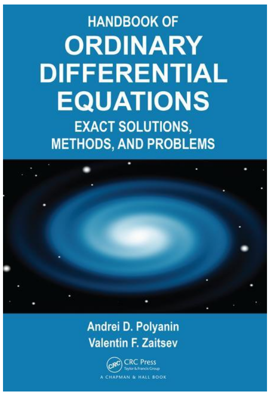

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

Andrei D. Polyanin, Valentin F. Zaitsev

Chapter 3: Methods for Second-Order Nonlinear Differential Equations

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Andrei D. Polyanin, Valentin F. Zaitsev
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Chapter 3

Methods for Second-Order Nonlinear Differential Equations

3.1 General Concepts. Cauchy Problem. Uniqueness and Existence Theorems

3.1.1 Equations Solved for the Derivative. General Solution

A *second-order ordinary differential equation* solved for the highest derivative has the form

$$y''_{xx} = f(x, y, y'_x). \tag{3.1.1.1}$$

A *solution of a differential equation* is a function $y(x)$ that, when substituted into the equation, turns it into an identity. The *general solution of a differential equation* is the set of all its solutions.

The general solution of this equation depends on two arbitrary constants, C_1 and C_2. In some cases, the general solution can be written in explicit form, $y = \varphi(x, C_1, C_2)$, but more often implicit or parametric forms of the general solution are encountered.

3.1.2 Cauchy Problem. Existence and Uniqueness Theorem

Cauchy problem: Find a solution of equation (3.1.1.1) satisfying the *initial conditions*

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1. \tag{3.1.2.1}$$

(At a point $x = x_0$, the value of the unknown function, y_0, and its derivative, y_1, are prescribed.)

EXISTENCE AND UNIQUENESS THEOREM. *Let $f(x, y, z)$ be a continuous function in all its arguments in a neighborhood of a point (x_0, y_0, y_1) and let f have bounded partial derivatives f_y and f_z in this neighborhood, or the Lipschitz condition is satisfied: $|f(x, y, z) - f(x, \bar{y}, \bar{z})| \le K\bigl(|y - \bar{y}| + |z - \bar{z}|\bigr)$, where K is some positive number. Then a solution of equation (3.1.1.1) satisfying the initial conditions (3.1.2.1) exists and is unique.*

⊙ *Literature for Section 3.1*: E. L. Ince (1956), G. M. Murphy (1960), L. E. El'sgol'ts (1961), P. Hartman (1964), N. M. Matveev (1967), I. G. Petrovskii (1970), G. F. Simmons (1972), E. Kamke (1977), G. Birkhoff and Rota (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997),

C. Chicone (1999), G. A. Korn and T. M. Korn (2000), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

3.2 Some Transformations. Equations Admitting Reduction of Order

3.2.1 Equations Not Containing y or x Explicitly. Related Equations

► **Equations not containing y explicitly.**

In the general case, an equation that does not contain y implicitly has the form

$$F(x, y'_x, y''_{xx}) = 0. \tag{3.2.1.1}$$

Such equations remain unchanged under an arbitrary translation of the dependent variable: $y \to y + \text{const}$. The substitution $y'_x = z(x)$, $y''_{xx} = z'_x(x)$ brings (3.2.1.1) to a first-order equation: $F(x, z, z'_x) = 0$.

► **Equations not containing x explicitly (autonomous equations).**

In the general case, an equation that does not contain x implicitly has the form

$$F(y, y'_x, y''_{xx}) = 0. \tag{3.2.1.2}$$

Such equations remain unchanged under an arbitrary translation of the independent variable: $x \to x + \text{const}$. Using the substitution $y'_x = w(y)$, where y plays the role of the independent variable, and taking into account the relations $y''_{xx} = w'_x = w'_y y'_x = w'_y w$, one can reduce (3.2.1.2) to a first-order equation: $F(y, w, ww'_y) = 0$.

Example 3.1. Consider the autonomous equation

$$y''_{xx} = f(y),$$

which often arises in the theory of heat and mass transfer and combustion. The change of variable $y'_x = w(y)$ leads to a separable first-order equation: $ww'_y = f(y)$. Integrating yields

$$w^2 = 2F(w) + C_1, \quad \text{where} \quad F(w) = \int f(w)\,dw.$$

where C_1 is an arbitrary constant. Solving for w and returning to the original variable, we obtain the separable equation $y'_x = \pm\sqrt{2F(w) + C_1}$. Its general solution is expressed as

$$\int \frac{dy}{\sqrt{2F(w) + C_1}} = \pm x + C_2,$$

or

$$\left[\int \frac{dy}{\sqrt{F(w) + c_1}}\right]^2 = 2(x + c_2)^2,$$

where C_2, c_1, and c_2 are arbitrary constants.

Remark 3.1. The equation $y''_{xx} = f(y + ax^2 + bx + c)$ is reduced by the change of variable $u = y + ax^2 + bx + c$ to an autonomous equation, $u''_{xx} = f(u) + 2a$.

► **Related equations.**

Consider equations of the form

$$F(ax + by, y'_x, y''_{xx}) = 0.$$

Such equations are invariant under simultaneous translations of the independent and dependent variables in accordance with the rule $x \to x + bc$, $y \to y - ac$, where c is an arbitrary constant.

For $b = 0$, see equation (3.2.1.1). For $b \neq 0$, the substitution $bw = ax + by$ leads to equation (3.2.1.2): $F(bw, w'_x - a/b, w''_{xx}) = 0$.

3.2.2 Homogeneous Equations

► **Equations homogeneous in the independent variable.**

The *equations homogeneous in the independent variable* remain unchanged under scaling of the independent variable, $x \to \alpha x$, where α is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$F(y, xy'_x, x^2 y''_{xx}) = 0. \qquad (3.2.2.1)$$

The substitution $z(y) = xy'_x$ leads to a first-order equation: $F(y, z, zz'_y - z) = 0$.

► **Equations homogeneous in the dependent variable.**

The *equations homogeneous in the dependent variable* remain unchanged under scaling of the variable sought, $y \to \alpha y$, where α is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$F(x, y'_x/y, y''_{xx}/y) = 0. \qquad (3.2.2.2)$$

The substitution $z(x) = y'_x/y$ leads to a first-order equation: $F(x, z, z'_x + z^2) = 0$.

► **Equations homogeneous in both variables.**

The *equations homogeneous in both variables* are invariant under simultaneous scaling (dilatation) of the independent and dependent variables, $x \to \alpha x$ and $y \to \alpha y$, where α is an arbitrary nonzero number. In the general case, such equations can be written in the form

$$F(y/x, y'_x, xy''_{xx}) = 0. \qquad (3.2.2.3)$$

The transformation $t = \ln|x|$, $w = y/x$ leads to an autonomous equation

$$F(w, w'_t + w, w''_{tt} + w'_t) = 0,$$

see Section 3.2.1.

Example 3.2. The homogeneous equation

$$xy''_{xx} - y'_x = f(y/x)$$

is reduced by the transformation $t = \ln|x|$, $w = y/x$ to the autonomous form: $w''_{tt} = f(w) + w$. For the solution of this equation, see Example 3.1 in Section 3.2.1 (the function on the right-hand side has to be changed there).

3.2.3 Generalized Homogeneous Equations

► **Equations of a special form.**

The *generalized homogeneous equations* remain unchanged under simultaneous scaling of the independent and dependent variables in accordance with the rule $x \to \alpha x$ and $y \to \alpha^k y$, where α is an arbitrary nonzero number and k is some number. Such equations can be written in the form

$$F(x^{-k}y, x^{1-k}y'_x, x^{2-k}y''_{xx}) = 0. \tag{3.2.3.1}$$

The transformation $t = \ln x$, $w = x^{-k}y$ leads to an autonomous equation (see Section 3.2.1):

$$F\bigl(w, w'_t + kw, w''_{tt} + (2k-1)w'_t + k(k-1)w\bigr) = 0.$$

► **Equations of the general form.**

The most general form of representation of generalized homogeneous equations is as follows:

$$\mathcal{F}(x^n y^m, xy'_x/y, x^2y''_{xx}/y) = 0. \tag{3.2.3.2}$$

The transformation $z = x^n y^m$, $u = xy'_x/y$ brings this equation to the first-order equation

$$\mathcal{F}\bigl(z, u, z(mu+n)u'_z - u + u^2\bigr) = 0.$$

Remark 3.2. For $m \neq 0$, equation (3.2.3.2) is equivalent to equation (3.2.3.1) in which $k = -n/m$. To the particular values $n = 0$ and $m = 0$ there correspond equations (3.2.2.1) and (3.2.2.2) homogeneous in the independent and dependent variables, respectively. For $n = -m \neq 0$, we have an equation homogeneous in both variables, which is equivalent to equation (3.2.2.3).

3.2.4 Equations Invariant under Scaling–Translation Transformations

► **Equations of the fist type.**

The equations of the form

$$F(e^{\lambda x}y, e^{\lambda x}y'_x, e^{\lambda x}y''_{xx}) = 0 \tag{3.2.4.1}$$

remain unchanged under simultaneous translation and scaling of variables, $x \to x + \alpha$ and $y \to \beta y$, where $\beta = e^{-\alpha\lambda}$ and α is an arbitrary number. The substitution $w = e^{\lambda x}y$ brings (3.2.4.1) to an autonomous equation: $F(w,\, w'_x - \lambda w,\, w''_{xx} - 2\lambda w'_x + \lambda^2 w) = 0$ (see Section 3.2.1).

► **Equations of the first type. Alterative representation.**

The equation

$$F(e^{\lambda x}y^n, y'_x/y, y''_{xx}/y) = 0 \tag{3.2.4.2}$$

is invariant under the simultaneous translation and scaling of variables, $x \to x + \alpha$ and $y \to \beta y$, where $\beta = e^{-\alpha\lambda/n}$ and α is an arbitrary number. The transformation $z = e^{\lambda x}y^n$, $w = y'_x/y$ brings (3.2.4.2) to a first-order equation: $F\bigl(z, w, z(nw+\lambda)w'_z + w^2\bigr) = 0$.

► **Equations of the second type.**

The equation

$$F(x^n e^{\lambda y}, xy'_x, x^2 y''_{xx}) = 0 \tag{3.2.4.3}$$

is invariant under the simultaneous scaling and translation of variables, $x \to \alpha x$ and $y \to y + \beta$, where $\alpha = e^{-\beta\lambda/n}$ and β is an arbitrary number. The transformation $z = x^n e^{\lambda y}$, $w = xy'_x$ brings (3.2.4.3) to a first-order equation: $F\big(z, w, z(\lambda w + n)w'_z - w\big) = 0$.

3.2.5 Exact Second-Order Equations

The second-order equation

$$F(x, y, y'_x, y''_{xx}) = 0 \tag{3.2.5.1}$$

is said to be exact if it is the total differential of some function, $F = \varphi'_x$, where $\varphi = \varphi(x, y, y'_x)$. If equation (3.2.5.1) is exact, then we have a first-order equation for y:

$$\varphi(x, y, y'_x) = C, \tag{3.2.5.2}$$

where C is an arbitrary constant.

If equation (3.2.5.1) is exact, then $F(x, y, y'_x, y''_{xx})$ must have the form

$$F(x, y, y'_x, y''_{xx}) = f(x, y, y'_x)y''_{xx} + g(x, y, y'_x). \tag{3.2.5.3}$$

Here f and g are expressed in terms of φ by the formulas

$$f(x, y, y'_x) = \frac{\partial\varphi}{\partial y'_x}, \quad g(x, y, y'_x) = \frac{\partial\varphi}{\partial x} + \frac{\partial\varphi}{\partial y}y'_x. \tag{3.2.5.4}$$

By differentiating (3.2.5.4) with respect to x, y, and $p = y'_x$, we eliminate the variable φ from the two formulas in (3.2.5.4). As a result, we have the following test relations for f and g:

$$\begin{aligned} f_{xx} + 2pf_{xy} + p^2 f_{yy} &= g_{xp} + pg_{yp} - g_y, \\ f_{xp} + pf_{yp} + 2f_y &= g_{pp}. \end{aligned} \tag{3.2.5.5}$$

Here the subscripts denote the corresponding partial derivatives.

If conditions (3.2.5.5) hold, then equation (3.2.5.1) with F of (3.2.5.3) is exact. In this case, we can integrate the first equation in (3.2.5.4) with respect to $p = y'_x$ to determine $\varphi = \varphi(x, y, y'_x)$:

$$\varphi = \int f(x, y, p)\,dp + \psi(x, y), \tag{3.2.5.6}$$

where $\psi(x, y)$ is an arbitrary function of integration. This function is determined by substituting (3.2.5.6) into the second equation in (3.2.5.4).

Example 3.3. The left-hand side of the equation

$$yy''_{xx} + (y'_x)^2 + 2axyy'_x + ay^2 = 0 \tag{3.2.5.7}$$

can be represented in the form (3.2.5.3), where $f = y$ and $g = p^2 + 2axyp + ay^2$. It is easy to verify that conditions (3.2.5.5) are satisfied. Hence, equation (3.2.5.7) is exact. Using (3.2.5.6), we obtain

$$\varphi = yp + \psi(x, y). \tag{3.2.5.8}$$

Substituting this expression into the second equation in (3.2.5.4) and taking into account the relation $g=p^2+2axyp+ay^2$, we find that $2axyp+ay^2=\psi_x+p\psi_y$. Since $\psi=\psi(x,y)$, we have $2axy=\psi_y$ and $ay^2=\psi_x$. Integrating yields $\psi=axy^2+\text{const}$. Substituting this expression into (3.2.5.8) and taking into account relation (3.2.5.2), we find a first integral of equation (3.2.5.7):

$$yp+axy^2=C_1, \qquad \text{where} \quad p=y'_x.$$

Setting $w=y^2$, we arrive at the first-order linear equation $w'_x+2axw=2C_1$, which is easy to integrate. Thus, we find the solution of the original equation in the form:

$$y^2=2C_1\exp\bigl(-ax^2\bigr)\int\exp\bigl(ax^2\bigr)\,dx+C_2\exp\bigl(-ax^2\bigr).$$

3.2.6 Nonlinear Equations Involving Linear Homogeneous Differential Forms

Consider the nonlinear differential equation

$$F\bigl(x,\mathrm{L}_1[y],\mathrm{L}_2[y]\bigr)=0, \tag{3.2.6.1}$$

where the $\mathrm{L}_n[y]$ are linear homogeneous differential forms,

$$\mathrm{L}_n[y]=\sum_{m=0}^{2}\varphi_{\mathrm{m}}^{(n)}(x)y_x^{(m)}, \qquad n=1,\,2.$$

Let $y_0=y_0(x)$ be a common particular solution of the two linear equations

$$\mathrm{L}_1[y_0]=0, \qquad \mathrm{L}_2[y_0]=0.$$

Then the substitution

$$w=\psi(x)\bigl[y_0(x)y'_x-y'_0(x)y\bigr] \tag{3.2.6.2}$$

with an arbitrary function $\psi(x)$ reduces by one the order of equation (3.2.6.1).

Example 3.4. Consider the second-order nonlinear equation

$$y''_{xx}=f(x)g(xy'_x-y).$$

It can be represented in the form (3.2.6.1) with

$$F(x,u,w)=w-f(x)g(u), \qquad u=\mathrm{L}_1[y]=xy'_x-y, \qquad w=\mathrm{L}_2[y]=y''_{xx}.$$

The linear equations $\mathrm{L}_n[y]=0$ are

$$xy'_x-y=0, \qquad y''_{xx}=0.$$

These equations have a common particular solution $y_0=x$. Therefore, the substitution $w=xy'_x-y$ (see formula (3.2.6.2) with $\psi(x)=1$) leads to a first-order equation with separable variables:

$$w'_x=xf(x)g(w).$$

For the solution of this equation, see Section 1.2.1.

3.2.7 Reduction of Quasilinear Equations to the Normal Form

Consider the quasilinear equation

$$y''_{xx} + f(x)y'_x + g(x)y = \Phi(x, y) \qquad (3.2.7.1)$$

with linear left-hand side and nonlinear right-hand side. Let $y_1(x)$ and $y_2(x)$ form a fundamental system of solutions of the truncated linear equation corresponding to $\Phi \equiv 0$. The transformation

$$\xi = \frac{y_2(x)}{y_1(x)}, \qquad u = \frac{y}{y_1(x)} \qquad (3.2.7.2)$$

brings equation (3.2.7.1) to the normal form:

$$u''_{\xi\xi} = \Psi(\xi, u), \qquad \text{where} \quad \Psi(\xi, u) = \frac{y_1^3(x)}{W^2(x)}\Phi\bigl(x, y_1(x)u\bigr).$$

Here, $W(x) = y_1 y_2' - y_2 y_1'$ is the Wronskian of the truncated equation; and the variable x must be expressed in terms of ξ using the first relation in (3.2.7.2).

Transformation (3.2.7.2) is convenient for the simplification and classification of equations having the form (3.2.7.1) with $\Phi(x, y) = h(x)y^k$, thus reducing the number of functions from three to one: $\{f, g, h\} \Longrightarrow \{0, 0, h_1\}$.

