\xi)W(\xi)} & \text{for } \xi \le x \le x_2, \end{cases} \tag{2.5.3.3}$$

where $W(x) = y_1(x)y_2'(x) - y_1'(x)y_2(x)$ is the Wronskian determinant.

2°. Formula (2.5.3.3) can also be used to construct Green's functions for the second, third, and mixed boundary value problems. To this end, one should find two linearly independent solutions, $y_1 = y_1(x)$ and $y_2 = y_2(x)$, of the homogeneous equation with $r = 0$; the former satisfies the corresponding homogeneous boundary condition at $x = x_1$ and the latter satisfies the one at $x = x_2$ (see also the paragraph "Modified Green's function" below).

3°. The solution of the nonhomogeneous equation (2.5.2.1) subject to the second, third, and mixed homogeneous boundary conditions is also expressed in terms of an appropriate Green's function by formula (2.5.3.1).

4°. Table 2.2 contains the simplest examples of Green's functions $G(x,\xi)$ for some linear boundary value problems for ODEs of the form (2.5.2.1). In all these examples, $G(x,\xi) = G(\xi,x)$, and therefore the Green's function is specified only in the domain $x \le \xi$. For equations with the operator $L[y] = [f(x)y_x']_x'$, it is assumed that $f(x) > 0$ and $\varphi(x) = \displaystyle\int_0^x \frac{dt}{f(t)}$.

5°. Formula (2.5.3.3) can also be used to construct the Green's functions for boundary value problems when the equation has a singular point at the boundary (i.e., when $p(x)$ becomes zero at $x = x_1$ or/and $x = x_2$ or when $q(x)$ becomes infinite at these point). In such cases, the relevant boundary condition must be replaced with a boundedness condition at the singular point (see rows 7, 8, and 9 of Table 2.2 for examples).

▶ **Modified Green's function. Representation in terms of particular solutions.**

Now let us look at equation (2.5.1.1) subject to homogeneous boundary conditions of the general form

$$\begin{aligned} m_1 y_x' + k_1 y &= 0 \quad \text{at} \quad x = x_1, \\ m_2 y_x' + k_2 y &= 0 \quad \text{at} \quad x = x_2. \end{aligned} \tag{2.5.3.4}$$

With suitably selected coefficients k_n and m_n, these conditions cover the first, second, third, and mixed boundary conditions are special cases (see Section 2.5.1).

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TABLE 2.2
Green's function for some boundary value problems for linear second-order ODEs $L[y] = r(x)$

No.	Differential operator, $L[y]$	Boundary conditions	Green's function, $G(x,\xi)$
1	y''_{xx}	$y(0) = y(a) = 0$	$x\left(\dfrac{\xi}{a} - 1\right)$
2	y''_{xx}	$y(0) = y'_x(a) = 0$	$-x$
3	y''_{xx}	$y'_x(0) = y(a) = 0$	$\xi - a$
4	y''_{xx}	$y(0) = 0$, $y(a) + ky'_x(a) = 0$	$\dfrac{x(\xi - a - k)}{a + k}$
5	$y''_{xx} + k^2 y$	$y(0) = y(1) = 0$	$-\dfrac{\sin(kx)\sin[k(1-\xi)]}{k\sin k}$
6	$y''_{xx} - k^2 y$	$y(0) = y(1) = 0$	$-\dfrac{\sinh(kx)\sinh[k(1-\xi)]}{k\sinh k}$
7	$xy''_{xx} + y'_x \equiv (xy'_x)'_x$	$y(0) \neq \infty,\ y(a) = 0$	$\ln\dfrac{\xi}{a}$
8	$(xy'_x)'_x - \dfrac{n^2}{x}y$ (Bessel's operator)	$y(0) \neq \infty,\ y(a) = 0$	$-\dfrac{1}{2n}\left(\dfrac{x}{a}\right)^n + \dfrac{(x\xi)^n}{2na^{2n}}$ $(n = 1,\ 2,\ \ldots)$
9	$[(1-x^2)y'_x]'_x - \dfrac{n^2}{1-x^2}y$ (Legendre's operator)	$y(-1) \neq \infty,\ y(1) \neq \infty$	$-\dfrac{1}{2n}\left(\dfrac{1+x}{1-x}\cdot\dfrac{1-\xi}{1+\xi}\right)^{n/2}$ $(n = 1,\ 2,\ \ldots)$
10	$[f(x)y'_x]'_x$	$y(0) = y(a) = 0$	$-\varphi(x) + \dfrac{\varphi(x)\varphi(\xi)}{\varphi(a)}$
11	$[f(x)y'_x]'_x$	$y(0) = y'_x(a) = 0$	$-\varphi(x)$
12	$[f(x)y'_x]'_x$	$y(0) = 0$, $y(a) + ky'_x(a) = 0$	$-\varphi(x) + \dfrac{f(a)\varphi(x)\varphi(\xi)}{f(a)\varphi(a) + k}$ $(k > 0)$

The solution of the nonhomogeneous equation (2.5.1.1) subject to homogeneous boundary conditions (2.5.3.4) is*

$$y(x) = \int_{x_1}^{x_2} \mathcal{G}(x,\xi)g(\xi)\,d\xi, \tag{2.5.3.5}$$

where $\mathcal{G}(x,\xi)$ is the *modified Green's function*

$$\mathcal{G}(x,\xi) = \begin{cases} \dfrac{y_1(x)y_2(\xi)}{f_2(\xi)W(\xi)} & \text{for } x_1 \le x \le \xi, \\ \dfrac{y_1(\xi)y_2(x)}{f_2(\xi)W(\xi)} & \text{for } \xi \le x \le x_2, \end{cases} \tag{2.5.3.6}$$

where $y_1 = y_1(x)$ and $y_2 = y_2(x)$ are linearly independent particular solutions of the

*The homogeneous boundary value problem, with $g(x) = 0$, is assumed to have only the trivial solution.

homogeneous equation (2.5.1.1), with $g = 0$, that satisfy the conditions

$$\begin{aligned} m_1(y_1)'_x + k_1 y_1 = 0 \quad &\text{at} \quad x = x_1, \\ m_2(y_2)'_x + k_2 y_2 = 0 \quad &\text{at} \quad x = x_2, \end{aligned} \tag{2.5.3.7}$$

and $W(x) = y_1(x)y'_2(x) - y'_1(x)y_2(x)$ is the Wronskian determinant.

Example 2.5. Consider the equation

$$y''_{xx} + ay'_x = g(x) \tag{2.5.3.8}$$

with the homogeneous mixed boundary conditions

$$y(0) = 0, \qquad y'(1) = 0. \tag{2.5.3.9}$$

The general solution of equation (2.5.3.8) with $g(x) = 0$ is

$$y = C_1 + C_2 e^{-ax}, \tag{2.5.3.10}$$

where C_1 and C_2 are arbitrary constants. Linearly independent particular solutions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ that satisfy the homogeneous conditions $y_1(0) = 0$ and $y'_2(1) = 0$ are

$$y_1(x) = 1 - e^{-ax}, \qquad y_2(x) = 1. \tag{2.5.3.11}$$

Substituting (2.5.3.11) into (2.5.3.6) and taking into account that $f_2(x) = 1$, we find the modified Green's function

$$\mathcal{G}(x,\xi) = \begin{cases} -\dfrac{1}{a}e^{a\xi}(1 - e^{-ax}) & \text{for } x_1 \le x \le \xi, \\ -\dfrac{1}{a}e^{ax}(1 - e^{-a\xi}) & \text{for } \xi \le x \le x_2, \end{cases} \tag{2.5.3.12}$$

The solution to the boundary value problem (2.5.3.8)–(2.5.3.9) is defined by formulas (2.5.3.5) and (2.5.3.12).

⊙ *Literature for Section 2.5*: L. E. El'sgol'ts (1961), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2008).

2.6 Eigenvalue Problems

2.6.1 Sturm–Liouville Problem

Consider the second-order homogeneous linear differential equation

$$[p(x)y'_x]'_x + [\lambda s(x) - q(x)]y = 0 \tag{2.6.1.1}$$

subject to linear boundary conditions of the general form

$$\begin{aligned} \alpha_1 y'_x + \beta_1 y = 0 \quad &\text{at} \quad x = x_1, \\ \alpha_2 y'_x + \beta_2 y = 0 \quad &\text{at} \quad x = x_2. \end{aligned} \tag{2.6.1.2}$$

It is assumed that the functions p, p'_x, s, and q are continuous, and p and s are positive on an interval $x_1 \le x \le x_2$. It is also assumed that $|\alpha_1| + |\beta_1| > 0$ and $|\alpha_2| + |\beta_2| > 0$.

The *Sturm–Liouville problem*: Find the values λ_n of the parameter λ at which problem (2.6.1.1), (2.6.1.2) has a nontrivial solution. Such λ_n are called *eigenvalues* and the corresponding solutions $y_n = y_n(x)$ are called *eigenfunctions* of the Sturm–Liouville problem (2.6.1.1), (2.6.1.2).

2.6.2 General Properties of the Sturm–Liouville Problem (2.6.1.1), (2.6.1.2)

1°. There are infinitely (countably) many eigenvalues. All eigenvalues can be ordered so that $\lambda_1 < \lambda_2 < \lambda_3 < \cdots$. Moreover, $\lambda_n \to \infty$ as $n \to \infty$; hence, there can only be a finite number of negative eigenvalues. Each eigenvalue has multiplicity 1.

2°. The eigenfunctions are defined up to a constant factor. Each eigenfunction $y_n(x)$ has precisely $n-1$ zeros on the open interval (x_1, x_2).

3°. Any two eigenfunctions $y_n(x)$ and $y_m(x)$, $n \neq m$, are orthogonal with weight $s(x)$ on the interval $x_1 \le x \le x_2$:

$$\int_{x_1}^{x_2} s(x) y_n(x) y_m(x)\, dx = 0 \quad \text{if} \quad n \neq m.$$

4°. An arbitrary function $F(x)$ that has a continuous derivative and satisfies the boundary conditions of the Sturm–Liouville problem can be decomposed into an absolutely and uniformly convergent series in the eigenfunctions

$$F(x) = \sum_{n=1}^{\infty} F_n y_n(x),$$

where the Fourier coefficients F_n of $F(x)$ are calculated by

$$F_n = \frac{1}{\|y_n\|^2} \int_{x_1}^{x_2} s(x) F(x) y_n(x)\, dx, \qquad \|y_n\|^2 = \int_{x_1}^{x_2} s(x) y_n^2(x)\, dx.$$

5°. If the conditions

$$q(x) \ge 0, \quad \alpha_1\beta_1 \le 0, \quad \alpha_2\beta_2 \ge 0 \tag{2.6.2.1}$$

hold true, there are no negative eigenvalues. If $q \equiv 0$ and $\beta_1 = \beta_2 = 0$, the least eigenvalue is $\lambda_1 = 0$, to which there corresponds an eigenfunction $y_1 = \text{const}$. In the other cases where conditions (2.6.2.1) are satisfied, all eigenvalues are positive.