Example 3.5. Consider the equation

$$y''_{xx} - y'_x = e^{2x} f(y). \qquad (3.2.7.3)$$

A fundamental system of solutions of the truncated linear equation with $f(y) \equiv 0$ are $y_1(x) = 1$ and $y_2(x) = e^x$. The transformation

$$\xi = e^x, \qquad u = y$$

brings equation (3.2.7.3) to the normal form:

$$u''_{xx} = f(u).$$

For solution of this autonomous equation, see Example 3.1 in Section 3.2.1.

3.2.8 Equations Defined Parametrically and Differential-Algebraic Equations

► **Preliminary remarks.**

In fluid dynamics, one often employs *von Mises* or *Crocco type transformations* to lower the order of boundary layer equations (and also some reduced equations that follow from the *Navier–Stokes equations*). Such transformations use suitable first- or second-order partial derivatives as new independent variables. The resulting equations sometimes admit exact solutions that are represented in implicit or parametric form. This leads to the problem: how to obtain exact solutions of the original hydrodynamic equations using these intermediate solutions.

To solve this problem, one has to be able to solve nonlinear ordinary differential equations defined parametrically. Due to their unusual form, such non-classical ODEs have been given very little attention.

► **General form of equations defined parametrically. Some examples.**

In general, second-order ordinary differential equations defined parametrically are defined by two coupled equations of the form

$$F_1(x, y, y'_x, y''_{xx}, t) = 0, \qquad F_2(x, y, y'_x, y''_{xx}, t) = 0, \tag{3.2.8.1}$$

where $y = y(x)$ is an unknown function, $t = t(x)$ is a functional parameter, $F_1(\dots)$ and $F_2(\dots)$ are given functions of their arguments. Below we consider two cases.

1°. *Degenerate case.* We assume that the derivative y''_{xx} can be eliminated from the equations (3.2.8.1) and the resulting equation can be solved for y'_x to obtain $y'_x = F(x, y, t)$. Using this expression, we eliminate the first derivative from one of the equations (3.2.8.1) to get $F_3(x, y, y''_{xx}, t) = 0$ and then solve this equation for y''_{xx}. The outlined procedure reduces the original equation (3.2.8.1) to the *canonical form*

$$y'_x = F(x, y, t), \qquad y''_{xx} = G(x, y, t). \tag{3.2.8.2}$$

Note that parametrically defined nonlinear differential equations (3.2.8.2) form a special class of coupled *differential-algebraic equations*.

Below we give a description of a method for integrating such equations and list a few simple equations of this kind whose general solutions can be obtained in parametric form; we deal with the general case where the parameter t cannot be eliminated from the equations (3.2.8.2).

On differentiating the first equation in (3.2.8.2) with respect to t, we obtain $(y'_x)'_t = F_x x'_t + F_y y'_t + F_t$. Taking into account the relations $y'_t = F x'_t$ and $(y'_x)'_t = x'_t y''_{xx}$, we find that

$$x'_t y''_{xx} = F_x x'_t + F F_y x'_t + F_t. \tag{3.2.8.3}$$

Eliminating the second derivative y''_{xx} with the help of equation (3.2.8.2), we arrive at the first-order equation

$$(G - F_x - F F_y) x'_t = F_t. \tag{3.2.8.4}$$

Taking into account that $y'_t = F x'_t$, we rewrite (3.2.8.4) in the form

$$(G - F_x - F F_y) y'_t = F F_t. \tag{3.2.8.5}$$

Equations (3.2.8.4) and (3.2.8.5) represent a system of first-order equations for $x = x(t)$ and $y = y(t)$. If we manage to solve this system, we thus obtain a solution to the original equation (3.2.8.2) in parametric form.

In some cases, it may be more convenient to use one of the equations (3.2.8.4) or (3.2.8.5) and the first equation (3.2.8.2).

Remark 3.3. With the above manipulations, isolated solutions may be lost, which satisfy the relation $G - F_x - F F_y = 0$ (this issue requires a further analysis).

Let us look at two special cases.

1°. If

$$G = F_x + F F_y + a(t) b(x) F_t,$$

where $a(t)$, $b(x)$, and $F = F(x, y, t)$ are arbitrary functions, the variables in equation (3.2.8.4) separate, thus resulting in the solution

$$\int b(x)\,dx = \int \frac{dt}{a(t)} + C_1$$

with C_1 is an arbitrary constant.

2°. If

$$G = F_x + FF_y + a(t)b(y)FF_t,$$

where $a(t)$, $b(y)$, and $F = F(x, y, t)$ are arbitrary functions, the variables in equation (3.2.8.5) separate, thus resulting in the solution

$$\int b(y)\,dy = \int \frac{dt}{a(t)} + C_1$$

with C_1 is an arbitrary constant.

Below are a few simple equations of the form (3.2.8.2) whose general solution can be obtained in parametric form.

Example 3.6. Consider the second-order parametric ODE

$$y'_x = \varphi(t), \qquad y''_{xx} = \psi(t), \tag{3.2.8.6}$$

where t is the parameter, while $\varphi(t)$ and $\psi(t)$ are given, sufficiently arbitrary functions.

In this case,

$$F = \varphi(t), \qquad G = \psi(t).$$

Substituting these expressions into (3.2.8.4) gives the equation $\psi(t)x'_t = \varphi'_t(t)$, whose general solution is

$$x = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \tag{3.2.8.7}$$

where C_1 is an arbitrary constant. Expression (3.2.8.7) together with the first equation (3.2.8.6) represent a first-order parametric ODE of the form (1.8.3.7) with

$$f(t) = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \qquad g(t) = \varphi(t). \tag{3.2.8.8}$$

Substituting (3.2.8.8) into (1.8.3.9) yields the general solution to ODE (3.2.8.6) in parametric form:

$$x = \int \frac{\varphi'_t(t)}{\psi(t)}\,dt + C_1, \quad y = \int \frac{\varphi(t)\varphi'_t(t)}{\psi(t)}\,dt + C_2, \tag{3.2.8.9}$$

where C_1 and C_2 are arbitrary constants.

Example 3.7. Consider equation (3.2.8.2) with

$$F = f(x)g(y)h(t), \quad G = f^2(x)g(y)g'_y(y)h^2(t) - f'_x(x)g(y)\lambda(t), \tag{3.2.8.10}$$

where $f(x)$, $g(y)$, $h(t)$, and $\lambda(t)$ are arbitrary functions. Equation (3.2.8.4) now becomes

$$f'_x(x)\big[h(t) + \lambda(t)\big]x'_t = -f(x)h'_t(t), \tag{3.2.8.11}$$

and its general solution is expressed as

$$f(x) = C_1 E(t), \qquad E(t) = \exp\biggl[-\int \frac{h'_t(t)\,dt}{h(t) + \lambda(t)}\biggr], \tag{3.2.8.12}$$

where C_1 is an arbitrary constant. Substituting expressions (3.2.8.10) and (3.2.8.12) into the first equation (3.2.8.2), we arrive at the separable first-order equation

$$y'_t = -g(y)\frac{f^2(x)}{f'_x(x)}\frac{h(t)h'_t(t)}{h(t)+\lambda(t)}, \tag{3.2.8.13}$$

in which x must be expressed via t using the integral (3.2.8.12).

In particular, if $f(x) = x$, the general solution to equation (3.2.8.13) is

$$\int \frac{dy}{g(y)} = -C_1^2 \int \frac{h(t)h'_t(t)E^2(t)}{h(t)+\lambda(t)}\,dt + C_2. \tag{3.2.8.14}$$

Formulas (3.2.8.12) and (3.2.8.14), where C_1 and C_2 are arbitrary constants, define the general solution to equation (3.2.8.2), (3.2.8.10) with $f(x) = x$.

Example 3.8. Consider a special case of equation (3.2.8.2) with

$$G = F_x + FF_y, \tag{3.2.8.15}$$

where $F = F(x, y, t)$ is an arbitrary function. In this case, the expressions in parentheses in (3.2.8.4) and (3.2.8.5) vanish and equation (3.2.8.2) admits the first integral

$$y'_x = F(x, y, C_1),$$

where C_1 is an arbitrary constant. In addition, there is a singular solution which is described by the parametric first-order equation

$$y'_x = F(x, y, t), \qquad F_t(x, y, t) = 0.$$

2°. *Degenerate case.* Suppose one of the two equations in (3.2.8.1) does not contain derivatives. If the other equation can be solved for y''_{xx}, we obtained a parametrically defined equation of the form

$$F(x, y, t) = 0, \qquad y''_{xx} = G(x, y, y'_x, t). \tag{3.2.8.16}$$

By differentiation of the first relation, such equations can be reduced to a nonlinear system of second-order equations. Without writing out this system, we give an example of such an equation whose solution can be obtained in parametric form.

Example 3.9. Consider the following second-order ODE defined parametrically:

$$y = \varphi(t), \qquad y''_{xx} = \psi(t). \tag{3.2.8.17}$$

Its solution is sought in the parametric form

$$x = \int f(t)\,dt + A, \qquad y = \varphi(t). \tag{3.2.8.18}$$

The derivatives are expressed as

$$y'_x = \frac{y'_t}{x'_t} = \frac{\varphi'_t}{f}, \qquad y''_{xx} = (y'_x)'_x = \frac{(y'_x)'_t}{x'_t} = \frac{(\varphi'_t/f)'_t}{f}. \tag{3.2.8.19}$$

By comparing the second derivatives in (3.2.8.17) and (3.2.8.19), we obtain a first-order equation for $f = f(t)$:

$$(\varphi'_t/f)'_t = \psi f. \tag{3.2.8.20}$$

The differentiation with respect to t in (3.2.8.20) results in a Bernoulli equation, whose general solution is expressed as

$$f(t) = \pm\varphi'_t(t)\left[2\int \psi(t)\varphi'_t(t)\,dt + B\right]^{-1/2}, \tag{3.2.8.21}$$

where B is an arbitrary constant. Formulas (3.2.8.18) and (3.2.8.21) define the general solution to to equation (3.2.8.17) in parametric form.

► **Reduction of standard differential equations to parametric differential equations**

A standard second-order ODE of the form

$$y''_{xx} = G(x, y, y'_x) \tag{3.2.8.22}$$

can be represented as a parametric ODE defined by two relations

$$\begin{aligned} y'_x &= t, \\ y''_{xx} &= G(x, y, t). \end{aligned} \tag{3.2.8.23}$$

This equation is a special case of equation (3.2.8.2) with $F(x, y, t) = t$; it can be reduced to the standard system of first-order ODEs

$$\begin{aligned} G(x, y, t)\, x'_t &= 1, \\ G(x, y, t)\, y'_t &= t. \end{aligned} \tag{3.2.8.24}$$

This system is obtained by substituting $F = t$ into equations (3.2.8.4)–(3.2.8.5).

System (3.2.8.24) is useful for the numerical solution of blow-up Cauchy problems or problems with a root singularity, in which the solution $y = y(x)$ or its derivative become infinite at a finite value $x = x_*$ (the value x_* is unknown in advance and has to be determined in the solution of the problem). In such and similar problems, the critical value $x = x_*$ for equation (3.2.8.22) corresponds to $t \to \pm\infty$ for system (3.2.8.24). For how one can use system (3.2.8.24) for the numerical integration of equations of the form (3.2.8.22) in blow-up problems, see Section 3.8.7.

⊙ *Literature for Section 3.2*: E. L. Ince (1956), G. M. Murphy (1960), L. E. El’sgol’ts (1961), P. Hartman (1964), N. M. Matveev (1967), I. G. Petrovskii (1970), G. F. Simmons (1972), E. Kamke (1977), G. Birkhoff and Rota (1978), M. Tenenbaum and H. Pollard (1985), A. N. Tikhonov, A. B. Vasil’eva, and A. G. Sveshnikov (1985), R. Grimshaw (1991), M. Braun (1993), D. Zwillinger (1997), C. Chicone (1999), G. A. Korn and T. M. Korn (2000), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007), A. D. Polyanin (2016), A. D. Polyanin and A. I. Zhurov (2016a, 2016b).

3.3 Boundary Value Problems. Uniqueness and Existence Theorems. Nonexistence Theorems

◆ *Nonlinear boundary value problems for ODEs are much more complex for mathematical analysis than initial value problems. This is because initial value problems (with well-behaved functions) have unique solutions (i.e., are “well-posed”), whereas boundary value problems (even with well-behaved functions) may have one solution, several solutions, or no solution at all. This section highlights characteristic features of different classes of nonlinear boundary value problem, states useful theorems on existence or nonexistence of solutions, and discusses examples of specific problems having nonunique solutions.*

3.3.1 Uniqueness and Existence Theorems for Boundary Value Problems

► **Preliminary remarks.**

► **First boundary value problems. Existence theorems.**

We will be looking at boundary value problems for second-order nonlinear differential equations of the form

$$y''_{xx} = f(x, y, y'_x) \tag{3.3.1.1}$$

defined on the unit interval $0 \le x \le 1$ (as shown in Section 2.5.2, any finite interval for the independent variable can reduced to a unit interval) and subject to the first-type boundary conditions*

$$y(0) = A, \quad y(1) = B. \tag{3.3.1.2}$$

EXISTENCE THEOREMS. *The first boundary value problem (3.3.1.1)–(3.3.1.2) has at least one solution if the function $f = f(x, y, z)$ is continuous in the domain $\Omega = \{0 \le x \le 1,\ -\infty < y, z < \infty\}$ and any of the following four assumptions holds:*

1. $f(x, y, z)$ is bounded;

2. For sufficiently large $|y|$, the inequality $f(x, y, z) < k|y|$ holds, where $k < \sqrt{3\pi^3} \approx 9.645$;

3. $\dfrac{f(x,y,z)}{|y|+|z|} \to 0$ uniformly on the interval $0 \le x \le 1$ as $|y| + |z| \to \infty$; in addition, on each finite interval, f satisfies the Lipschitz condition

$$|f(x, y, z) - f(x, \bar{y}, \bar{z})| \le K|y - \bar{y}| + L|z - \bar{z}|, \tag{3.3.1.3}$$

where K and L are some positive numbers (Lipschitz constants);

4. f satisfies the Lipschitz condition (3.3.1.3) and has the form $f = \varphi(x, y) + \psi(x, y, z)$, where φ is continuous and monotonically increasing with respect to y, and $\dfrac{\psi(x,y,z)}{|y|+|z|} \to 0$ uniformly on the interval $0 \le x \le 1$ as $|y| + |z| \to \infty$.

UNIQUENESS AND EXISTENCE THEOREMS.

1. Let the function $f = f(x, y, z)$ be continuous in the domain $\Omega = \{0 \le x \le 1,\ -\infty < y, z < \infty\}$ and satisfy the Lipschitz condition (3.3.1.3). Then problem (3.3.1.1)–(3.3.1.2) has one and only one solution if the inequality $\frac{1}{8}K + \frac{1}{2}L < 1$ holds, where K and L are Lipschitz constants.

2. Let the function $f = f(x, y, z)$ be continuous in the domain $\Omega_N = \{0 \le x \le 1,\ -N \le y \le N,\ -4N \le z \le 4N\}$ and satisfy the Lipschitz condition (3.3.1.3) in Ω_N. In addition, let

$$m = \max_{0 \le x \le 1} |f(x, 0, 0)|, \qquad M = \max_{x,y,z \in \Omega_N} |f(x, y, z)|.$$

Then if

$$\alpha = \tfrac{1}{8}K + \tfrac{1}{2}L < 1$$

*First-, second-, third-, and mixed-type boundary conditions for second-order nonlinear differential equations are stated in exactly the same way as for linear equations; see Section 2.5.1.

and any of the two inequalities

$$\text{(i)}\quad m \le 8N(1-\alpha),$$
$$\text{(ii)}\quad M \le 8N$$

hold, then problem (3.3.1.1)–(3.3.1.2) *has one and only one solution* $y = y(x)$ *such that*

$$|y| \le N, \qquad |y'_x| \le 4N \qquad (0 \le x \le 1).$$

Remark 3.4. Under certain conditions, the unique solution to problem (3.3.1.1)–(3.3.1.2) can be obtained with Picard's method of successive approximations by solving the equations

$$y''_n = f(x, y_{n-1}, y'_{n-1}),$$

where each y_n is chosen so as to satisfy the boundary conditions (3.3.1.2); the desired solution is $y = \lim_{n\to\infty} y_n$. For the iterative process to converge, it suffices that the Lipschitz conditions (3.3.1.3) hold.

EXISTENCE THEOREMS (FOR EQUATIONS OF A SPECIAL FORM). *The first boundary value problem*

$$y''_{xx} = f(x, y); \qquad y(0) = A, \qquad y(1) = B \tag{3.3.1.4}$$

has at least one solution if $f = f(x, y)$ *is continuous in the domain* $\Omega = \{0 \le x \le 1,\ -\infty < y < \infty\}$ *and any of the following two assumptions holds:*

1. The function f *is monotonically increasing (nondecreasing) with respect to* y *and satisfies the Lipschitz condition* $|f(x, y) - f(x, \bar{y})| \le K|y - \bar{y}|$ *on each finite interval (or if* f_y *is bounded on each finite interval).*

2. If $A = B = 0$ *and the inequality*

$$\int_0^y f(x, t)\, dt \ge -c_1 y^2 - c_0$$

holds, where $c_0 \ge 0$ *and* $0 < c_1 < \frac{1}{2}\pi^2$.