6°. The following asymptotic formula is valid for eigenvalues as $n \to \infty$:

$$\lambda_n = \frac{\pi^2 n^2}{\Delta^2} + O(1), \qquad \Delta = \int_{x_1}^{x_2} \sqrt{\frac{s(x)}{p(x)}}\, dx. \tag{2.6.2.2}$$

Sections 2.6.3 through 2.6.6 will describe special properties of the Sturm–Liouville problem that depend on the specific form of the boundary conditions.

Remark 2.3. Equation (2.6.1.1) can be reduced to the case where $p(x) \equiv 1$ and $s(x) \equiv 1$ by the change of variables

$$\zeta = \int \sqrt{\frac{s(x)}{p(x)}}\, dx, \qquad u(\zeta) = \big[p(x)s(x)\big]^{1/4} y(x).$$

In this case, the boundary conditions are transformed to boundary conditions of similar form.

Remark 2.4. The second-order linear equation

$$\varphi_2(x) y''_{xx} + \varphi_1(x) y'_x + [\lambda + \varphi_0(x)] y = 0$$

can be represented in the form of equation (2.6.1.1) where $p(x)$, $s(x)$, and $q(x)$ are given by

$$\begin{gathered} p(x) = \exp\left[\int \frac{\varphi_1(x)}{\varphi_2(x)}\, dx\right], \qquad s(x) = \frac{1}{\varphi_2(x)} \exp\left[\int \frac{\varphi_1(x)}{\varphi_2(x)}\, dx\right], \\ q(x) = -\frac{\varphi_0(x)}{\varphi_2(x)} \exp\left[\int \frac{\varphi_1(x)}{\varphi_2(x)}\, dx\right]. \end{gathered} \tag{2.6.2.3}$$

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2.6.3 Problems with Boundary Conditions of the First Kind

Let us note some special properties of the Sturm–Liouville problem that is the first boundary value problem for equation (2.6.1.1) with the boundary conditions

$$y = 0 \quad \text{at} \quad x = x_1, \qquad y = 0 \quad \text{at} \quad x = x_2. \tag{2.6.3.1}$$

1°. For $n \to \infty$, the asymptotic relation (2.6.2.2) can be used to estimate the eigenvalues λ_n. In this case, the asymptotic formula

$$\frac{y_n(x)}{\|y_n\|} = \left[\frac{4}{\Delta^2 p(x)s(x)}\right]^{1/4} \sin\left[\frac{\pi n}{\Delta}\int_{x_1}^{x}\sqrt{\frac{s(x)}{p(x)}}\,dx\right] + O\Big(\frac{1}{n}\Big), \quad \Delta = \int_{x_1}^{x_2}\sqrt{\frac{s(x)}{p(x)}}\,dx$$

holds true for the eigenfunctions $y_n(x)$.

2°. If $q \ge 0$, the following upper estimate holds for the least eigenvalue (*Rayleigh–Ritz principle*):

$$\lambda_1 \le \frac{\displaystyle\int_{x_1}^{x_2}\big[p(x)(z'_x)^2 + q(x)z^2\big]\,dx}{\displaystyle\int_{x_1}^{x_2} s(x)z^2\,dx}, \tag{2.6.3.2}$$

where $z = z(x)$ is any twice differentiable function that satisfies the conditions $z(x_1) = z(x_2) = 0$. The equality in (2.6.3.2) is attained if $z = y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1. One can take $z = (x - x_1)(x_2 - x)$ or $z = \sin\left[\dfrac{\pi(x - x_1)}{x_2 - x_1}\right]$ in (2.6.3.2) to obtain specific estimates.

It is significant to note that the left-hand side of (2.6.3.2) usually gives a fairly precise estimate of the first eigenvalue (see Table 2.3).

TABLE 2.3
Example estimates of the first eigenvalue λ_1 in Sturm–Liouville problems with boundary conditions of the first kind $y(0) = y(1) = 0$ obtained using the Rayleigh–Ritz principle [the right-hand side of relation (2.6.3.2)]

Equation	Test function	λ_1, approximate	λ_1, exact
$y''_{xx} + \lambda(1 + x^2)^{-2}y = 0$	$z = \sin \pi x$	15.337	15.0
$y''_{xx} + \lambda(4 - x^2)^{-2}y = 0$	$z = \sin \pi x$	135.317	134.837
$[(1 + x)^{-1}y'_x]'_x + \lambda y = 0$	$z = \sin \pi x$	7.003	6.772
$\big(\sqrt{1 + x}\,y'_x\big)'_x + \lambda y = 0$	$z = \sin \pi x$	11.9956	11.8985
$y''_{xx} + \lambda(1 + \sin \pi x)y = 0$	$z = \sin \pi x$ $z = x(1 - x)$	$0.54105\,\pi^2$ $0.55204\,\pi^2$	$0.54032\,\pi^2$ $0.54032\,\pi^2$

3°. The extension of the interval $[x_1, x_2]$ leads to decreasing in eigenvalues.