Remark 3.5. Problem (3.3.1.4) has a unique solution if $f = f(x, y)$ is continuous in the domain Ω and satisfies the Lipschitz condition with the Lipschitz constant $K < \pi^2$.

► First boundary value problems. Lower and upper solution. Nagumo theorem.

Definition 1. Twice differentiable functions $u = u(x)$ and $v = v(x)$ are said to be *a lower and an upper solution to the boundary value problem* (3.3.1.4) if the following inequalities hold:

$$\begin{gathered} u''_{xx} - f(x, u) \ge 0 \quad \text{at} \quad 0 < x < 1; \\ v''_{xx} - f(x, v) \le 0 \quad \text{at} \quad 0 < x < 1; \\ u(0) \le A \le v(0), \quad u(1) \le B \le v(1). \end{gathered} \tag{3.3.1.5}$$

Here, $u(0) = \lim_{x\to 0} u(x)$; the values $v(0)$, $u(1)$, and $v(1)$ are defined likewise.

NAGUMO-TYPE THEOREM (FOR EQUATIONS OF A SPECIAL FORM). *Let the boundary value problem* (3.3.1.4) *have a lower solution* $u = u(x)$ *and an upper solution* $v = v(x)$*,* *with* $u(x) \le v(x)$ *for* $0 \le x \le 1$*. In addition, let* $f(x, y)$ *be continuous and satisfy the Lipschitz condition on* $0 \le x \le 1$ *with* $u(x) \le y \le v(x)$*. Then there exists a solution* $y = y(x)$ *to problem* (3.3.1.4) *satisfying the inequalities*

$$u(x) \le y \le v(x) \qquad (0 \le x \le 1). \tag{3.3.1.6}$$

This theorem allows one to effectively determine the domain of existence of solutions to some classes of nonlinear boundary value problems. The linear functions $u = C_1 + D_1 x$ and $v = C_2 + D_2 x$ can be used as lower and upper solutions, with the coefficients C_i and D_i chosen so as to satisfy the inequalities (3.3.1.5).

Example 3.10. Consider the first boundary value problem for the Emden–Fowler equation

$$y''_{xx} = x^n y^m; \qquad y(0) = A, \quad y(1) = B. \tag{3.3.1.7}$$

Let $n \ge 0$, $m > 1$, $A \ge 0$, and $B > 0$. In this case, $u(x) \equiv 0$ is a lower solution. Any constant C such that $C \ge \max[A, B]$ can be taken to be the upper solution, $v(x) = C$. Then, by the Nagumo-type theorem, there is a nonnegative solution to the boundary value problem (3.3.1.7) satisfying the inequalities

$$0 \le y(x) \le \max[A, B].$$

Example 3.11. Consider the first boundary value problem for the equation with a cubic nonlinearity

$$y''_{xx} = y[y + g(x)][y - h(x)]; \qquad y(0) = A, \quad y(1) = B, \tag{3.3.1.8}$$

where $g(x) > 0$ and $h(x) > 0$ are continuous functions in the domain $0 \le x \le 1$.

Let $A \ge 0$ and $B > 0$. In this case, $u(x) \equiv 0$ is a lower solution. Let $h_{\max} = \max\limits_{0 \le x \le 1} h(x)$.

We will show that any constant C such that $C \ge \max[A, B, h_{\max}]$ can be taken as the upper solution, $v(x) = C$. Indeed, we have $f(x, v) \ge 0$ and, therefore, $v''_{xx} - f(x, v) \le 0$. Then, by the Nagumo-type theorem, there exists a nonnegative solution to the boundary value problem (3.3.1.8) satisfying the inequalities

$$0 \le y(x) \le \max[A, B, h_{\max}]. \tag{3.3.1.9}$$

The estimate (3.3.1.9) can be improved. To this end, the lower solution can be taken in the form $u = \delta > 0$, where $\delta \le \min[A, B, h_{\min}]$ with $h_{\min} = \min\limits_{0 \le x \le 1} h(x)$. The upper solution will be left unchanged. It follows that there exists a nonnegative solution to the boundary value problem (3.3.1.8) satisfying the inequalities

$$\min[A, B, h_{\min}] \le y(x) \le \max[A, B, h_{\max}].$$

Definition 2. The function $f(x, y, z)$ will be said to belong to the *class of Nagumo functions* on a set $(x, y) \in D$ if there is a positive continuous function $\varphi(z)$ satisfying the following two conditions:

(i) $|f(x, y, z)| \le \varphi(|z|)$ for all $(x, y) \in D$ and $-\infty < z < \infty$;

(ii) $\displaystyle\int_0^\infty \frac{z\,dz}{\varphi(z)} = \infty$.

NAGUMO THEOREM. *Let* $u(x)$ *be a lower solution and* $v(x)$ *an upper solution to the first boundary value problem* (3.3.1.1)–(3.3.1.2) *such that*

1. *The inequality* $u(x) < v(x)$ *holds for* $0 \le x \le 1$.

2. *The function* $f(x, y, z)$ *belongs to the class of Nagumo functions on the set* $D = \{0 \le x \le 1,\ u(x) < y < v(x)\}$.

3. The function $f(x,y,z)$ is continuous in x and continuously differentiable with respect to y and z in the domain $0 \le x \le 1$, $u(x) < y < v(x)$, $-\infty < z < \infty$.

Then there exists at least one twice continuously differentiable solution $y = y(x)$ to problem (3.3.1.1)–(3.3.1.2) satisfying the inequalities

$$u(x) < y < v(x) \qquad (0 \le x \le 1).$$

► **Third boundary value problems.**

Let us consider the equation (3.3.1.1) with third-type boundary conditions

$$\alpha_0 y - \alpha_1 y'_x = A \quad \text{at} \quad x = 0, \tag{3.3.1.10}$$

$$\beta_0 y - \beta_1 y'_x = B \quad \text{at} \quad x = 1, \tag{3.3.1.11}$$

where α_0, α_1, β_0, and β_1 are nonnegative constants with $\alpha_0 + \alpha_1 > 0$, $\beta_0 + \beta_1 > 0$, and $\alpha_0 + \beta_0 > 0$.

EXISTENCE AND UNIQUENESS THEOREM. *There exists a unique solution $y = y(x)$ of the boundary value problem (3.3.1.1), (3.3.1.11) if the following conditions hold:*

1. The function $f(x,y,z)$ is continuous on the set $\Omega = \{0 \le x < \infty,\ -\infty < y, z < \infty\}$.
2. There exists an $M > 0$ such that $|f(x,y,z_2) - f(x,y,z_1)| \le M|z_2 - z_1|$, on Ω.
3. The function $f(x,y,z)$ is nondecreasing with respect to y on the set Ω.

3.3.2 Reduction of Boundary Value Problems to Integral Equations. Integral Identity. Jentzch Theorem

► **Reduction of boundary value problems to integral equations.**

We will be looking at boundary value problems for second-order nonlinear differential equations of the form*

$$y''_{xx} + \lambda f(x, y, y'_x) = 0 \tag{3.3.2.1}$$

with parameter λ and homogeneous boundary conditions of a different kind on the unit interval $0 \le x \le 1$.

Assuming $f(x, y(x), y'_x(x))$ to be a known function of x and using formula (2.5.3.1) with $r(x) = -\lambda f(x, y(x), y'_x(x))$ as well as suitable Green's functions for the operator $L[y] = y''_{xx}$ (see the first four rows of Table 2.2 with $a = 1$), we can represent boundary value problems for equation (3.3.2.1) subject to boundary conditions of the first or mixed kind as a nonlinear integral equation with constant limits of integration:

$$y(x) = \lambda \int_0^1 |G(x,\xi)| f(\xi, y(\xi), y'_\xi(\xi))\, d\xi. \tag{3.3.2.2}$$

The modulus of the Green's function is used to stress that the kernel of the integral operator is positive.

Table 3.1 lists a few Green's functions $|G(x,\xi)|$, which appear in the integral equation (3.3.2.2), for several boundary value problems on the unit interval $0 \le x \le 1$. Note that Table 3.1 contains a new Green's function (for the third boundary value problem) as compared to Table 2.2.

*Note that equations (3.3.1.1) and (3.3.2.1) differ in form.

TABLE 3.1
Kernel of the integral operator $G(x,\xi) = |G(x,\xi)|$ appearing on the right-hand side of equation (3.3.2.2) for some boundary value problems ($0 \le x \le 1$, $0 \le \xi \le 1$)

No.	Boundary value problem	Boundary conditions	Green's function, $G(x,\xi)$
1	First	$y(0) = y(1) = 0$	$x(1-\xi)$ if $x \le \xi$ $\xi(1-x)$ if $\xi \le x$
2	Mixed	$y(0) = y'_x(1) = 0$	x if $x \le \xi$ ξ if $\xi \le x$
3	Mixed	$y'_x(0) = y(1) = 0$	$1-\xi$ if $x \le \xi$ $1-x$ if $\xi \le x$
4	Mixed	$y(0) = 0$, $y(1) + ky'_x(1) = 0$ (with $k \ne -1$)	$\dfrac{x(k+1-\xi)}{k+1}$ if $x \le \xi$ $\dfrac{\xi(k+1-x)}{k+1}$ if $\xi \le x$
5	Third	$\alpha y(0) - \beta y'_x(0) = 0$, $\gamma y(1) + \delta y'_x(1) = 0$ (with $\alpha, \beta, \gamma, \delta \ge 0$ and $\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0$)	$\dfrac{1}{\rho}(\beta + \alpha x)(\gamma + \delta - \gamma\xi)$ if $x \le \xi$ $\dfrac{1}{\rho}(\beta + \alpha\xi)(\gamma + \delta - \gamma x)$ if $\xi \le x$

POSITIVE PROPERTY SOLUTIONS. If $\lambda > 0$ and $f > 0$ (f can be zero at isolated points $x = x_k$) and a boundary value problem for the nonlinear ODE (3.3.2.1) from Table 3.1 has a solution, then the right-hand side of the integral equation (3.3.2.2) is positive, and hence, the desired function $y = y(x)$ (on the left-hand side) is positive in the domain $0 < x < 1$.

► Integral identity.

Let us multiply the differential equation (3.3.2.1) by a test function $u = u(x)$ and then integrate with respect to x from 0 to 1 while using the identity $uy''_{xx} = (uy'_x)'_x - (yu'_x)'_x + yu''_{xx}$ to obtain

$$u(1)y'_x(1) - y(1)u'_x(1) - u(0)y'_x(0) + y(0)u'_x(0) + \int_0^1 y(x)u''_{xx}(x)\,dx + \lambda \int_0^1 u(x)f(x, y(x), y'_x(x))\,dx = 0. \quad (3.3.2.3)$$

By choosing different test functions $u = u(x)$ in (3.3.2.3), we will be analyzing important qualitative features of some nonlinear boundary value problems in subsequent paragraphs.

► Properties of integral equations with positive kernel. Jentzch theorem.

A number σ is called a *characteristic value* of the linear integral equation

$$u(x) - \sigma \int_a^b K(x,t)u(t)\,dt = f(x)$$

if there exist nontrivial solutions of the corresponding homogeneous equation (with $f(x) \equiv 0$). The nontrivial solutions themselves are called the *eigenfunctions* of the integral equation corresponding to the characteristic value σ.

A kernel $K(x,t)$ of an integral operator $I[u] = \int_a^b K(x,\xi)u(\xi)\,d\xi$ is said to be *positive definite* if for all functions $\varphi(x)$ that are not identically zero we have

$$\int_a^b \int_a^b K(x,\xi)\varphi(x)\varphi(\xi)\,dx\,d\xi > 0,$$

and the above quadratic functional vanishes for $\varphi(x) = 0$ only. Such a kernel has positive characteristic values only. It is allowed that the kernel may vanish at isolated points (on a set of zero measure) of the domain $a \le x, t \le b$.

GENERALIZED JENTZCH THEOREM. *If a continuous or polar kernel $K(x,t)$ is positive, then its characteristic values σ_0 with the smallest modulus is positive and simple, and the corresponding eigenfunction $u_0(x)$ does not change sign on the interval $a \le x \le b$.*

3.3.3 Theorem on Nonexistence of Solutions to the First Boundary Value Problem. Theorems on Existence of Two Solutions

► **Theorem on nonexistence of solutions to the first boundary value problem.**

KEY ASSUMPTIONS:

1°. Let $\lambda > 0$ and $f(x,y,z) > 0$ be a continuous function in the domain $0 < x < 1$, $-\infty < y, z < \infty$ (f can be zero at finitely many isolated points $x = x_k$).

2°. Suppose that Assumption 1 holds and the function appearing in equation (3.3.2.1) possesses the property

$$f(x,y,z) > ay, \quad \text{where} \quad a > 0, \ \ y > 0. \tag{3.3.3.1}$$

Consider the nonlinear boundary value problem for equation (3.3.2.1) with the homogeneous boundary conditions of the first kind

$$y(0) = 0, \quad y(1) = 0. \tag{3.3.3.2}$$

We assume that the problem has at least one solution. Let us take

$$u(x) = \sin(\pi x) \tag{3.3.3.3}$$

to be the test function, which possesses the properties

$$u(0) = u(1) = 0, \quad u(x) > 0 \ \text{ for } \ 0 < x < 1, \quad u''_{xx}(x) = -\pi^2 u(x). \tag{3.3.3.4}$$

By virtue of conditions (3.3.3.2) and (3.3.3.4), the first line of the integral identity (3.3.2.3) is zero. Using the last relation from (3.3.3.4), we rewrite (3.3.2.3) in the form

$$\begin{aligned}\int_0^1 y(x)u''_{xx}(x)\,dx + \lambda\int_0^1 u(x)f(x,y(x),y'_x(x))\,dx \\ = \int_0^1 u(x)\big[\lambda f(x,y(x),y'_x(x)) - \pi^2 y(x)\big]\,dx = 0.\end{aligned} \tag{3.3.3.5}$$

Using the key assumptions above, we obtain the estimate

$$\int_0^1 u(x)\big[\lambda f(x,y(x),y'_x(x))-\pi^2 y\big]\,dx > \int_0^1 (\lambda a-\pi^2)u(x)y(x)\,dx. \qquad (3.3.3.6)$$

Since $u(x)$ and $y(x)$ are both positive on $0<x<1$ (see the positive property solutions at the end of Section 3.3.2 and (3.3.3.4)), the second integral in (3.3.3.6) must also be positive, provided that $\lambda>\pi^2/a$. On the other hand, if the first integral in (3.3.3.6) is zero, the second integral must be negative. This contradiction, obtained under the assumption that the problem has a solution, allows us to state the following theorem.

NONEXISTENCE THEOREM (FIRST BOUNDARY VALUE PROBLEM). *If the key assumptions (see the beginning of this section) are valid and λ is a sufficiently large number such that*

$$\lambda>\pi^2/a, \qquad (3.3.3.7)$$

the first boundary value problem for equation (3.3.2.1) *subject to the boundary conditions* (3.3.3.2) *does not have solutions.*

Examples of mixed boundary value problems that do not have solutions can be found in Section 3.3.4.

► On the evaluation of the constant a appearing in condition (3.3.3.1).

Let us look at the nonlinear boundary value problem for the autonomous equation

$$y''_{xx}+\lambda f(y)=0 \qquad (3.3.3.8)$$

subject to the boundary conditions of the first kind (3.3.3.2). Note that equation (3.3.3.8) coincides, up to notation, with the autonomous equation considered in Example 3.1, which admits order reduction and is easy to integrate.

We assume that the conditions

$$f>0 \ \text{ for } \ -\infty<y<\infty, \qquad f'_y\ge 0 \ \text{ for } \ y\ge 0, \qquad \lim_{y\to\infty} f'_y=\infty$$

hold. The constant a appearing in (3.3.3.1) can be evaluated as

$$a=\min_{0\le y<\infty}\frac{f(y)}{y}. \qquad (3.3.3.9)$$

Differentiating $f(y)/y$ with respect to y yields an algebraic (transcendental) equation for the minimum point y°:

$$f(y^\circ)-y^\circ f'_y(y^\circ)=0. \qquad (3.3.3.10)$$

Then a can be found using either formula

$$a=\frac{f(y^\circ)}{y^\circ} \quad \text{or} \quad a=f'_y(y^\circ). \qquad (3.3.3.11)$$

Example 3.12. In the first boundary value problem for the equation

$$y''_{xx}+\lambda(\alpha+\beta|y|^k)=0, \qquad \alpha,\beta>0, \quad k>1,$$

subject to the boundary conditions (3.3.3.2), the constant a appearing in (3.3.3.1) is found as

$$a=\beta k\Big[\frac{\alpha}{\beta(k-1)}\Big]^{\frac{k-1}{k}}.$$

► Theorems on existence of two solutions for the first boundary value problem.

Let us look at the nonlinear boundary value problem with homogeneous boundary conditions of the first kind

$$y''_{xx} + f(x,y) = 0 \quad (0 < x < 1); \qquad y(0) = y(1) = 0. \tag{3.3.3.12}$$

Let the function $f(x,y) \geq 0$ be continuous in the domain $\Omega = \{0 \leq x \leq 1,\ 0 \leq y < \infty\}$ and let $f(x,y) \not\equiv 0$ on any subinterval of $0 \leq x \leq 1$ for $y > 0$. We use the notation: $\|y\| = \sup_{0 \leq x \leq 1} |y(x)|$.

ERBE–HU–WANG THEOREM 1 (A SPECIAL CASE). *Suppose the following two assumptions are valid:*

1. *The limits relations*

$$\lim_{y\to 0}\min_{0\leq x\leq 1}\frac{f(x,y)}{y} = \lim_{y\to\infty}\min_{0\leq x\leq 1}\frac{f(x,y)}{y} = \infty \tag{3.3.3.13}$$

hold.

2. *There is a constant $p > 0$ such that*

$$f(x,y) \leq 6p \quad \text{for} \quad 0 \leq x \leq 1,\ 0 \leq y \leq p. \tag{3.3.3.14}$$

Then the first boundary value problem (3.3.3.12) has at least two positive solutions, $y_1 = y_1(x)$ and $y_2 = y_2(x)$, such that

$$0 < \|y_1\| < p < \|y_2\|.$$

Example 3.13. Consider the first boundary value problem

$$y''_{xx} + 1 + y^2 = 0 \quad (0 < x < 1); \qquad y(0) = y(1) = 0. \tag{3.3.3.15}$$

Condition (3.3.3.13) for this equation holds. Condition (3.3.3.14) becomes

$$1 + y^2 \leq 6p \quad \text{for} \quad 0 \leq y \leq p.$$

The maximum allowed value of p is determined from the quadratic equation $p^2 - 6p + 1 = 0$, which gives $p_{\rm m} = 3 + 2\sqrt{2} \approx 5.828$. Hence, by virtue of the Erbe–Hu–Wang theorem (see above), problem (3.3.3.12) has at least two positive solutions y_1 and y_2 such that $0 < \|y_1\| < p_{\rm m} < \|y_2\|$.

ERBE–HU–WANG THEOREM 2 (A SPECIAL CASE). *Let the following two assumptions be valid:*

1. *The limits relations*

$$\lim_{y\to 0}\max_{0\leq x\leq 1}\frac{f(x,y)}{y} = \lim_{y\to\infty}\max_{0\leq x\leq 1}\frac{f(x,y)}{y} = 0 \tag{3.3.3.16}$$

hold.

2. *There is a constant $q > 0$ such that*

$$f(x,y) \geq \tfrac{32}{3}q \quad \text{for} \quad \tfrac{1}{4} \leq x \leq \tfrac{3}{4},\ \tfrac{1}{4}q \leq y \leq q. \tag{3.3.3.17}$$

Then the boundary value problem (3.3.3.12) has at least two positive solutions $y_1 = y_1(x)$ and $y_2 = y_2(x)$ such that

$$0 < \|y_1\| < q < \|y_2\|.$$

Remark 3.6. The above Erbe–Hu–Wang theorems are special cases of more general theorems for boundary value problems of the third kind, which are stated below in Section 3.3.7.

3.3.4 Examples of Existence, Nonuniqueness, and Nonexistence of Solutions to First Boundary Value Problems

Below we exemplify the above qualitative features of nonlinear boundary value problems with boundary conditions of the first kind by looking at a few specific problems admitting exact analytical solutions.

► A nonlinear boundary value problem arising in combustion theory.

Example 3.14. Consider the nonlinear boundary value problem described by the equation

$$y''_{xx} + \lambda e^y = 0 \tag{3.3.4.1}$$

subject to the homogeneous boundary conditions of the first kind (3.3.3.2). Equation (3.3.4.1) arises in combustion theory, when the Frank-Kamenetskii approximation is used for the kinetic function, with y denoting dimensionless excess temperature, x dimensionless distance, and $\lambda \geq 0$ is the dimensionless rate of reaction. Equation (3.3.4.1) is a special case of equation (3.3.3.8).

Let us analyze the qualitative features of problem (3.3.4.1), (3.3.3.2) for different values of the determining parameters λ, which is assumed positive.

Equation (3.3.4.1) is a special case of the autonomous second-order equation considered in Example 3.1, which admits order reduction and is easy to integrate. With $\lambda > 0$, the general solution to equation (3.3.4.1) is

$$y = \ln\left[\frac{2c^2}{\lambda\cosh^2(cx+b)}\right], \tag{3.3.4.2}$$

where b and c are arbitrary constants. From the boundary conditions (3.3.3.2), we obtain a system of transcendental equations for b and c,

$$2c^2 = \lambda\cosh^2 b, \quad 2c^2 = \lambda\cosh^2(c+b),$$

which is convenient to rewrite in the equivalent form

$$\lambda = \frac{8b^2}{\cosh^2 b}, \qquad c = -2b. \tag{3.3.4.3}$$

The first equation serves to determine b, after which the evaluation of c is elementary.

The function $p(b) = 8b^2/\cosh^2 b$ is positive if $b \neq 0$, it tends to zero as $b \to 0$ and $b \to \infty$, and it has the only maximum equal to $\lambda_{\rm f}^* = \max p(b) = 3.5138$. It follows that if

$$\lambda > \lambda_{\rm f}^*,$$

the first equation in (3.3.4.3) has no solution; hence, the original boundary value problem (3.3.4.1), (3.3.3.2) has no solution either (the critical value $\lambda = \lambda_{\rm f}^*$ corresponds to a thermal explosion). For $0 < \lambda < \lambda_{\rm f}^*$, the first equation in (3.3.4.3) has two distinct positive roots, b_1 and b_2, which generate two different solutions of the original boundary value problem (3.3.4.1), (3.3.3.2). When $\lambda = \lambda_{\rm f}^*$, the roots b_1 and b_2 become the same, $b_1 = b_2 = b_{\rm f}^* \approx 1.1997$, to give a single solution to the original problem.

Let us assess the accuracy of the critical value $\lambda_{\rm f}^*$ by using the above theorem on nonexistence of solutions to the first boundary value problem. In this case, $f(x, y, y'_x) = e^y$. It is not difficult to show that $e^y \geq ey$ for $y > 0$, which suggests that $a = e$. Substituting this value into (3.3.3.7) gives an approximate estimate for the boundary of the nonexistence domain with respect to the parameter λ:

$$\lambda > \lambda_{\rm f}^{\rm ap} = \pi^2/e \approx 3.6311.$$

This value, provided by the nonexistence theorem, differs from the exact value $\lambda_{\rm f}^*$ by only 3.3% (which is a very high accuracy for a qualitative analysis).

Now let us estimate the boundaries of the existence domain for the two solutions using Erbe–Hu–Wang theorem 1. The first condition of the theorem, (3.3.3.13), clearly holds, since

$$\lim_{y\to 0}(e^y/y) = \lim_{y\to\infty}(e^y/y) = \infty.$$

The second condition, (3.3.3.14), can be rewritten in the form

$$\lambda \le 6pe^{-y} \quad \text{for} \quad 0 \le y \le p.$$

It follows that $\lambda \le 6pe^{-p}$. The left-hand side of this inequality attains a maximum at $p = 1$; hence, the condition $\lambda \le 6/e \approx 2.207$ must hold to ensure that the two solutions exist. This estimate is lower than the exact boundary of the existence domain of two solutions by 37.2%.

Remark 3.7. The second boundary value problem for equation (3.3.4.1) subject to the boundary conditions $y'_x(0) = y'_x(1) = 0$ for any $\lambda > 0$ has no solution. This is easy to see from the general solution (3.3.4.2).

► A problem on an electron beam passing between two electrodes.

Example 3.15. Consider the autonomous equation

$$y''_{xx} = \lambda y^{-1/2} \qquad (0 < x < 1) \tag{3.3.4.4}$$

subject to the nonhomogeneous boundary conditions

$$y(0) = 1, \quad y(1) = 1. \tag{3.3.4.5}$$

The following notation is used here: y is dimensionless potential, x is dimensionless distance, and $\lambda \ge 0$ is dimensionless electric current density (Zinchenko, 1958).

Remark 3.8. Problem (3.3.4.4)–(3.3.4.5) is quite interesting because it can be reduced, with the change of variable $u = 1 - y$, to a problem of the form (3.3.3.8), (3.3.3.2) for which the conditions of the theorems stated in Section 3.3.3 do not hold.

Problem (3.3.4.4)–(3.3.4.5) is symmetric about the mid-point $x = 1/2$. Therefore, it reaches a maximum at $x = 1/2$, with $y'_x(1/2) = 0$. With this in mind, we integrate equation (3.3.4.4) multiplied by $2y'_x$ from x to $1/2$ to obtain

$$(y'_x)^2 = 4\lambda\big(\sqrt{y} - C\big), \tag{3.3.4.6}$$

where $C = \sqrt{y}|_{x=1/2}$ is an arbitrary constant. Integrating again from x to $1/2$ and rearranging, we arrive at the solution in implicit form

$$(\sqrt{y} - C)(\sqrt{y} + 2C)^2 = \tfrac{9}{64}\lambda(2x-1)^2. \tag{3.3.4.7}$$

Formula (3.3.4.7) describes a family of third-order curves with respect to $\sqrt{y}$. The constant C depends on λ and satisfies the cubic equation

$$(1 - C)(1 + 2C)^2 = \tfrac{9}{64}\lambda, \tag{3.3.4.8}$$

which is obtained by inserting the boundary conditions (3.3.4.5) into equation (3.3.4.7) (both boundary conditions result in the same equation for C).

Since $\lambda \ge 0$, it follows from equation (3.3.4.8) that $C \le 1$. On the other hand, from the first integral (3.3.4.6) we get $C = \sqrt{y}_{\min} \ge 0$. In the range $0 \le C \le 1$, the maximum of the left-hand side of equation (3.3.4.8) is attained at $C = \frac{1}{2}$, which gives $\lambda_{\max} = \frac{128}{9} \approx 14.22$. Hence, problem (3.3.4.4)–(3.3.4.5) has no solution for $\lambda > \lambda_{\max}$.

A more detailed analysis of the curve (3.3.4.7) shows that three different situations are possible depending on the value of λ:

(i) If $0 \le \lambda < \lambda_*$ ($\lambda_* = \frac{64}{9} \approx 7.11$), problem (3.3.4.4)–(3.3.4.5) has only one solution, which corresponds to the only root of the cubic equation (3.3.4.8) in the domain $\frac{\sqrt{3}}{2} < C \le 1$. For small λ, equation (3.3.4.8) provides the asymptotic behavior

$$C = 1 - \tfrac{1}{64}\lambda + o(\lambda) \qquad (\lambda \to 0).$$

(ii) If $\lambda_* \le \lambda < 2\lambda_* = \lambda_{\max} \approx 14.22$, problem (3.3.4.4)–(3.3.4.5) has two solutions, which correspond to two distinct roots of the cubic equation (3.3.4.8) in the domain $0 \le C \le \frac{\sqrt{3}}{2} \approx 0.866$. For the upper curve, which has a physical meaning (the other solution has no physical meaning), the value of C gradually decreases as λ increases. When $\lambda = \lambda_{\max}$, which corresponds to $C_{1,2} = \frac{1}{2}$, the two solutions become the same.

(iii) If $\lambda > \lambda_{\max}$, the problem has no solution.

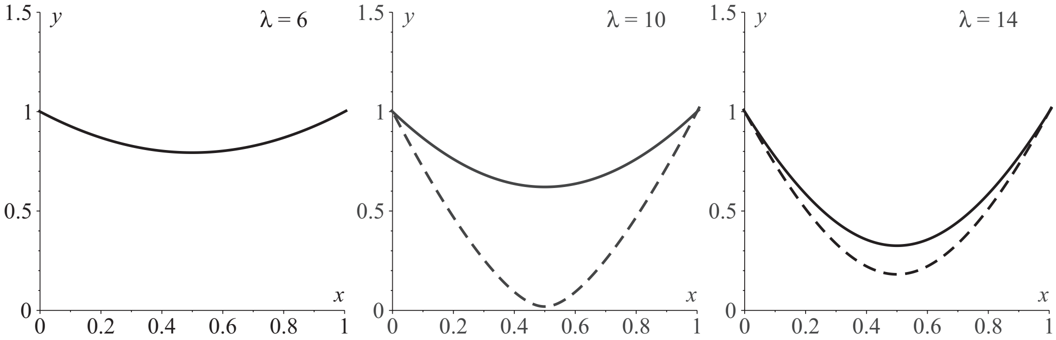

Figure 3.1: Solutions to problem (3.3.4.4)–(3.3.4.5) for different values of λ.

Figure 3.1 displays solutions to problem (3.3.4.4)–(3.3.4.5) for different values of the parameter: $\lambda = 6,\ 10,\ 14$; the dashed lines correspond to the second solution (which has no physical meaning). Figure 3.2 shows the dependence of the roots $C_{1,2}$ of the cubic equation (3.3.4.8) on the parameter λ (the root C_1 corresponds to the solution having a physical meaning).

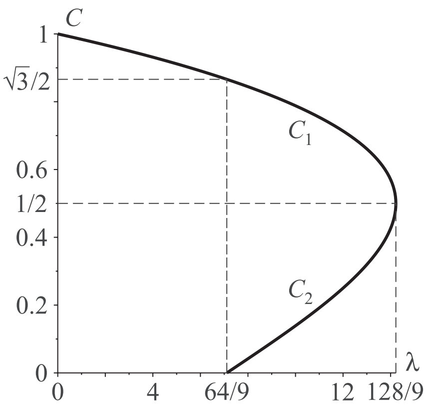

Figure 3.2: Dependence of the roots of the cubic equation (3.3.4.8) on the parameter λ (the root C_1 corresponds to the solution having a physical meaning).

Remark 3.9. For $\lambda < 0$, the boundary value problem (3.3.4.4)–(3.3.4.5) has no solution.

► A model boundary value problem with the modulus of the unknown.

Example 3.16. Consider the nonlinear boundary value problem

$$y''_{xx} + k^2|y| = 0 \qquad (0 < x < a); \tag{3.3.4.9}$$

$$y(0) = 0, \quad y(a) = -b, \tag{3.3.4.10}$$

where a, b, and k are all positive numbers.

Depending on the sign of y, the nonlinear equation (3.3.4.9) reduces to two linear equations, $y''_{xx} \pm k^2 y = 0$, whose solutions are expressed in terms of trigonometric and hyperbolic functions. For $ak > \pi$, problem (3.3.4.9) has two solutions:

$$y_1 = -\frac{b}{\sinh(ka)}\sinh(kx); \tag{3.3.4.11}$$

$$y_2 = \begin{cases} \dfrac{b}{\sinh(ka-\pi)}\sin(kx) & \text{if } 0 \le x \le \pi/k, \\ -\dfrac{b}{\sinh(ka-\pi)}\sinh(kx-\pi) & \text{if } \pi/k \le x \le a. \end{cases} \tag{3.3.4.12}$$

Here, $y_1 = y_1(x)$ is a monotonically decreasing function such that $y_1(x) \le 0$. The function $y_2 = y_2(x)$ monotonically increases for $0 \le x < \pi/(2k)$, attains a maximum at $x = \pi/(2k)$ and monotonically decreases for $\pi/(2k) < x \le a$. It is positive for $0 < x < \pi/k$, becomes zero at $x = \pi/k$, and is negative for $x > \pi/k$. For all $0 < x < a$, the inequality $y_2 > y_1$ holds.

◆ *See also Section 8.3.3.*

3.3.5 Theorems on Nonexistence of Solutions for the Mixed Problem. Theorems on Existence of Two Solutions

► Theorems on nonexistence of solutions for the mixed problem.

Let us look at the nonlinear boundary value problem for equation (3.3.2.1) subject to homogeneous mixed boundary conditions of the form

$$y'_x(0) = 0, \quad y(1) = 0. \tag{3.3.5.1}$$

It is assumed to have at least one solution.

Suppose that the key assumptions stated at the beginning of Section 3.3.3 are valid. This means that the function appearing in equation (3.3.2.1) has the property (3.3.3.1). Just as previously, we use the integral identity (3.3.2.3). We take

$$u(x) = \cos\left(\tfrac{\pi}{2}x\right) \tag{3.3.5.2}$$

as the test function; it possesses the properties

$$u'_x(0) = u(1) = 0, \quad u(x) > 0 \ \text{ for } \ 0 < x < 1, \quad u''_{xx}(x) = -\tfrac{1}{4}\pi^2 u(x). \tag{3.3.5.3}$$

By virtue of conditions (3.3.5.1) and (3.3.5.3), the first line of the integral identity (3.3.2.3) is zero. Now using the last relation in (3.3.5.3), we rewrite (3.3.2.3) in the form

$$\int_0^1 y(x)u''_{xx}(x)\,dx + \lambda\int_0^1 u(x)f(x,y(x),y'_x(x))\,dx = \int_0^1 u(x)\bigl[\lambda f(x,y(x),y'_x(x)) - \tfrac{1}{4}\pi^2 y(x)\bigr]\,dx = 0. \tag{3.3.5.4}$$

In view of inequality (3.3.3.1), it follows that

$$\int_0^1 u(x)\big[\lambda f(x, y(x), y'_x(x)) - \tfrac{1}{4}\pi^2 y\big]\,dx > \int_0^1 (\lambda a - \tfrac{1}{4}\pi^2)u(x)y(x)\,dx. \qquad (3.3.5.5)$$

Since $u(x)$ and $y(x)$ are both positive on $0<x<1$ (see positive property solutions at the end of Section 3.3.2 and (3.3.3.4)), the second integral in (3.3.5.5) must be positive, provided that $\lambda > \frac{1}{4}\pi^2/a$. On the other hand, the first integral in (3.3.5.5) is zero, suggesting that the second integral must be negative. This contradiction, obtained under the assumption that the problem has at least one solution, allows one to state the following theorem.