4°. Let the inequalities

$$0 < p_{\min} \le p(x) \le p_{\max}, \quad 0 < s_{\min} \le s(x) \le s_{\max}, \quad 0 < q_{\min} \le q(x) \le q_{\max}$$

be satisfied. Then the following bilateral estimates hold:

$$\frac{p_{\min}}{s_{\max}}\frac{\pi^2 n^2}{(x_2-x_1)^2}+\frac{q_{\min}}{s_{\max}}\le\lambda_n\le\frac{p_{\max}}{s_{\min}}\frac{\pi^2 n^2}{(x_2-x_1)^2}+\frac{q_{\max}}{s_{\min}}.$$

5°. In engineering calculations for eigenvalues, the approximate formula

$$\lambda_n=\frac{\pi^2 n^2}{\Delta^2}+\frac{1}{x_2-x_1}\int_{x_1}^{x_2}\frac{q(x)}{s(x)}\,dx,\qquad \Delta=\int_{x_1}^{x_2}\sqrt{\frac{s(x)}{p(x)}}\,dx \tag{2.6.3.3}$$

may be quite useful. This formula provides an exact result if $p(x)s(x)=\text{const}$ and $q(x)/s(x)=\text{const}$ (in particular, for constant equation coefficients, $p=p_0$, $q=q_0$, and $s=s_0$) and gives a correct asymptotic behavior of (2.6.2.2) for any $p(x)$, $q(x)$, and $s(x)$. In addition, relation (2.6.3.3) gives two correct leading asymptotic terms as $n\to\infty$ if $p(x)=\text{const}$ and $s(x)=\text{const}$ [and also if $p(x)s(x)=\text{const}$].

6°. Suppose $p(x)=s(x)=1$ and the function $q=q(x)$ has a continuous derivative. The following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n\to\infty$:

$$\sqrt{\lambda_n}=\frac{\pi n}{x_2-x_1}+\frac{1}{\pi n}Q(x_1,x_2)+O\Big(\frac{1}{n^2}\Big),$$

$$y_n(x)=\sin\frac{\pi n(x-x_1)}{x_2-x_1}-\frac{1}{\pi n}\Big[(x_1-x)Q(x,x_2)+(x_2-x)Q(x_1,x)\Big]\cos\frac{\pi n(x-x_1)}{x_2-x_1}+O\Big(\frac{1}{n^2}\Big),$$

where

$$Q(u,v)=\frac{1}{2}\int_u^v q(x)\,dx. \tag{2.6.3.4}$$

7°. Let us consider the eigenvalue problem for the equation with a small parameter

$$y''_{xx}+[\lambda+\varepsilon q(x)]y=0\qquad(\varepsilon\to 0)$$

subject to the boundary conditions (2.6.3.1) with $x_1=0$ and $x_2=1$. We assume that $q(x)=q(-x)$.

This problem has the following eigenvalues and eigenfunctions:

$$\lambda_n=\pi^2 n^2-\varepsilon A_{nn}+\frac{\varepsilon^2}{\pi^2}\sum_{k\neq n}\frac{A_{nk}^2}{n^2-k^2}+O(\varepsilon^3),\quad A_{nk}=2\int_0^1 q(x)\sin(\pi nx)\sin(\pi kx)\,dx;$$

$$y_n(x)=\sqrt{2}\,\sin(\pi nx)-\varepsilon\frac{\sqrt{2}}{\pi^2}\sum_{k\neq n}\frac{A_{nk}}{n^2-k^2}\sin(\pi kx)+O(\varepsilon^2).$$

Here the summation is carried out over k from 1 to ∞. The next term in the expansion of y_n can be found in Nayfeh (1973).

2.6.4 Problems with Boundary Conditions of the Second Kind

Let us note some special properties of the Sturm–Liouville problem that is the second boundary value problem for equation (2.6.1.1) with the boundary conditions

$$y'_x=0\quad\text{at}\quad x=x_1,\qquad y'_x=0\quad\text{at}\quad x=x_2.$$

1°. If $q>0$, the upper estimate (2.6.3.2) is valid for the least eigenvalue, with $z=z(x)$ being any twice-differentiable function that satisfies the conditions $z'_x(x_1)=z'_x(x_2)=0$. The equality in (2.6.3.2) is attained if $z=y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1.

2°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. The following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(n-1)}{x_2 - x_1} + \frac{1}{\pi(n-1)} Q(x_1, x_2) + O\Big(\frac{1}{n^2}\Big),$$

$$y_n(x) = \cos \frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + \frac{1}{\pi(n-1)} \Big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\Big] \sin \frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is given by (2.6.3.4).

2.6.5 Problems with Boundary Conditions of the Third Kind

We consider the third boundary value problem for equation (2.6.1.1) subject to condition (2.6.1.2) with $\alpha_1 = \alpha_2 = 1$. We assume that $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative.

The following asymptotic formulas hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(n-1)}{x_2 - x_1} + \frac{1}{\pi(n-1)} \big[Q(x_1, x_2) - \beta_1 + \beta_2\big] + O\Big(\frac{1}{n^2}\Big),$$

$$y_n(x) = \cos \frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + \frac{1}{\pi(n-1)} \Big\{(x_1 - x)\big[Q(x, x_2) + \beta_2\big] + (x_2 - x)\big[Q(x_1, x) - \beta_1\big]\Big\} \sin \frac{\pi(n-1)(x-x_1)}{x_2 - x_1} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is defined by (2.6.3.4).

2.6.6 Problems with Mixed Boundary Conditions

Let us note some special properties of the Sturm–Liouville problem that is the mixed boundary value problem for equation (2.6.1.1) with the boundary conditions

$$y'_x = 0 \quad \text{at} \quad x = x_1, \qquad y = 0 \quad \text{at} \quad x = x_2.$$

1°. If $q \geq 0$, the upper estimate (2.6.3.2) is valid for the least eigenvalue, with $z = z(x)$ being any twice-differentiable function that satisfies the conditions $z'_x(x_1) = 0$ and $z(x_2) = 0$. The equality in (2.6.3.2) is attained if $z = y_1(x)$, where $y_1(x)$ is the eigenfunction corresponding to the eigenvalue λ_1.