NONEXISTENCE THEOREM 1 (MIXED BOUNDARY VALUE PROBLEM). *If the key assumptions from Section 3.3.3 are valid and λ is a sufficiently large number such that*

$$\lambda > \tfrac{1}{4}\pi^2/a, \qquad (3.3.5.6)$$

the mixed boundary value problem for equation (3.3.2.1) *with the boundary conditions* (3.3.5.1) *has no solution.*

See Section 3.3.6 for examples of mixed boundary value problems having no solution.

► Generalization of nonexistence theorem 1 for the mixed problem.

Suppose that the function $f(x, y, z)$ appearing in (3.3.2.1) satisfies the inequality

$$f(x, y, z) \geq \varphi(x)y \qquad (0 < x < 1,\ y > 0), \qquad (3.3.5.7)$$

where $\varphi(x) > 0$ is a continuous function.

To be specific, we will consider a boundary value problem for equation (3.3.2.1) subject to the mixed boundary conditions

$$y'_x(0) = y(1) = 0. \qquad (3.3.5.8)$$

The problem is assumed to have at least one solution. Let us impose conditions on the test function $u(x)$ such that the first line of the integral identity (3.3.2.2) is zero. These are

$$u(0) = u'_x(1) = 0. \qquad (3.3.5.9)$$

As a result, equation (3.3.2.3) becomes

$$\int_0^1 y(x)u''_{xx}(x)\,dx + \lambda \int_0^1 u(x)f(x, y(x), y'_x(x))\,dx = 0. \qquad (3.3.5.10)$$

Let $u = u(x)$ satisfy the linear equation

$$u''_{xx} + \sigma\varphi(x)u = 0. \qquad (3.3.5.11)$$

where $\varphi(x)$ is the function appearing in inequality (3.3.5.7) and σ is some (spectral) parameter. The boundary value problem (3.3.5.11), (3.3.5.9) is equivalent to the integral equation

$$u(x) = \sigma \int_0^1 |G_2(x, \xi)|\varphi(\xi)u(\xi)\,d\xi, \qquad (3.3.5.12)$$

where $|G_2(x,\xi)|$ is the modulus of the Green's function shown in the second row of Table 3.1.

Since the kernel of the integral operator (3.3.5.12) is positive, it follows from the Jentzch theorem (see Section 3.3.2) that the least eigenvalue is positive, $\sigma_0 > 0$, and the corresponding eigenfunction $u_0(x)$ does not change its sign on $0 \le x \le 1$. In equations (3.3.5.10) and (3.3.5.11), we first set $u = u_0(x)$ and $\sigma = \sigma_0$ and then eliminate the second derivative $(u_0)''_{xx}$ from (3.3.5.10) with the help of (3.3.5.11). The resulting expression can be written as

$$\frac{\sigma_0}{\lambda} = \frac{\int_0^1 u_0(x) f(x, y(x), y'_x(x))\, dx}{\int_0^1 u_0(x)\varphi(x) y(x)\, dx}. \tag{3.3.5.13}$$

Since, by assumption, inequality (3.3.5.7) holds, it follows from (3.3.5.13) that $\sigma_0/\lambda \ge 1$. However, for sufficiently large $\lambda > \sigma_0$, this estimate cannot be ensured. For such values of λ, the boundary value problem (3.3.2.1), (3.3.2.1) surely has no solution. In the class of boundary value problems concerned, there is a critical value of the parameter, λ_*, that delimits the domains of existence and nonexistence of solutions. For $\lambda > \lambda_*$ with $\lambda_* < \sigma_0$, there are no solutions (σ_0 provides an upper estimate for the critical value λ_* beyond which there are no solutions).

These results allow us to state the following theorem on nonexistence of solutions to the mixed problem.

NONEXISTENCE THEOREM 2 (MIXED BOUNDARY VALUE PROBLEM). *If inequalities* (3.3.5.7) *hold and λ is sufficiently large, $\lambda > \lambda_* > 0$, the mixed boundary value problem for equation* (3.3.2.1) *subject to the boundary conditions* (3.3.5.8) *has no solution. The critical value satisfies the inequality $\lambda_* < \sigma_0$, where σ_0 is the least eigenvalue of the linear eigenvalue problem* (3.3.5.11), (3.3.5.9).

Remark 3.10. The nonexistence theorem can be elaborated further if the boundary value problem is for a nonlinear equation of the special form

$$y''_{xx} + \lambda\big[\varphi(x)g(y) + h(x,y,y'_x)\big] = 0$$

with the initial conditions (3.3.5.8). If the conditions

$$\varphi(x) > 0, \quad g(y) > 0, \quad \lim_{y\to\infty} g'_y(y) = \infty, \quad h(x,y,y'_x) \ge 0 \qquad (0 < x < 1,\ y > 0)$$

hold, the problem has no solution for sufficiently large $\lambda > \lambda_* > 0$.

► Theorems on existence of two solutions for the mixed boundary value problem.

Let us look at the nonlinear boundary value problem with homogeneous boundary conditions of the first kind

$$y''_{xx} + f(x,y) = 0 \quad (0 < x < 1); \qquad y(0) = y'_x(1) = 0. \tag{3.3.5.14}$$

Let the function $f(x,y) \ge 0$ be continuous in the domain $\Omega = \{0 \le x \le 1,\ 0 \le y < \infty\}$ and $f(x,y) \not\equiv 0$ on any subinterval of $0 \le x \le 1$ for $y > 0$. We use the notation: $\|y\| = \sup_{0\le x\le 1}|y(x)|$.

ERBE–HU–WANG THEOREM 1 (A SPECIAL CASE). *Let the following two assumptions hold:*

1. $\lim\limits_{y\to 0}\min\limits_{0\le x\le 1}\dfrac{f(x,y)}{y}=\lim\limits_{y\to\infty}\min\limits_{0\le x\le 1}\dfrac{f(x,y)}{y}=\infty.$
2. *There is a constant* $p>0$ *such that*

$$f(x,y)\le 2p \quad \text{for} \quad 0\le x\le 1,\ 0\le y\le p.$$

Then the first boundary value problem (3.3.5.14) *has at least two positive solutions,* $y_1=y_1(x)$ *and* $y_2=y_2(x)$*, such that*

$$0<\|y_1\|<p<\|y_2\|.$$

ERBE–HU–WANG THEOREM 2 (A SPECIAL CASE). *Let the following two assumptions hold:*

1. $\lim\limits_{y\to 0}\max\limits_{0\le x\le 1}\dfrac{f(x,y)}{y}=\lim\limits_{y\to\infty}\max\limits_{0\le x\le 1}\dfrac{f(x,y)}{y}=0.$
2. *There is a constant* $q>0$ *such that*

$$f(x,y)\ge \tfrac{32}{7}q \quad \text{for} \quad \tfrac14\le x\le\tfrac34,\ \tfrac14 q\le y\le q.$$

Then boundary value problem (3.3.5.14) *has at least two positive solutions,* $y_1=y_1(x)$ *and* $y_2=y_2(x)$*, such that*

$$0<\|y_1\|<q<\|y_2\|.$$

Remark 3.11. The above Erbe–Hu–Wang theorems are special cases of more general theorems for boundary value problems of the third kind, which are stated below in Section 3.3.7.

3.3.6 Examples of Existence, Nonuniqueness, and Nonexistence of Solutions to Mixed Boundary Value Problems

In this section, we exemplify the above qualitative features of nonlinear boundary value problems with mixed boundary conditions by looking at a few specific problems admitting exact analytical solutions.

► Plane problem I arising in combustion theory (Frank-Kamenetskii approximation).

Example 3.17. Consider a *one-dimensional problem on thermal explosion* in a plane channel described by equation (3.3.4.1) subject to the mixed boundary conditions (3.3.5.1):

$$y''_{xx}+\lambda e^y=0; \qquad y'_x(0)=y(1)=0, \tag{3.3.6.1}$$

where $y=y(x)$ is dimensionless excess temperature.

We proceed from the general solution to the equation, which is given by formula (3.3.4.2). Using the boundary conditions, we get the equations for the constants b and c:

$$b=0, \qquad \lambda=\frac{2c^2}{\cosh^2 c}. \tag{3.3.6.2}$$

The function $q(c)=2c^2/\cosh^2 c$ is positive for $c\ne 0$, it tends to zero as $c\to 0$ and $c\to\infty$, and it has the only maximum equal to $\lambda^*_{\rm m}=\max q(c)\approx 0.8785$. Consequently, if

$$\lambda>\lambda^*_{\rm m},$$

the second equation in (3.3.6.2) has no solution; it follows that the original boundary value problem (3.3.6.1) has no solution either. For $0 < \lambda < \lambda^*_{\rm m}$, the second equation in (3.3.6.2) has two distinct roots, c_1 and c_2, which determine two different solutions of problem (3.3.6.1). If $\lambda = \lambda^*_{\rm m}$, the roots c_1 and c_2 merge to become one, $c_1 = c_2 = c^*_{\rm m} \approx 1.1997$, which corresponds to a single solution of the problem. The critical value $\lambda = \lambda^*_{\rm m}$ corresponds to a heat explosion.

By comparing the critical values of the parameter λ determining the boundary of a nonexistence domain for solutions to the first and mixed boundary value problems, we obtain the simple relation

$$\lambda^*_{\rm f} = 4\lambda^*_{\rm m}.$$

This relation is exact; it follows from the equation $p(b) = 4q(b)$, which is valid for all b.

The maximum value of the dimensionless excess temperature is attained at $x = 0$; it is given by the formula $y(0) = \ln(2c^2/\lambda)$, which is derived from (3.3.4.2) and (3.3.6.2). The critical values $\lambda^*_{\rm m}$ and $c^*_{\rm m}$ correspond to a thermal explosion. Substituting these values in the formula for the temperature at $x = 0$ yields the critical temperature $y_*(0) \approx 1.1868$ leading to the thermal explosion.

Now let us assess the accuracy of the critical value $\lambda^{\rm ap}_{\rm f}$ provided by theorem 1 on nonexistence of solutions (see the previous section). In this case, $f(x, y, y'_x) = e^y$. So we have $e^y \ge ey$ for $y > 0$; hence, $a = e$. Substituting this value into (3.3.5.6) yields an approximate estimate for the boundary of nonexistence of solutions with respect to λ:

$$\lambda > \lambda^{\rm ap}_{\rm f} = \tfrac{1}{4}\pi^2/e \approx 0.9077.$$

One can see that, in this problem, the difference between $\lambda^{\rm ap}_{\rm f}$, estimated using the nonexistence theorem, and the exact value $\lambda^*_{\rm f}$ is just over 3%.

► Plane problem II arising in combustion theory (Arrhenius law-based model).

Example 3.18. Let us look at a more realistic model of thermal explosion than that considered in Example 3.17, in which the kinetic function describing heat release is now bounded and determined by the *Arrhenius law*. In terms of suitable dimensionless variables, the corresponding nonlinear boundary value problem is

$$y''_{xx} + \lambda \exp\Big(\frac{y}{1+\sigma y}\Big) = 0; \qquad y'_x(0) = y(1) = 0, \tag{3.3.6.3}$$

where $\lambda \ge 0$ and $\sigma > 0$.

The general solution to the equation of (3.3.6.3) can be obtained by quadrature (e.g., using formulas from Example 3.1); however, this solution cannot be expressed in terms of elementary functions. In the limit case of $\sigma = 0$, problem (3.3.6.3) becomes (3.3.6.1).

It can be shown that, for $\sigma > 0$, problem (3.3.6.3) has at least one solution for any $\lambda \ge 0$. Furthermore, for sufficiently small σ, there is a domain of λ with three solutions (the curve $y_0 = y_0(\lambda)$, with $y_0 = y(0)$, has an S-shaped portion).

A numerical analysis of problem (3.3.6.3) shows that at $\sigma = 0.2$, there are two critical values, $\lambda^*_1 \approx 0.877$ and $\lambda^*_2 \approx 1.162$, called *hysteresis parameters*, such that

(i) there is only one solution for $0 < \lambda_1$ and $\lambda > \lambda_2$,
(ii) there are three solutions for $\lambda_1 < \lambda < \lambda_2$, and
(iii) there are two solutions for $\lambda = \lambda_1$ and $\lambda = \lambda_2$.

► An axisymmetric problem arising in combustion theory (Frank-Kamenetskii approximation).

Example 3.19. Now consider the one-dimensional problem on thermal explosion in a cylindrical vessel described by the following equation and mixed boundary conditions:

$$y''_{xx} + \frac{1}{x}y'_x + \lambda e^y = 0; \qquad y'_x(0) = y(1) = 0, \tag{3.3.6.4}$$

where x is a dimensional radial coordinate.

Problem (3.3.6.4) is solved explicitly in terms of elementary functions:

$$y = -2\ln\bigl[b + (1-b)x^2\bigr],$$
$$\lambda = 8b(1-b), \quad b = e^{-y_0/2}, \quad y_0 = y(0). \tag{3.3.6.5}$$

One can see that for $0 < \lambda < \lambda^*_{\rm m} = 2$, there are two solutions corresponding to two distinct values of y_0 (the solution having a physical meaning must satisfy the condition $0 \le y_0 < y^*_{\rm m} \approx 1.3863$). When $\lambda = \lambda^*_{\rm m}$, the two solution merge to become one. For $\lambda > \lambda^*_{\rm m}$, there are no solutions. The critical value $\lambda = \lambda^*_{\rm m} = 2$ corresponds to thermal explosion.

► A problem on bending of a flexible electrode in an electrostatic field.

Example 3.20. Consider the nonlinear boundary value problem on the interval $0 \le x \le 1$ with mixed boundary conditions

$$y''_{xx} + \frac{\lambda}{(1-y)^2} = 0; \qquad y'_x(0) = y(1) = 0. \tag{3.3.6.6}$$

This problem describes the shape of a flexible electrode bending under the action of electrostatic forces due to potential difference between electrodes, with y denoting dimensionless deflexion of the electrode, x denoting dimensionless distance, and λ being a dimensionless parameter proportional to the squared potential difference between electrodes.

The equation of (3.3.6.6) is a special case of the autonomous second-order equations considered in Example 3.1, which admits order reduction and so is easy to integrate. The solution to problem (3.3.6.6) can be written in implicit form as

$$x = \frac{1}{\varphi(a)}\left[\sqrt{(1-a)(a-y)} + (1-a)\ln\frac{\sqrt{1-y}+\sqrt{a-y}}{\sqrt{1-a}}\right],$$
$$\lambda = \frac{1}{2}(1-a)\varphi^2(a), \qquad \varphi(a) = \sqrt{a} + (1-a)\ln\frac{1+\sqrt{a}}{\sqrt{1-a}}, \tag{3.3.6.7}$$

where $a = y(0)$, $0 \le a < 1$, $0 \le y \le a$, and $\lambda > 0$.

The function $y = y(x)$ is convex; at $x = 0$, it has a maximum equal to a and monotonically decreases with x to zero at $x = 1$. An analysis of formula (3.3.6.7) shows that for $0 < \lambda < \lambda^*_{\rm m}$, there are two solutions corresponding to two distinct values of a; the physically realizable (stable) solution corresponds to $0 < a < y^*_{\rm m} \approx 0.3883$. When $\lambda = \lambda^*_{\rm m}$, the two solutions merge to become one. For $\lambda > \lambda^*_{\rm m}$, there are no solutions.

► A model problem having three solutions.

Example 3.21. Let us look at the nonlinear boundary value problem

$$y''_{xx} + \lambda\frac{\sinh(ky)}{\cosh^3(ky)} = 0; \qquad y'_x(0) = y(1) = 0, \tag{3.3.6.8}$$

where $k > 0$ and $\lambda > 0$. The function $f(y) = \sinh(ky)/\cosh^3(ky)$ is nonmonotonic and it changes sign; it vanishes ar $y = 0$ and tends to zero as $y \to \pm\infty$. Its extrema are at the points $y_{\rm m} = \pm 0.6585/k$ and are equal to $f_{\rm m} = \pm\frac{2}{3\sqrt{3}} \approx \pm 0.3849$.

Problem (3.3.6.8) admits the trivial solution $y = 0$ for any λ.

If y is a solution to problem (3.3.6.8), then $-y$ is also a solution to the problem.

The positive solution is determined implicitly by the formula

$$\arcsin\left[\frac{\sinh(ky)}{\sinh(ka)}\right] = \frac{\pi}{2}(1-x), \qquad \lambda = \frac{\pi^2}{4k}\cosh^2(ka), \tag{3.3.6.9}$$

where $a = y(0) > 0$ and $y > 0$ for $0 < x < 1$. This solution can be represented in the explicit form

$$y = \frac{1}{k}\operatorname{arsinh}\left\{\sinh(ka)\sin\left[\frac{\pi}{2}(1-x)\right]\right\}, \qquad \lambda = \frac{\pi^2}{4k}\cosh^2(ka), \tag{3.3.6.10}$$

where $\operatorname{arsinh} z = \ln\bigl(z + \sqrt{z^2+1}\bigr)$.

The negative solution to problem (3.3.6.8) is given by formula (3.3.6.10) where a must be replaced with $-a$.

◆ *See also Section 8.3.3.*

3.3.7 Theorems on Existence of Two Solutions for the Third Boundary Value Problem

► Statement of the problem. Initial assumptions.

Consider a boundary value problem for the nonlinear equation

$$y''_{xx} + f(x,y) = 0 \qquad (0 < x < 1) \tag{3.3.7.1}$$

subject to the boundary conditions of the third kind

$$\begin{aligned} \alpha y(0) - \beta y'_x(0) &= 0, \\ \gamma y(1) + \delta y'_x(1) &= 0. \end{aligned} \tag{3.3.7.2}$$

The following conditions will be assumed to hold throughout this section:

(i) The function $f(x,y) \ge 0$ is continuous in the domain $\Omega = \{0 \le x \le 1,\ 0 \le y < \infty\}$ with $f(x,y) \not\equiv 0$ on any subinterval of $0 \le x \le 1$ for $y > 0$.

(ii) The coefficients $\alpha, \beta, \gamma, \delta \ge 0$ and $\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0$.

► Erbe–Hu–Wang theorems on nonuniqueness of a solution to the boundary value problem.

THEOREM 1. *Let conditions (i) and (ii) hold and the following assumptions be valid:*

(iii) $\lim\limits_{y\to 0}\min\limits_{0\le x\le 1}\dfrac{f(x,y)}{y} = \lim\limits_{y\to\infty}\min\limits_{0\le x\le 1}\dfrac{f(x,y)}{y} = \infty.$

(iv) *There is a constant $p > 0$ such that*

$$f(x,y) \le \eta p \quad \text{for} \quad 0 \le x \le 1,\ 0 \le y \le p,$$

where $\eta = \left[\int_0^1 G(\xi,\xi)\,d\xi\right]^{-1} = \frac{6\rho}{\alpha\gamma+3\alpha\delta+3\beta\gamma+6\beta\delta}$. (Here $G(x,\xi)$ is the Green's function for the equation $y''_{xx} = 0$ with respect to the boundary conditions (3.3.7.2); the expression of this Green's function can be found at the end of Table 3.1.)

Then the boundary value problem (3.3.7.1)–(3.3.7.2) has at least two positive solutions, $y_1 = y_1(x)$ and $y_2 = y_2(x)$, such that

$$0 < \|y_1\| < p < \|y_2\|.$$

Here $\|y\| = \sup\limits_{0\le x\le 1}|y(x)|$.

THEOREM 2. *Let conditions (i) and (ii) hold and the following assumptions be valid:*

(v) $\lim\limits_{y\to 0}\max\limits_{0\le x\le 1}\dfrac{f(x,y)}{y}=\lim\limits_{y\to\infty}\max\limits_{0\le x\le 1}\dfrac{f(x,y)}{y}=0.$

(vi) *There is a constant* $q>0$ *such that*

$$f(x,y)\ge \mu q \quad \text{for} \quad \tfrac{1}{4}\le x\le \tfrac{3}{4},\ \sigma q\le y\le q,$$

where $\mu=\bigl[\int_{1/4}^{3/4}G(\frac{1}{2},\xi)\,d\xi\bigr]^{-1}=\frac{32\rho}{3\alpha\gamma+7\alpha\delta+7\beta\gamma+16\beta\delta}$ *and* $\sigma=\min\bigl[\frac{\alpha+4\beta}{4(\alpha+\beta)},\frac{\gamma+4\delta}{4(\gamma+\delta)}\bigr]$.

Then the boundary value problem (3.3.7.1)–(3.3.7.2) *has at least two positive solutions,* $y_1=y_1(x)$ *and* $y_2=y_2(x)$*, such that*

$$0<\|y_1\|<q<\|y_2\|.$$

3.3.8 Boundary Value Problems for Linear Equations with Nonlinear Boundary Conditions

► Statements of problems. Solution procedure.

In this section, we consider a few boundary value problems for linear homogeneous second-order differential equations

$$y''_{xx}+f_1(x)y'_x+f_0(x)y=0 \tag{3.3.8.1}$$

subject to a nonlinear boundary condition

$$y'_x=\varphi(y) \quad \text{at} \quad x=x_1 \tag{3.3.8.2}$$

and a linear homogeneous boundary condition at $x=x_2$.

Such problems are solved successively in a few stages. First, one obtains the general solution to equation (3.3.8.1). Then, one finds a particular solution $y=\bar{y}(x)$, satisfying the boundary condition at the right end, $x=x_2$. Finally, one seeks the solution to the problem in the form

$$y=A\bar{y}(x), \tag{3.3.8.3}$$

where A is a constant determined from the algebraic (transcendental) equation

$$A\bar{y}'_x(x_1)=\varphi(A\bar{y}(x_1)), \tag{3.3.8.4}$$

obtained by substituting (3.3.8.3) into the nonlinear boundary condition at the left end (3.3.8.2).

► Qualitative features of some problems with nonlinear boundary conditions.

Solutions to boundary value problems for linear equations satisfying nonlinear boundary conditions can significantly differ from those satisfying linear boundary conditions.

Example 3.22. Consider a boundary value problem for a linear equation subject to a nonlinear boundary condition at $x=0$ and a homogeneous linear condition of the first kind at $x=a$:

$$y''_{xx}+k^2y=0; \tag{3.3.8.5}$$

$$y'_x=\varphi(y) \quad \text{at} \quad x=0, \qquad y=0 \quad \text{at} \quad x=a. \tag{3.3.8.6}$$

The general solution of the linear equation with constant coefficients (3.3.8.5) is given by

$$y = C_1 \sin(kx) + C_2 \cos(kx), \tag{3.3.8.7}$$

where C_1 and C_2 are arbitrary constants. In order to find a particular solution $\bar{y}$ satisfying the second boundary condition (3.3.8.6), we can set $C_1 = -\cos(ak)$ and $C_2 = \sin(ak)$ in (3.3.8.7) to obtain

$$\bar{y} = -\cos(ak)\sin(kx) + \sin(ak)\cos(kx) = \sin[k(a-x)].$$

The solution to problem (3.3.8.5)–(3.3.8.6) is sought in the form

$$y = A\bar{y} = A\sin[k(a-x)]. \tag{3.3.8.8}$$

For any A, this solution satisfies equation (3.3.8.5) and the second boundary condition (3.3.8.6). Substituting (3.3.8.8) into the first boundary condition (3.3.8.6) yields an algebraic (or transcendental) equation for A:

$$Ak\cos(ak) + \varphi(A\sin(ak)) = 0. \tag{3.3.8.9}$$

Let us dwell on the first boundary condition (3.3.8.6) having a power-law nonlinearity

$$\varphi(y) = by^m. \tag{3.3.8.10}$$

Equation (3.3.8.9) then becomes

$$Ak\cos(ak) + bA^m \sin^m(ak) = 0. \tag{3.3.8.11}$$

For any $m > 0$, this equation has the trivial solution $A = 0$ (or $k = 0$). Let us look at different special cases.

1°. To get a linear boundary condition of the third kind, one should set $m = 1$ in (3.3.8.10)–(3.3.8.11). The corresponding eigenvalue problem gives solution (3.3.8.8) with A being an arbitrary constant and a and k linked to each other by the discrete relations

$$ak = \frac{\pi}{2} - \theta_0 + \pi n, \quad \theta_0 = \arctan\frac{b}{k}, \quad n = 0,\ 1,\ 2,\ \dots \tag{3.3.8.12}$$

To boundary conditions of the first and second kind there correspond the limit cases $b = \infty$ ($\theta_0 = \frac{\pi}{2}$) and $b = 0$ ($\theta_0 = 0$).

2°. In the case of a quadratic nonlinearity, with $m = 2$, equation (3.3.8.11) has a nontrivial solution

$$A = -\frac{k\cos(ak)}{b\sin^2(ak)}$$

for any a, b, and k ($abk \neq 0$)

3°. In the case of a cubic nonlinearity, $m = 3$, equation (3.3.8.11) can have two nontrivial solutions or no solutions at all depending on the sign of the expression $bk\tan(ak)$:

$$\begin{aligned} A_{1,2} = \pm\left[-\frac{k\cos(ak)}{b\sin^3(ak)}\right]^{1/2} &\quad \text{if } bk\tan(ak) < 0, \\ \text{no nontrivial solutions} &\quad \text{if } bk\tan(ak) > 0. \end{aligned}$$

4°. In the case of a fractional nonlinearity with $m = \frac{1}{2}$, equation (3.3.8.11) can have one nontrivial solution or no solution at all depending on the sign of the expression $bk\tan(ak)$:

$$\begin{aligned} A = \frac{b^2\sin(ak)}{k^2\cos^2(ak)} &\quad \text{if } bk\tan(ak) < 0, \\ \text{no nontrivial solutions} &\quad \text{if } bk\tan(ak) > 00. \end{aligned}$$

It is apparent from Items 1°–4° that the solutions to boundary value problems with linear and nonlinear boundary conditions can significantly differ from each other; specifically, in linear problems, for nontrivial solutions to exist, the parameters a and k must be connected with each other by discrete relations of the form (3.3.8.11) with A being an arbitrary number, while in nonlinear problems, a and k can change independently from each other, with A expressed via them (under certain conditions, several nontrivial solutions can exist or a nontrivial solution can be absent at all).

► **A problem of convective mass transfer with a heterogeneous chemical reaction.**

Consider the equation

$$y''_{xx} + ax^n y'_x = 0 \tag{3.3.8.13}$$

subject to the boundary condition

$$y'_x = -k\Phi(y) \quad \text{at} \quad x = 0, \qquad y \to 0 \quad \text{as} \quad x \to \infty. \tag{3.3.8.14}$$

Problem (3.3.8.13)–(3.3.8.14), written in terms of dimensionless variables, describes convective mass transfer about the critical point of a drop (for $n = 1$) or a solid particle (for $n = 2$) with a heterogeneous chemical reaction on the surface. In (3.3.8.13) and (3.3.8.14), y is concentration, $\Phi(y)$ is the kinetic function, satisfying the condition $\Phi(1) = 0$, k is the rate of chemical reaction, and a is a positive constant. For a reaction of order m, we have $\Phi(y) = (1 - y)^m$. To the limit case $k \to \infty$ there corresponds the diffusion mode of the surface reaction with $y(0) = 1$.

The solution to equation (3.3.8.13) satisfying the second boundary condition (3.3.8.14) is given by

$$y = A \int_x^\infty \exp\left(-\frac{a}{n+1}\xi^{n+1}\right) d\xi, \tag{3.3.8.15}$$

where A is a constant. Substituting (3.3.8.15) into the first boundary condition (3.3.8.14) yields an algebraic (or transcendental) equation for A:

$$A = k\Phi(Ac), \tag{3.3.8.16}$$

where

$$c = \int_0^\infty \exp\left(-\frac{a}{n+1}\xi^{n+1}\right) d\xi = a^{-\frac{1}{n+1}}(n+1)^{-\frac{n}{n+1}}\Gamma\left(\frac{1}{n+1}\right)$$

and $\Gamma(z)$ is the gamma function. In particular, for a reaction with the fractional order $m = 1/2$, we have $\Phi(y) = (1 - y)^{1/2}$; hence, the solution to equation (3.3.8.16) is $A = -\frac{1}{2}ck^2 + \sqrt{\frac{1}{4}c^2k^4 + k^2}$.

⊙ *Literature for Section 3.3*: K. Akô (1967, 1968), P. B. Bailey, L. F. Shampine, and P. E. Waltman (1968), J. Bebernes and R. Gaines (1968), E. Kamke (1977), L. K. Jackson and P. K. Palamides (1984), D. A. Frank-Kamenetskii (1987), V. F. Zaitsev and A. D. Polyanin (1993, 1994), L. H. Erbe, S. Hu, and H. Wang (1994), L. H. Erbe and H. Wang (1994), S.-H. Wang (1994), W.-C. Lian, F.-H. Wong, and C.-C. Yen (1996), P. Korman and Y. Li (1999, 2010), P. Korman, Y. Li, and T. Ouyang (2005), A. B. Vasil'eva and H. H. Nefedov (2006), S. I. Faddeev and V. V. Kogan (2008), G. L. Karakostas (2012).

3.4 Method of Regular Series Expansions with Respect to the Independent Variable. Padé Approximants

3.4.1 Method of Expansion in Powers of the Independent Variable

A solution of the Cauchy problem

$$y''_{xx} = f(x, y, y'_x), \tag{3.4.1.1}$$

$$y(x_0) = y_0, \quad y'_x(x_0) = y_1 \tag{3.4.1.2}$$

can be sought in the form of a Taylor series in powers of the difference $(x-x_0)$, specifically:

$$y(x) = y(x_0) + y'_x(x_0)(x-x_0) + \frac{y''_{xx}(x_0)}{2!}(x-x_0)^2 + \frac{y'''_{xxx}(x_0)}{3!}(x-x_0)^3 + \cdots . \quad (3.4.1.3)$$

The first two coefficients $y(x_0)$ and $y'_x(x_0)$ in solution (3.4.1.3) are defined by the initial conditions (3.4.1.2). The values of the subsequent derivatives of y at the point $x = x_0$ are determined from equation (3.4.1.1) and its derivative equations (obtained by successive differentiation of the equation) taking into account the initial conditions (3.4.1.2). In particular, setting $x = x_0$ in (3.4.1.1) and substituting (3.4.1.2), we obtain the value of the second derivative:

$$y''_{xx}(x_0) = f(x_0, y_0, y_1). \quad (3.4.1.4)$$

Further, differentiating (3.4.1.1) yields

$$y'''_{xxx} = f_x(x, y, y'_x) + f_y(x, y, y'_x)y'_x + f_{y'_x}(x, y, y'_x)y''_{xx}. \quad (3.4.1.5)$$

On substituting $x = x_0$, the initial conditions (3.4.1.2), and the expression of $y''_{xx}(x_0)$ of (3.4.1.4) into the right-hand side of equation (3.4.1.5), we calculate the value of the third derivative:

$$y'''_{xxx}(x_0) = f_x(x_0, y_0, y_1) + f_y(x_0, y_0, y_1)y_1 + f(x_0, y_0, y_1)f_{y'_x}(x_0, y_0, y_1).$$

The subsequent derivatives of the unknown are determined likewise.

The thus obtained solution (3.4.1.3) can only be used in a small neighborhood of the point $x = x_0$.

Example 3.23. Consider the following Cauchy problem for a second-order nonlinear equation:

$$y''_{xx} = yy'_x + y^3; \quad (3.4.1.6)$$

$$y(0) = y'_x(0) = 1. \quad (3.4.1.7)$$

Substituting the initial values of the unknown and its derivative (3.4.1.7) into equation (3.4.1.6) yields the initial value of the second derivative:

$$y''_{xx}(0) = 2. \quad (3.4.1.8)$$

Differentiating equation (3.4.1.6) gives

$$y'''_{xxx} = yy''_{xx} + (y'_x)^2 + 3y^2y'_x. \quad (3.4.1.9)$$

Substituting here the initial values from (3.4.1.7) and (3.4.1.8), we obtain the initial condition for the third derivative:

$$y'''_{xxx}(0) = 6. \quad (3.4.1.10)$$

Differentiating (3.4.1.9) followed by substituting (3.4.1.7), (3.4.1.8), and (3.4.1.10), we find that

$$y''''_{xxxx}(0) = 24. \quad (3.4.1.11)$$

On substituting the initial data (3.4.1.7), (3.4.1.8), (3.4.1.10), and (3.4.1.11) into (3.4.1.3), we arrive at the Taylor series expansion of the solution about $x = 0$:

$$y = 1 + x + x^2 + x^3 + x^4 + \cdots . \quad (3.4.1.12)$$

This geometric series is convergent only for $|x| < 1$.

3.4.2 Padé Approximants

Suppose the $k+1$ leading coefficients in the Taylor series expansion of a solution to a differential equation about the point $x=0$ are obtained by the method presented in Section 3.4.1, so that

$$y_{k+1}(x) = a_0 + a_1x + \cdots + a_kx^k. \tag{3.4.2.1}$$

The partial sum (3.4.2.1) pretty well approximates the solution at small x but is poor for intermediate and large values of x, since the series can be slowly convergent or even divergent. This is also related to the fact that $y_k \to \infty$ as $x \to \infty$, while the exact solution can well be bounded.

In many cases, instead of the expansion (3.4.2.1), it is reasonable to consider a Padé approximant $P_M^N(x)$, which is the ratio of two polynomials of degree N and M, specifically,

$$P_M^N(x) = \frac{A_0 + A_1x + \cdots + A_Nx^N}{1 + B_1x + \cdots + B_Mx^M}, \quad \text{where} \quad N+M=k. \tag{3.4.2.2}$$

The coefficients $A_1, \ldots, A_N$ and $B_1, \ldots, B_M$ are selected so that the $k+1$ leading terms in the Taylor series expansion of (3.4.2.2) coincide with the respective terms of the expansion (3.4.2.1). In other words, the expansions (3.4.2.1) and (3.4.2.2) must be asymptotically equivalent as $x \to 0$.