2°. Suppose $p(x) = s(x) = 1$ and the function $q = q(x)$ has a continuous derivative. The following asymptotic relations hold for eigenvalues λ_n and eigenfunctions $y_n(x)$ as $n \to \infty$:

$$\sqrt{\lambda_n} = \frac{\pi(2n-1)}{2(x_2 - x_1)} + \frac{2}{\pi(2n-1)} Q(x_1, x_2) + O\Big(\frac{1}{n^2}\Big),$$

$$y_n(x) = \cos \frac{\pi(2n-1)(x-x_1)}{2(x_2 - x_1)} + \frac{2}{\pi(2n-1)} \Big[(x_1 - x)Q(x, x_2) + (x_2 - x)Q(x_1, x)\Big] \sin \frac{\pi(2n-1)(x-x_1)}{2(x_2 - x_1)} + O\Big(\frac{1}{n^2}\Big),$$

where $Q(u, v)$ is defined by (2.6.3.4).

⊙ *Literature for Section 2.6*: L. Collatz (1963), E. Kamke (1977), A. G. Kostyuchenko and I. S. Sargsyan (1979), V. A. Marchenko (1986), B. M. Levitan and I. S. Sargsyan (1988), V. A. Vinokurov and V. A. Sadovnichii (2000), A. D. Polyanin (2002), A. D. Polyanin and V. F. Zaitsev (2003).

2.7 Theorems on Estimates and Zeros of Solutions

2.7.1 Theorem on Estimates of Solutions

Let $f_n(x)$ and $g_n(x)$ $(n = 1,\ 2)$ be continuous functions on the interval $a \le x \le b$.

THEOREM. *Let the following inequalities hold:*

$$0 \le f_1(x) \le f_2(x), \qquad 0 \le g_1(x) \le g_2(x).$$

If $y_n = y_n(x)$ are some solutions to the linear equations

$$y_n'' = f_n(x)y_n + g_n(x) \qquad (n = 1,\ 2)$$

and $y_1(a) \le y_2(a)$ and $y_1'(a) \le y_2'(a)$, then $y_1(x) \le y_2(x)$ and $y_1'(x) \le y_2'(x)$ on each interval $a \le x \le a_1$, where $y_2(x) > 0$.

2.7.2 Sturm Comparison Theorem on Zeros of Solutions

Consider the equation

$$[f(x)y']' + g(x)y = 0 \qquad (a \le x \le b), \tag{2.7.2.1}$$

where the function $f(x)$ is positive and continuously differentiable, and the function $g(x)$ is continuous.

COMPARISON THEOREM (STURM). *Let $y_n = y_n(x)$ be nonzero solutions of the linear equations*

$$[f_n(x)y_n']' + g_n(x)y_n = 0 \qquad (n = 1,\ 2)$$

and let the inequalities $f_1(x) \ge f_2(x) > 0$ and $g_1(x) \le g_2(x)$ hold. Then the function y_2 has at least one zero lying between any two adjacent zeros, x_1 and x_2, of the function y_1 (it is assumed that the identities $f_1 \equiv f_2$ and $g_1 \equiv g_2$ are not satisfied on any interval simultaneously).

COROLLARY 1. *If $g(x) \le 0$ or there exists a constant k_1 such that*

$$f(x) \ge k_1 > 0, \quad g(x) < k_1\left(\frac{\pi}{b-a}\right)^2,$$

then every nontrivial solution to equation (2.7.2.1) has no more than one zero on the interval $[a, b]$.

COROLLARY 2. *If there exists a constant k_2 such that*

$$0 < f(x) \le k_2, \quad g(x) > k_2\left(\frac{\pi m}{b-a}\right)^2, \quad \text{where} \quad m = 1,\ 2,\ \ldots,$$

then every nontrivial solution to equation (2.7.2.1) has at least m zeros on the interval $[a, b]$.

2.7.3 Qualitative Behavior of Solutions as $x \to \infty$

Consider the equation

$$y'' + f(x)y = 0, \tag{2.7.3.1}$$

where $f(x)$ is a continuous function for $x \geq a$.

1°. For $f(x) \leq 0$, every nonzero solution has no more than one zero, and hence $y \neq 0$ for sufficiently large x.

If $f(x) \leq 0$ for all x and $f(x) \not\equiv 0$, then $y \equiv 0$ is the only solution bounded for all x.

2°. Suppose $f(x) \geq k^2 > 0$. Then every nontrivial solution $y(x)$ and its derivative $y'(x)$ have infinitely many zeros, with the distance between any adjacent zeros remaining finite.

If $f(x) \to k^2 > 0$ for $x \to \infty$ and $f' \geq 0$, then the solutions of the equation for large x behave similarly to those of the equation $y'' + k^2 y = 0$.

3°. Let $f(x) \to -\infty$ for $|x| \to \infty$. Then every nonzero solution has only finitely many zeros, and $|y'/y| \to \infty$ as $|x| \to \infty$. There are two linearly independent solutions, y_1 and y_2, such that $y_1 \to 0$, $y_1' \to 0$, $y_2 \to \infty$, and $y_2' \to -\infty$ as $x \to -\infty$, and there are two linearly independent solutions, $\bar{y}_1$ and $\bar{y}_2$, such that $\bar{y}_1 \to 0$, $\bar{y}_1' \to 0$, $\bar{y}_2 \to \infty$, and $\bar{y}_2' \to \infty$ as $x \to \infty$.

4°. If the function f in equation (2.7.3.1) is continuous, monotonic, and positive, then the amplitude of each solution decreases (resp., increases) as f increases (resp., decreases).

⊙ *Literature for Section 2.7*: E. Kamke (1977), A. D. Polyanin and A. V. Manzhirov (2007).

2.8 Numerical Methods

Linear problems can be solved using the numerical methods outlined in Section 3.8, which are designed for solving more complex, nonlinear problems. Due to their specific properties, linear problems are easier and more efficient to solve with special methods described below.