In practice, one usually takes $N = M$ (the diagonal sequence). It often turns out that formula (3.4.2.2) pretty well approximates the exact solution on the entire range of x (for sufficiently large N).

Example 3.24. Consider the Cauchy problem (3.4.1.6)–(3.4.1.7) again. The Taylor series expansion of the solution about $x = 0$ has the form (3.4.1.12). This geometric series is convergent only for $|x| < 1$.

The diagonal sequence of Padé approximants corresponding to series (3.4.1.12) is

$$P_1^1(x) = \frac{1}{1-x}, \quad P_2^2(x) = \frac{1}{1-x}, \quad P_3^3(x) = \frac{1}{1-x}. \tag{3.4.2.3}$$

It is not difficult to verify that the function $y(x) = \dfrac{1}{1-x}$ is the exact solution of the Cauchy problem (3.4.1.6)–(3.4.1.7). Hence, in this case, the diagonal sequence of Padé approximants recovers the exact solution from only a few terms in the Taylor series.

Example 3.25. Consider the Cauchy problem for a second-order nonlinear equation:

$$y''_{xx} = 2yy'_x; \qquad y(0) = 0, \quad y'_x(0) = 1. \tag{3.4.2.4}$$

Following the method presented in Section 3.4.1, we obtain the Taylor series expansion of the solution to problem (3.4.2.4) in the form

$$y(x) = x + \tfrac{1}{3}x^3 + \tfrac{2}{15}x^5 + \tfrac{17}{315}x^7 + \cdots. \tag{3.4.2.5}$$

The exact solution of problem (3.4.2.4) is given by $y(x) = \tan x$. Hence it has singularities at $x = \pm\frac{1}{2}(2n+1)\pi$. However, any finite segment of the Taylor series (3.4.2.5) does not have any singularities.

With series (3.4.2.5), we construct the diagonal sequence of Padé approximants:

$$P_2^2(x) = \frac{3x}{3-x^2}, \quad P_3^3(x) = \frac{x(x^2-15)}{3(2x^2-5)}, \quad P_4^4(x) = \frac{5x(21-2x^2)}{x^4-45x^2+105}. \tag{3.4.2.6}$$

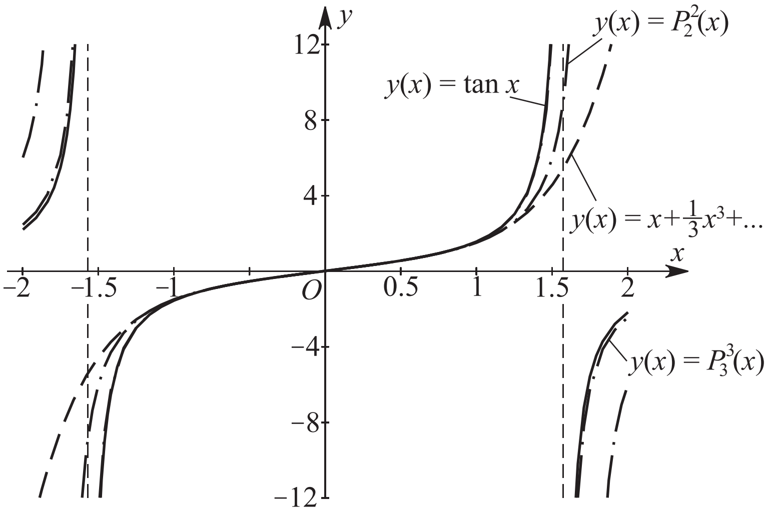

Figure 3.3: Comparison of the exact solution to problem (3.4.2.4) with the approximate truncated series solution (3.4.2.5) and associated Padé approximants (3.4.2.6).

These Padé approximants have singularities (at the points where the denominators vanish):

$$x \simeq \pm 1.732 \qquad \text{for } P_2^2(x),$$
$$x \simeq \pm 1.581 \qquad \text{for } P_3^3(x),$$
$$x \simeq \pm 1.571 \text{ and } x \simeq \pm 6.522 \qquad \text{for } P_4^4(x).$$

It is apparent that the Padé approximants are attempting to recover the singularities of the exact solution at $x = \pm\pi/2$ and $x = \pm 3\pi/2$.

In Fig. 3.3, the solid line shows the exact solution of problem (3.4.2.4), the dashed line corresponds to the four-term Taylor series solution (3.4.2.5), and the dot-and-dash line depicts the Padé approximants (3.4.2.6). It is evident that the Padé approximant $P_4^4(x)$ gives an accurate numerical approximation of the exact solution on the interval $|x| \le 2$; everywhere the error is less than 1%, except for a very small neighborhood of the point $x = \pm\pi/2$ (the error is 1% for $x = \pm 1.535$ and 0.84% for $x = \pm 2$).

⊙ *Literature for Section 3.4*: A. H. Nayfeh (1973, 1981), G. A. Baker (Jr.) and P. Graves–Morris (1981), D. Zwillinger (1997), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

3.5 Movable Singularities of Solutions of Ordinary Differential Equations. Painlevé Equations

3.5.1 Preliminary Remarks. Singular Points of Solutions

► Fixed and movable singular points of solutions to ODEs.

Singular points of solutions to ordinary differential equations can be *fixed* or *movable*. The coordinates of fixed singular points remain the same for different solutions of an equation.*

*Solutions of linear ordinary differential equations can only have fixed singular points, and their positions are determined by the singularities of the equation coefficients.

The coordinates of movable singular points vary depending on the particular solution selected (i.e., they depend on the initial conditions).

Listed below are simple examples of first-order ordinary differential equations and their solutions having movable singularities:

Equation	*Solution*	*Solution's singularity type*
$y'_z = -y^2$	$y = 1/(z - z_0)$	movable pole
$y'_z = 1/y$	$y = 2\sqrt{z - z_0}$	algebraic branch point
$y'_z = e^{-y}$	$y = \ln(z - z_0)$	logarithmic branch point
$y'_z = -y\ln^2 y$	$y = \exp[1/(z - z_0)]$	essential singularity

Algebraic branch points, logarithmic branch points, and essential singularities are called *movable critical points*.

► Classification of second-order ODEs. Painlevé equations.

The Painlevé equations arise from the classification of the following second-order differential equations over the complex plane:

$$y''_{zz} = R(z, y, y'_z),$$

where $R = R(z, y, w)$ is a function rational in y and w and analytic in z. It was shown by P. Painlevé (1897–1902) and B. Gambier (1910) that all equations of this type whose solutions do not have movable critical points (but are allowed to have fixed singular points and movable poles) can be reduced to 50 classes of equations. Moreover, 44 classes out of them are integrable by quadrature or admit reduction of order. The remaining 6 equations are irreducible; these are known as the *Painlevé equations*, and their solutions are known as the *Painlevé transcendental functions* or *Painlevé transcendents*.

Remark 3.12. The Painlevé equations are sometimes referred to as the Painlevé transcendents, but in this section this term will be used only for their solutions.

The canonical forms of the Painlevé equations are given below in Sections 3.5.2 through 3.5.7. Solutions of the first, second, and fourth Painlevé equations have movable poles (no fixed singular points). Solutions of the third and fifth Painlevé equations have two fixed logarithmic branch points, $z = 0$ and $z = \infty$. Solutions of the sixth Painlevé transcendent have three fixed logarithmic branch points, $z = 0$, $z = 1$, and $z = \infty$.

It is significant that the Painlevé equations often arise in mathematical physics.

3.5.2 First Painlevé Equation

► Form of the first Painlevé equation. Solutions in the vicinity of a movable pole.

The *first Painlevé equation* has the form

$$y''_{zz} = 6y^2 + z. \qquad (3.5.2.1)$$

The solutions of the first Painlevé equation are single-valued functions of z.

The solutions of equation (3.5.2.1) can be presented, in the vicinity of a movable pole z_p, in terms of the series

$$y = \frac{1}{(z-z_\mathrm{p})^2} + \sum_{n=2}^{\infty} a_n (z-z_\mathrm{p})^n,$$
$$a_2 = -\tfrac{1}{10} z_\mathrm{p}, \quad a_3 = -\tfrac{1}{6}, \quad a_4 = C, \quad a_5 = 0, \quad a_6 = \tfrac{1}{300} z_\mathrm{p}^2,$$

where z_p and C are arbitrary constants; the coefficients a_j ($j \ge 7$) are uniquely defined in terms of z_p and C.

Remark 3.13. The first Painlevé equation (3.5.2.1) is invariant under scaling of variables, $z = \lambda \bar{z}$, $y = \lambda^3 \bar{y}$, where $\lambda^5 = 1$.

► Solutions in the form of a Taylor series.

In a neighborhood of a fixed point $z = z_0$, the solution of the Cauchy problem for the first Painlevé equation (3.5.2.1) can be represented by the Taylor series (see Section 3.4.1):

$$y = A + B(z-z_0) + \tfrac{1}{2}(6A^2 + z_0)(z-z_0)^2 + \tfrac{1}{6}(12AB+1)(z-z_0)^3$$
$$+ \tfrac{1}{2}(6A^3 + B^2 + Az_0)(z-z_0)^4 + \cdots,$$

where A and B are initial data of the Cauchy problem, so that $y|_{z=z_0} = A$ and $y'_z|_{z=z_0} = B$.

Remark 3.14. The solutions of the Cauchy problems for the second and fourth Painlevé equations can be expressed likewise (fixed singular points should be excluded from consideration for the remaining Painlevé equations).

► Asymptotic formulas and some properties.

1°. There are solutions of equation (3.5.2.1) such that

$$y(x) = -\big(\tfrac{1}{6}|x|\big)^{1/2} + a_1|x|^{-1/8} \sin\big[\phi(x) - b_1\big] + o\big(|x|^{-1/8}\big) \quad \text{as} \quad x \to -\infty, \tag{3.5.2.2}$$

where

$$\phi(x) = (24)^{1/4}\big(\tfrac{4}{5}|x|^{5/4} - \tfrac{5}{8} a_1^2 \ln|x|\big),$$

and a_1 and b_1 are some constants (there are also solutions such that $a_1 = 0$).

2°. For given initial conditions $y(0) = 0$ and $y'_x(0) = k$, with k real, $y(x)$ has at least one pole on the real axis. There are two special values, $k_1 \approx -0.45143$ and $k_2 \approx 1.85185$, such that:

(a) If $k < k_1$, then $y(x) > 0$ for $x_\mathrm{p} < x < 0$, where x_p is the first pole on the negative real axis.

(b) If $k_1 < k < k_2$ then $y(x)$ oscillates about, and is asymptotic to, $-\big(\tfrac{1}{6}|x|\big)^{1/2}$ as $x \to -\infty$ (see formula (3.5.2.2)).

(c) If $k_2 < k$ then $y(x)$ changes sign once, from positive to negative, as x passes from x_p to 0.

3°. For large values of $|z| \to \infty$, the following asymptotic formula holds:

$$y \sim z^{1/2} \wp\big(\tfrac{4}{5} z^{5/4} - a_2;\ 12,\ b_2\big),$$

where the elliptic Weierstrass function $\wp(\zeta; 12, b_2)$ is defined implicitly by the integral

$$\zeta = \int \frac{d\wp}{\sqrt{4\wp^3 - 12\wp - b_2}},$$

and a_2 and b_2 are some constants.

3.5.3 Second Painlevé Equation

► Form of the 2nd Painlevé equation. Solutions in the vicinity of a movable pole.

The *second Painlevé equation* has the form

$$y''_{zz} = 2y^3 + zy + \alpha. \tag{3.5.3.1}$$

The solutions of the second Painlevé equation are single-valued functions of z.

The solutions of equation (3.5.3.1) can be represented, in the vicinity of a movable pole z_p, in terms of the series

$$y = \frac{m}{z - z_\text{p}} + \sum_{n=1}^{\infty} b_n (z - z_\text{p})^n,$$

$$b_1 = -\tfrac{1}{6} m z_\text{p},\ \ b_2 = -\tfrac{1}{4}(m + \alpha),\ \ b_3 = C,\ \ b_4 = \tfrac{1}{72} z_\text{p}(m + 3\alpha),$$
$$b_5 = \tfrac{1}{3024}\big[(27 + 81\alpha^2 - 2z_\text{p}^3)m + 108\alpha - 216 C z_\text{p}\big],$$

where z_p and C are arbitrary constants, $m = \pm 1$, and the coefficients b_n $(n \ge 6)$ are uniquely defined in terms of z_p and C.

► Relations between solutions. Bäcklund transformations.

For fixed α, denote the solution by $y(z, \alpha)$. Then the following relation holds:

$$y(z, -\alpha) = -y(z, \alpha), \tag{3.5.3.2}$$

while the solutions $y(z, \alpha)$ and $y(z, \alpha - 1)$ are related by the Bäcklund transformations:

$$\begin{aligned} y(z, \alpha - 1) &= -y(z, \alpha) + \frac{2\alpha - 1}{2y'_z(z, \alpha) - 2y^2(z, \alpha) - z}, \\ y(z, \alpha) &= -y(z, \alpha - 1) - \frac{2\alpha - 1}{2y'_z(z, \alpha - 1) + 2y^2(z, \alpha - 1) + z}. \end{aligned} \tag{3.5.3.3}$$

Therefore, in order to study the general solution of equation (3.5.3.1) with arbitrary α, it is sufficient to construct the solution for all α out of the band $0 \le \operatorname{Re}\alpha < \frac{1}{2}$.

Three solutions corresponding to α and $\alpha \pm 1$ are related by the rational formulas

$$y_{\alpha+1} = -\frac{(y_{\alpha-1} + y_\alpha)(4y_\alpha^3 + 2zy_\alpha + 2\alpha + 1) + (2\alpha - 1)y_\alpha}{2(y_{\alpha-1} + y_\alpha)(2y_\alpha^2 + z) + 2\alpha - 1},$$

where y_α stands for $y(z, \alpha)$.

The solutions $y(z,\alpha)$ and $y(z,-\alpha-1)$ are related by the Bäcklund transformations:

$$y(z,-\alpha-1) = y(z,\alpha) + \frac{2\alpha+1}{2y'_z(z,\alpha) + 2y^2(z,\alpha) + z},$$
$$y(z,\alpha) = y(z,-\alpha-1) - \frac{2\alpha+1}{2y'_z(z,-\alpha-1) + 2y^2(z,-\alpha-1) + z}.$$

► Rational particular solutions.

For $\alpha = 0$, equation (3.5.3.1) has the trivial solution $y = 0$. Taking into account this fact and relations (3.5.3.2) and (3.5.3.3), we find that the second Painlevé equation with $\alpha = \pm 1, \pm 2, \ldots$ has the rational particular solutions

$$y(z,\pm 1) = \mp\frac{1}{z}, \quad y(z,\pm 2) = \pm\left(\frac{1}{z} - \frac{3z^2}{z^3+4}\right),$$
$$y(z,\pm 3) = \pm\left[\frac{3z^2}{z^3+4} - \frac{6z^2(z^3+10)}{z^6+20z^3-80}\right],$$
$$y(z,\pm 4) = \pm\left[-\frac{1}{z} + \frac{6z^2(z^3+10)}{z^6+20z^3-80} - \frac{9z^5(z^3+40)}{z^9+60z^6+11200}\right], \quad \ldots$$

► Solutions in terms of Bessel functions.

For $\alpha = \frac{1}{2}$, equation (3.5.3.1) admits the one-parameter family of solutions:

$$y(z,\tfrac{1}{2}) = -\frac{w'_z}{w}, \quad \text{where} \quad w = \sqrt{z}\left[C_1 J_{1/3}\left(\tfrac{\sqrt{2}}{3}z^{3/2}\right) + C_2 Y_{1/3}\left(\tfrac{\sqrt{2}}{3}z^{3/2}\right)\right]. \quad (3.5.3.4)$$

(Here the function w is a solution of the Airy equation, $w''_{zz} + \frac{1}{2}zw = 0$.)

It follows from (3.5.3.2)–(3.5.3.4) that the second Painlevé equation for all $\alpha = n + \frac{1}{2}$ with $n = 0, \pm 1, \pm 2, \ldots$ has a one-parameter family of solutions that can be expressed in terms of Bessel functions.

► Asymptotic formulas and some properties with $\alpha = 0$.

1°. Any nontrivial real solution of (3.5.3.1) with $\alpha = 0$ that satisfies the boundary condition

$$y \to 0 \quad \text{as} \quad x \to +\infty$$

is asymptotic to $k\,\mathrm{Ai}(x)$, for some nonzero real k, where Ai denotes the Airy function (see Section S.4.8).

Conversely, for any nonzero real k, there is a unique solution $y_k(x)$ of (3.5.3.1) with $\alpha = 0$ that is asymptotic to $k\,\mathrm{Ai}(x)$ as $x \to +\infty$. The asymptotic behavior of this solution as $x \to -\infty$ depends on $|k|$; three possible situations are highlighted below.