In the numerical methods discussed below, the second derivative is approximated with the following finite-difference expression:

$$y''_{xx} \approx \frac{y_{k+1} - 2y_k + y_{k-1}}{h^2},$$

where $y_k = y(x_k)$, $x_k = x_0 + kh$ ($x_0 \leq x \leq x_*$), and h is the mesh increment.

2.8.1 Numerov's Method (Cauchy Problem)

The Cauchy problem for linear differential equations of the form

$$y''_{xx} + f(x)y = g(x) \tag{2.8.1.1}$$

can be solved using the recurrence formula

$$u_{k+1} = 2u_k - u_{k-1} + \left[-f_k y_k + g_k + \tfrac{1}{12}(g_{k+1} - 2g_k + g_{k-1})\right]h^2, \tag{2.8.1.2}$$

where

$$u_k = y_k\left(1 + \tfrac{1}{12} f_k h^2\right), \quad f_k = f(x_k), \quad g_k = g(x_k).$$

2.8.2 Modified Shooting Method (Boundary Value Problems)

Let us look at the linear boundary value problem defined by the equation

$$y''_{xx} + f_1(x)y'_x + f_0(x)y = g(x) \tag{2.8.2.1}$$

and general homogeneous boundary conditions of the third kind

$$a_1 y'_x + b_1 y = 0 \quad \text{at} \quad x = 0, \tag{2.8.2.2}$$

$$a_2 y'_x + b_2 y = 0 \quad \text{at} \quad x = l. \tag{2.8.2.3}$$

We assume that a solution to problem (2.8.2.1)–(2.8.2.3) exists and is unique.

First, we find an auxiliary function $y_1 = y_1(x)$ that solves the first auxiliary Cauchy problem for the nonhomogeneous equation (2.8.2.1) with the initial conditions

$$y = a_1 \quad \text{at} \quad x = 0; \qquad y'_x = -b_1 \quad \text{at} \quad x = 0. \tag{2.8.2.4}$$

By virtue of (2.8.2.1), the function $y_1 = y_1(x)$ satisfies the left boundary condition (2.8.2.2). Then, we find an auxiliary function $y_0 = y_0(x)$ that solves the second auxiliary Cauchy problem for the homogeneous equation (2.8.2.1) with $g(x) = 0$ and the boundary conditions (2.8.2.4). By virtue of the linearity of the problem and homogeneous boundary conditions, the function $Cy_0(x)$ is also a solution to equation (2.8.2.1) satisfying the left boundary condition (2.8.2.2). Therefore, the solution of the original boundary value problem (2.8.2.1)–(2.8.2.3) can be sought as the sum

$$y(x) = y_1(x) + Cy_0(x). \tag{2.8.2.5}$$

The constant C is determined from the requirement that function (2.8.2.5) must satisfy the right boundary condition (2.8.2.3):

$$a_2 y'_1(l) + b_2 y_1(l) + C[a_2 y'_0(l) + b_2 y_0(l)] = 0. \tag{2.8.2.6}$$

Thus, solving the original boundary value problem is reduced to solving two auxiliary Cauchy problems; this can be done using, for example, the Runge–Kutta method (see Section 3.8). The case of nonhomogeneous boundary condition can be considered likewise.

Example 2.6. Let us look at the special case of equation (2.8.2.1)

$$y''_{xx} + f(x)y = g(x) \tag{2.8.2.7}$$

subject to the nonhomogeneous boundary conditions of the first kind

$$y = a \quad \text{at} \quad x = 0; \qquad y = b \quad \text{at} \quad x = l. \tag{2.8.2.8}$$

The mesh version of the above shooting method for this problem is as follows. By setting the initial values*

$$y_0^1 = a, \quad y_1^1 = \beta_1; \quad y_0^0 = 0, \quad y_1^0 = \beta_2, \tag{2.8.2.9}$$

we successively find $y_2^1, \ldots, y_n^1$ and $y_2^0, \ldots, y_n^0$ from the difference equations

$$\frac{y_{k+1}^1 - 2y_k^1 + y_{k-1}^1}{h^2} + f_k y_k^1 = g_k,$$

$$\frac{y_{k+1}^0 - 2y_k^0 + y_{k-1}^0}{h^2} + f_k y_k^0 = 0,$$

where $f_k = f(x_k)$, $g_k = g(x_k)$, and h is the mesh increment. Then, we find C from the equation $y_n^1 + Cy_n^0 = b$ and set $y_k = y_k^1 + Cy_k^0$; the function y_k is the desired solution.

*The numbers β_1 and $\beta_2 \neq 0$ can generally be any; in particular, we can set $\beta_1 = a$ and $\beta_2 = h$.

2.8.3 Sweep Method (Boundary Value Problems)

Below we outline the sweep method for the following system of difference equations:

$$A_k y_{k-1} - C_k y_k + B_k y_{k+1} = D_k, \qquad k = 1, \dots, n-1, \tag{2.8.3.1}$$

$$y_0 = \alpha y_1 + \beta, \qquad y_n = \gamma y_{n-1} + \delta. \tag{2.8.3.2}$$

Relation (2.8.3.1) approximates a linear differential equation, while relations (2.8.3.2) represent boundary conditions of the third kind (or the first kind if $\alpha = \gamma = 0$).

Provided that all of the conditions

$$A_k > 0,\ B_k > 0,\ C_k > 0;\ \ C_k \geq A_k + B_k;\ \ 0 \leq \alpha < 1;\ \ 0 \leq \gamma < 1 \tag{2.8.3.3}$$

hold true, problem (2.8.3.1)–(2.8.3.2) is solvable and has a unique solution.

Remark 2.5. Problem (2.8.2.7)–(2.8.2.8) is approximated by the difference equation (2.8.3.1) and boundary conditions (2.8.3.2) with

$$A_k = B_k = 1,\ \ C_k = 2 - h^2 f_k,\ \ D_k = h^2 g_k,\ \ \alpha = \gamma = 0,\ \ \beta = a,\ \ \delta = b.$$

We will look for numbers α_k and β_k, called *sweep coefficients*, such that for all $k = 1,\ 2,\ \dots,\ n$ the relations

$$y_{k-1} = \alpha_k y_k + \beta_k \tag{2.8.3.4}$$

hold. Substituting (2.8.3.4) into (2.8.3.1) yields

$$(A_k\alpha_k - C_k)y_k + B_k y_{k+1} + A_k\beta_k - D_k = 0.$$

By expressing y_k in terms of y_{k+1} using formula (2.8.3.4), we obtain

$$\big[(A_k\alpha_k - C_k)\alpha_{k+1} + B_k\big]y_{k+1} + \big[(A_k\alpha_k - C_k)\beta_{k+1} + A_k\beta_k - D_k\big] = 0.$$

Equating the expressions in square brackets with zero for all $k = 1,\ 2,\ \dots,\ n-1$, we arrive at recurrence relations to determine the coefficients α_{k+1} and β_{k+1} once $\alpha = \alpha_1$ and $\beta = \beta_1$ are known (*forward sweep*):

$$\alpha_{k+1} = \frac{B_k}{C_k - A_k\alpha_k}, \qquad \beta_{k+1} = \frac{A_k\beta_k - D_k}{C_k - A_k\alpha_k}. \tag{2.8.3.5}$$

If conditions (2.8.3.3) hold, the numerators in formulas (2.8.3.5) are positive and $0 \leq \alpha_k < 1$.

From formula (2.8.3.4) with $k = n$ and the second boundary condition in (2.8.3.2) we find the last value of the unknown:

$$y_n = \frac{\gamma\beta_n + \delta}{1 - \gamma\alpha_n}, \tag{2.8.3.6}$$

where $1 - \gamma\alpha_n > 0$. Now, by formula (2.8.3.4), we can successively determine the unknowns y_{k-1} with $k = n,\ n-1,\ \dots,\ 1$ (*backward sweep*).

Remark 2.6. In the above sweep method, the coefficients are first determined starting from the left boundary condition and then the solution is recovered from right to left by formula (2.8.3.4). Quite similarly, the reverse scheme can be used where the coefficients are first determined starting from the right boundary condition and then the solution is recovered with the sweep from left to right.

2.8.4 Method of Accelerated Convergence in Eigenvalue Problems

Consider the Sturm–Liouville problem for the second-order nonhomogeneous linear equation

$$[f(x)y'_x]'_x + [\lambda g(x) - h(x)]y = 0 \qquad (2.8.4.1)$$

with linear homogeneous boundary conditions of the first kind

$$y(0) = y(1) = 0. \qquad (2.8.4.2)$$

It is assumed that the functions $f,\ f'_x,\ g,\ h$ are continuous and $f > 0,\ g > 0$.

First, using the Rayleigh–Ritz principle, one finds an upper estimate for the first eigenvalue λ_1^0 [this value is determined by the right-hand side of relation (2.6.3.2)]. Then, one solves numerically the Cauchy problem for the auxiliary equation

$$[f(x)y'_x]'_x + [\lambda_1^0 g(x) - h(x)]y = 0 \qquad (2.8.4.3)$$

with the boundary conditions

$$y(0) = 0, \qquad y'_x(0) = 1. \qquad (2.8.4.4)$$

The function $y(x, \lambda_1^0)$ satisfies the condition $y(x_0, \lambda_1^0) = 0$, where $x_0 < 1$. The criterion of closeness of the exact and approximate solutions, λ_1 and λ_1^0, has the form of the inequality $|1 - x_0| \le \delta$, where δ is a sufficiently small given constant. If this inequality does not hold, one constructs a refinement for the approximate eigenvalue on the basis of the formula

$$\lambda_1^1 = \lambda_1^0 - \varepsilon_0 f(1)\frac{[y'_x(1)]^2}{\|y\|^2}, \qquad \varepsilon_0 = 1 - x_0, \qquad (2.8.4.5)$$

where $\|y\|^2 = \int_0^1 g(x)y^2(x)\,dx$. Then the value λ_1^1 is substituted for λ_1^0 in the Cauchy problem (2.8.4.3)–(2.8.4.4). As a result, a new solution y and a new point x_1 are found; and one has to check whether the criterion $|1 - x_1| \le \delta$ holds. If this inequality is violated, one refines the approximate eigenvalue by means of the formula

$$\lambda_1^2 = \lambda_1^1 - \varepsilon_1 f(1)\frac{[y'_x(1)]^2}{\|y\|^2}, \qquad \varepsilon_1 = 1 - x_1, \qquad (2.8.4.6)$$

and repeats the above procedure.

Remark 2.7. Formulas of the type (2.8.4.5) are obtained by a perturbation method based on a transformation of the independent variable x (see Section 3.6.3). If $x_n > 1$, the functions f, g, and h are smoothly extended to the interval $(1, \xi]$, where $\xi \ge x_n$.

Remark 2.8. The algorithm described above possesses the property of accelerated convergence, $\varepsilon_{n+1} \sim \varepsilon_n^2$, which ensures that the relative error of the approximate solution becomes 10^{-4} to 10^{-8} after two or three iterations for $\varepsilon_0 \sim 0.1$. This method is quite effective for high-precision calculations, is fail-safe, and guarantees against accumulation of roundoff errors.

Remark 2.9. In a similar way, one can compute subsequent eigenvalues λ_m, $m = 2, 3, \ldots$ (to that end, a suitable initial approximation λ_m^0 should be chosen).

Remark 2.10. A similar computation scheme can also be used in the case of boundary conditions of the second and the third kinds, periodic boundary conditions, etc. (see the reference below).

Example 2.7. The eigenvalue problem for the equation

$$y''_{xx} + \lambda(1 + x^2)^{-2} y = 0$$

with the boundary conditions (2.8.4.2) admits an exact analytic solution and has eigenvalues $\lambda_1 = 15$, $\lambda_2 = 63, \dots, \lambda_n = 16n^2 - 1$.

According to the Rayleigh–Ritz principle, formula (2.6.3.2) for $z = \sin(\pi x)$ yields the approximate value $\lambda_1^0 = 15.33728$. The solution of the Cauchy problem (2.8.4.3)–(2.8.4.4) with $f(x) = 1$, $g(x) = \lambda(1 + x^2)^{-2}$, $h(x) = 0$ yields $x_0 = 0.983848$, $1 - x_0 = 0.016152$, $\|y\|^2 = 0.024585$, $y'_x(x_0) = -0.70622822$.

The first iteration for the first eigenvalue is determined by (2.8.4.5) and results in the value $\lambda_1^1 = 14.99245$ with the relative error $\Delta\lambda/\lambda_1^1 = 5 \times 10^{-4}$.

The second iteration results in $\lambda_1^2 = 14.999986$ with the relative error $\Delta\lambda/\lambda_1^2 < 10^{-6}$.

Example 2.8. Consider the eigenvalue problem for the equation

$$(\sqrt{1 + x}\, y'_x)'_x + \lambda y = 0$$

with the boundary conditions (2.8.4.2).

The Rayleigh–Ritz principle yields $\lambda_1^0 = 11.995576$. The next two iterations result in the values $\lambda_1^1 = 11.898578$ and $\lambda_1^2 = 11.898458$. For the relative error we have $\Delta\lambda/\lambda_1^2 < 10^{-5}$.

2.8.5 Well-Conditioned and Ill-Conditioned Problems

Numerical methods can only be applied to *well-conditioned* linear problems, in which small perturbations in the initial data (or the right-hand side of the equation, which determines its nonhomogeneity) lead to small changes in the solution. Otherwise, when the problem is *ill-conditioned*, small perturbations in the initial data (or the right-hand side of the equation) or equivalent small errors of the numerical method can significantly distort the solution.

Example 2.9. Let us look at the linear second-order ordinary differential equation

$$y''_{xx} + (1 + a)y'_x + ay = 0 \tag{2.8.5.1}$$

subject to the initial conditions

$$y(0) = 1, \quad y'_x(0) = -1, \tag{2.8.5.2}$$

where a is a free parameter ($a \neq 1$).

The solution of problem (2.8.5.1)–(2.8.5.2) is

$$y = e^{-x}. \tag{2.8.5.3}$$

Now let us suppose that the boundary conditions of equation (2.8.5.1) are slightly changed:

$$y(0) = 1 + \varepsilon, \quad y'_x(0) = -1, \tag{2.8.5.4}$$

where ε is a small positive number.

The solution of problem (2.8.5.1), (2.8.5.4) is

$$y_\varepsilon = \left(1 - \frac{a\varepsilon}{1 - a}\right) e^{-x} + \frac{\varepsilon}{1 - a} e^{-ax}. \tag{2.8.5.5}$$

Solution (2.8.5.5) behaves qualitatively differently depending on the value of the parameter a. Consider the possible situations.

If $a > 0$, solution (2.8.5.5) decays exponentially as $x \to \infty$. The difference between solutions (2.8.5.3) and (2.8.5.5) vanishes as $\varepsilon \to 0$ for all $x \geq 0$. If $a = 0$, the difference between solutions (2.8.5.3) and (2.8.5.5) is a small constant quantity equal to ε. If $a \geq 0$, problem (2.8.5.1)–(2.8.5.2) is well-conditioned.

Remark 2.11. For $0 < a < 1$ and fixed $\varepsilon > 0$, the second term in solution (2.8.5.5), proportional to e^{-ax}, dominates as $x \to \infty$ and the relative disturbance, $|y_\varepsilon - y|/y$, due to the small perturbation in the initial conditions, tends to infinity.

If $a < 0$, solution (2.8.5.5) increases without bound as $x \to \infty$. In this case, for any $\varepsilon > 0$, solutions (2.8.5.3) and (2.8.5.5) diverge indefinitely far apart as $x \to \infty$. If $a < 0$, problem (2.8.5.1)–(2.8.5.2) is ill-conditioned.

Remark 2.12. It is easy to show that for $a < 0$, the solution to the equation

$$y''_{xx} + (1+a)y'_x + ay = \varepsilon \qquad (\varepsilon \ll 1)$$

subject to the initial conditions (2.8.5.2) increases indefinitely as $x \to \infty$. This means that for $a < 0$, problem (2.8.5.1)–(2.8.5.2) is ill-conditioned with respect to perturbations of the right-hand side.

For more details about iteration and numerical methods, see the list of references given below.

⊙ *Literature for Section 2.8*: J. D. Lambert (1973), N. N. Kalitkin (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), J. C. Butcher (1987), W. E. Schiesser (1994), L. F. Shampine (1994), L. D. Akulenko and S. V. Nesterov (1996, 1997, 2005), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and A. V. Manzhirov (2007), S. C. Chapra and R. P. Canale (2010).

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