If $|k| < 1$, then

$$y_k(x) = b|x|^{-1/4}\sin\bigl[\phi(x) - c\bigr] + o\bigl(|x|^{-1/4}\bigr) \quad \text{as} \quad x \to -\infty,$$

where

$$\phi(x) = \tfrac{2}{3}|x|^{3/2} - \tfrac{3}{4}b^2\ln|x|, \quad b = -\tfrac{1}{\pi}\ln(1-k^2),$$

with c is a real constant.

If $|k| = 1$, then

$$y_k(x) \sim \left(\tfrac{1}{2}|x|\right)^{1/2} \operatorname{sign} k \quad \text{as} \quad x \to -\infty.$$

If $|k| < 1$, then $y_k(x)$ has a pole at a finite point $x = x_{\rm p}$, dependent on k, and

$$y_k(x) \sim \frac{\operatorname{sign} k}{x - x_{\rm p}} \quad \text{at} \quad x \to x_{\rm p}^{+}.$$

2°. Replacement of y by iy in (3.5.3.1) with $\alpha = 0$ gives the modified second Painlevé equation

$$y''_{zz} = -2y^3 + zy. \tag{3.5.3.5}$$

Any nontrivial real solution of (3.5.3.5) satisfies

$$y(x) = b|x|^{-1/4} \sin\bigl[\phi(x) - c\bigr] + O\bigl(|x|^{-5/4}\bigr) \quad \text{as} \quad x \to -\infty,$$

where

$$\phi(x) = \tfrac{2}{3}|x|^{3/2} + \tfrac{3}{4}b^2 \ln|x|,$$

with $b \neq 0$ and c are real constants.

3.5.4 Third Painlevé Equation

► Form of the third Painlevé equation.

The *third Painlevé equation* has the form

$$y''_{zz} = \frac{(y'_z)^2}{y} - \frac{y'_z}{z} + \frac{1}{z}(\alpha y^2 + \beta) + \gamma y^3 + \frac{\delta}{y}. \tag{3.5.4.1}$$

In terms of the new independent variable ζ defined by $z = e^{\zeta}$, the solutions of the transformed equation will be single-valued functions of ζ.

In some special cases, equation (3.5.4.1) can be integrated by quadrature.

If $\gamma\delta \neq 0$ in (3.5.4.1), then set $\gamma = 1$ and $\delta = -1$, without loss of generality, by rescaling y and z if necessary. If $\gamma = 0$ and $\alpha\delta \neq 0$ in (3.5.4.1), then set $\alpha = 1$ and $\delta = -1$, without loss of generality. Lastly, if $\delta = 0$ and $\beta\gamma \neq 0$, then set $\beta = -1$ and $\gamma = 1$, without loss of generality.

► Rational particular solutions.

Let $y = y(z, \alpha, \beta, \gamma, \delta)$ be a solution of equation (3.5.4.1). Then special rational solutions of the third Painlevé equation are

$$y(z, \mu, -\mu k^2, \lambda, -\lambda k^4) = k,$$
$$y(z, 0, -\mu, 0, \mu k) = kz,$$
$$y(z, 2k+3, -2k+1, 1, -1) = \frac{z+k}{z+k+1},$$

where k, λ, and μ are arbitrary constants.

In the general case $\gamma\delta \neq 0$, we may set $\gamma = 1$ and $\delta = -1$. Then equation (3.5.4.1) has rational solutions iff

$$\alpha \pm \beta = 4n,$$

where n is integers. These solutions have the form $y = P_m(z)/Q_m(z)$, where $P_m(z)$ and $Q_m(z)$ are polynomials of degree m, with no common zeros. For examples see Milne et al. (1997) and Clarkson (2003).

▶ Elementary nonrational particular solutions I.

Elementary nonrational solutions of equation (3.5.4.1) are

$$y(z, \mu, 0, 0, -\mu k^3) = kz^{1/2},$$
$$y(z, 0, -2k, 0, 4k\mu - \lambda^2) = z(k \ln^2 z + \lambda z + \mu),$$
$$y(z, \nu^2\lambda, 0, \nu^2(\lambda^2 - 4k\mu), 0) = \frac{z^{\nu-1}}{kz^{2\nu} + \lambda z^\nu + \mu},$$

where k, λ, μ, and ν are arbitrary constants.

Let $\gamma = 0$ and $\alpha\delta \neq 0$. In this case we assume $\alpha = 1$ and $\delta = -1$ (without loss of generality). Then equation (3.5.4.1) has algebraic solution iff

$$\beta = 2n, \qquad n \in \mathbb{Z}.$$

These are rational solutions in $\zeta = z^{1/3}$ of the form

$$y = P_{n^2+1}(\zeta)/Q_{n^2}(\zeta),$$

where $P_{n^2+1}(\zeta)$ and $Q_{n^2}(\zeta)$ are polynomials of degree $n^2 + 1$ and n^2, respectively, with no common zeros. Similar results hold when $\delta = 0$ and $\beta\gamma \neq 0$.

▶ Elementary nonrational particular solutions II.

In some special cases, equation (3.5.4.1) can be integrated by quadrature. Rewrite equation (3.5.4.1) in the form of integro-differential relations in two ways:

$$\left(\frac{y'_\zeta}{y}\right)^2 + \left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + 2\left(\frac{\beta}{y} - \alpha y\right)e^\zeta = 2\int\left[\left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + \left(\frac{\beta}{y} - \alpha y\right)e^\zeta\right] d\zeta; \tag{3.5.4.2}$$

$$\frac{y'_\zeta}{y} = \int\left[\left(\frac{\delta}{y^2} + \gamma y^2\right)e^{2\zeta} + \left(\frac{\beta}{y} + \alpha y\right)e^\zeta\right] d\zeta, \qquad z = e^\zeta. \tag{3.5.4.3}$$

It is obvious from (3.5.4.2) that for $\alpha = \beta = \gamma = \delta = 0$, the general solution has the form: $y = C_1 z^{C_2}$.

Adding (3.5.4.3) multiplied by 2 to (3.5.4.2), we obtain

$$\left(\frac{y'_\zeta}{y}\right)^2 + 2\frac{y'_\zeta}{y} + \left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + 2\left(\frac{\beta}{y} - \alpha y\right)e^\zeta = 4\int \frac{\delta e^{2\zeta} + \beta e^\zeta y}{y^2}\, d\zeta. \tag{3.5.4.4}$$

Subtracting (3.5.4.3) times 2 from (3.5.4.2) yields

$$\left(\frac{y'_\zeta}{y}\right)^2 - 2\frac{y'_\zeta}{y} + \left(\frac{\delta}{y^2} - \gamma y^2\right)e^{2\zeta} + 2\left(\frac{\beta}{y} - \alpha y\right)e^\zeta = -4\int (\gamma e^{2\zeta} y^2 + \alpha e^\zeta y)\, d\zeta. \tag{3.5.4.5}$$

Substituting $\delta=\beta=0$ into equation (3.5.4.4) and $\gamma=\alpha=0$ into equation (3.5.4.5), we arrive at

$$\left(\frac{y'_\zeta}{y}\right)^2+2\frac{y'_\zeta}{y}-2\alpha ye^\zeta-\gamma y^2e^{2\zeta}=C_1, \tag{3.5.4.6}$$

$$\left(\frac{y'_\zeta}{y}\right)^2-2\frac{y'_\zeta}{y}+\frac{\delta}{y^2}e^{2\zeta}+\frac{2\beta}{y}e^\zeta=C_2. \tag{3.5.4.7}$$

Equations (3.5.4.6) and (3.5.4.7) are integrable by elementary functions. Substituting $y=e^{-\zeta}/v$ into (3.5.4.6), we obtain an autonomous equation:

$$(v'_\zeta)^2=2\alpha v+\gamma+(1+C_1)v^2. \tag{3.5.4.8}$$

As a result, we find:

$$y=\begin{cases}\dfrac{2\alpha}{z(\alpha^2\ln^2 z+2\alpha C\ln z+C^2-\gamma)} & \text{if } C_1=-1,\ \beta=\delta=0;\\ \dfrac{1}{z\bigl(\sqrt{\gamma}\ln z+C\bigr)} & \text{if } C_1=-1,\ \alpha=\beta=\delta=0;\\ \dfrac{z^{m-1}}{C_2z^{2m}+K_1z^m+K_2} & \text{if } C_1\neq-1,\ \beta=\delta=0,\end{cases}$$

where $C_2\neq 0,\ K_1=-\dfrac{\alpha}{C_1+1},\ K_2=\dfrac{\alpha^2-\gamma(1+C_1)}{4C_2(1+C_1)^2},\ m^2=1+C_1.$

Accordingly, equation (3.5.4.7) is reduced to equation (3.5.4.8) with the substitution $y=ve^\zeta$.

If $\beta=-\alpha$ and $\delta=-\gamma$, the substitution $y=e^{-iw}$ brings equation (3.5.4.1) to the following form: $w''_{zz}+\dfrac{1}{z}w'_z=\dfrac{2\alpha}{z}\sin w+2\gamma\sin 2w$.

► A solution in terms of Bessel functions.

Any solution of the Riccati equation

$$y'_z=ky^2+\frac{\alpha-k}{kz}y+c, \tag{3.5.4.9}$$

where $k^2=\gamma,\ c^2=-\delta,\ k\beta+c(\alpha-2k)=0$, is a solution of equation (3.5.4.1). Substituting $z=\lambda\tau,\ y=-\dfrac{u'_z}{ku}$, where $\lambda^2=\dfrac{1}{kc}$, into (3.5.4.9), we obtain a linear equation

$$u''_{\tau\tau}+\frac{k-\alpha}{k\tau}u'_\tau+u=0,$$

whose general solution is expressed in terms of Bessel functions:

$$u=\tau^{\frac{\alpha}{2k}}\Bigl[C_1J_{\frac{\alpha}{2k}}(\tau)+C_2Y_{\frac{\alpha}{2k}}(\tau)\Bigr].$$

► Asymptotic formulas and some properties.

Let $\alpha=-\beta=2\nu$ ($\nu\in\mathbb{R}$) and $\gamma=-\delta=1$. Then

$$y(x)-1\sim -c_1 2^{-2\nu}\Gamma\bigl(\nu+\tfrac12\bigr)x^{-(2\nu+1)/2}e^{-2x}\quad\text{as}\quad x\to+\infty, \tag{3.5.4.10}$$

where c_1 is an arbitrary constant such that $-1/\pi < c_1 < 1/\pi$, and

$$y(x) \sim c_2 x^{\sigma} \quad \text{at} \quad x \to 0, \tag{3.5.4.11}$$

where c_2 and σ are constants such that $c_2 \neq 0$ and $|\operatorname{Re}\sigma| < 1$. The connection formulas relating (3.5.4.10) and (3.5.4.11) are

$$\sigma = \frac{2}{\pi}\arcsin(\pi c_1), \qquad c_2 = 2^{-2\sigma}\frac{\Gamma^2\big(\frac{1}{2}-\frac{1}{2}\sigma\big)\Gamma\big(\frac{1}{2}+\frac{1}{2}\sigma+\nu\big)}{\Gamma^2\big(\frac{1}{2}+\frac{1}{2}\sigma\big)\Gamma\big(\frac{1}{2}-\frac{1}{2}\sigma+\nu\big)}.$$

3.5.5 Fourth Painlevé Equation

► Form of the fourth Painlevé equation. Solutions in the vicinity of a movable pole.

The *fourth Painlevé equation* has the form

$$y''_{zz} = \frac{(y'_z)^2}{2y} + \frac{3}{2}y^3 + 4zy^2 + 2(z^2-\alpha)y + \frac{\beta}{y}. \tag{3.5.5.1}$$

The solutions of the fourth Painlevé equation are single-valued functions of z.

The Laurent-series expansion of the solution of equation (3.5.5.1) in the vicinity of a movable pole $z_{\rm p}$ is given by

$$y = \frac{m}{z-z_{\rm p}} - z_{\rm p} - \frac{m}{3}(z_{\rm p}^2 + 2\alpha - 4m)(z-z_{\rm p}) + C(z-z_{\rm p})^2 + \sum_{j=3}^{\infty} a_j (z-z_{\rm p})^j,$$

where $m = \pm 1$; $z_{\rm p}$ and C are arbitrary constants; and the a_j ($j \geq 3$) are uniquely defined in terms of α, β, $z_{\rm p}$, and C.

Remark 3.15. Equation (3.5.5.1) is invariant under the transformation $y = \lambda\bar{y}$, $z = \lambda\bar{z}$, $\alpha = \bar{\alpha}\lambda^2$, $\beta = \bar{\beta}$, where $\lambda^4 = 1$.

► Rational particular solutions.

Let $y = y(z, \alpha, \beta)$ be a solution of equation (3.5.5.1). Then special rational solutions of the fourth Painlevé equation are

$$y_1(z, \pm 2, -2) = \pm 1/z, \qquad y_2(z, 0, -2) = -2z, \qquad y_3(z, 0, -\tfrac{2}{9}) = -\tfrac{2}{3}z.$$

There are also three more complex families of solutions of equation (3.5.5.1) of the form

$$\begin{aligned} y_1(z,\alpha_1,\beta_1) &= P_{1,n-1}(z)/Q_{1,n}(z), \\ y_2(z,\alpha_2,\beta_2) &= -2z + \big[P_{2,n-1}(z)/Q_{2,n}(z)\big], \\ y_3(z,\alpha_3,\beta_3) &= -\tfrac{2}{3}z + \big[P_{3,n-1}(z)/Q_{3,n}(z)\big], \end{aligned}$$

where $P_{j,n-1}(z)$ and $Q_{j,n}(z)$ are polynomials of degrees $n-1$ and n, respectively, with no common zeros.

Some rational particular solutions:

$$y(z,-m,-2(m-1)^2) = -\frac{H'_{m-1}(z)}{H_{m-1}(z)}, \qquad m = 1,\ 2,\ 3,\ \dots,$$
$$y(z,-m,-2(m+1)^2) = -2z + \frac{H'_m(z)}{H_m(z)}, \qquad m = 0,\ 1,\ 2,\ \dots,$$

where $H_m(z)$ are the Hermite polynomials.

In general, equation (3.5.5.1) has rational solutions iff either

$$\alpha = m, \qquad \beta = -2(1+2n-m)^2,$$

or

$$\alpha = m, \qquad \beta = -2(\tfrac{1}{3}+2n-m)^2,$$

with $m, n \in \mathbb{Z}$.

► **Relation between solutions of two equations. Bäcklund transformations.**

Two solutions of equation (3.5.5.1) corresponding to different values of the parameters α and β are related to each other by the Bäcklund transformations:

$$\widetilde{y} = \frac{1}{2sy}(y'_z - q - 2szy - sy^2), \qquad q^2 = -2\beta,$$
$$y = -\frac{1}{2s\widetilde{y}}(\widetilde{y}'_z - p + 2sz\widetilde{y} + s\widetilde{y}^2), \qquad p^2 = -2\widetilde{\beta},$$
$$2\beta = -(\widetilde{\alpha}s - 1 - \tfrac{1}{2}p)^2, \qquad 4\alpha = -2s - 2\widetilde{\alpha} - 3sp,$$

where $y = y(z,\alpha,\beta)$, $\widetilde{y} = \widetilde{y}(z,\widetilde{\alpha},\widetilde{\beta})$, and s is an arbitrary parameter.

► **A solution in terms of solutions of the Riccati equation.**

If the condition

$$\beta = -2(1+\epsilon\alpha)^2 \quad \text{with} \quad \epsilon = \pm 1$$

is satisfied, then every solution of the Riccati equation

$$y'_z = \epsilon y^2 + 2\epsilon zy - 2(1+\epsilon\alpha) \tag{3.5.5.2}$$

is simultaneously a solution of the fourth Painlevé equation (3.5.5.1). The general solution of equation (3.5.5.2) can be expressible in terms of parabolic cylinder functions.

For $\alpha = 1$ and $\epsilon = -1$, equation (3.5.5.2) has a solution

$$y = \frac{2\exp(-z^2)}{\sqrt{\pi}\,(C - \operatorname{erfc} z)},$$

where C is an arbitrary constant and $\operatorname{erfc} z$ is the complementary error function.

Remark 3.16. In general, equation (3.5.5.2) has solutions expressible in terms of parabolic cylinder functions iff either

$$\beta = -2(2n+1+\epsilon\alpha)^2 \quad \text{or} \quad \beta = -2n^2,$$

with $n \in \mathbb{Z}$ and $\epsilon = \pm 1$.

► **Symmetric forms.**

Let

$$\begin{aligned} f_1' + f_1(f_2 - f_3) + 2\mu_1 = 0, \\ f_2' + f_2(f_3 - f_1) + 2\mu_2 = 0, \\ f_3' + f_3(f_1 - f_2) + 2\mu_2 = 0, \end{aligned}$$

where μ_1, μ_2, μ_3 are constants, f_1, f_2, f_3 are functions of z (the prime denotes differentiation with respect to z), with

$$\begin{aligned} \mu_1 + \mu_2 + \mu_3 = 1, \\ f_1 + f_2 + f_3 = -2z. \end{aligned}$$

Then the function $y = f_1(z)$ satisfies equation (3.5.5.1) with

$$\alpha = \mu_3 - \mu_2, \qquad \beta = -2\mu_1^2.$$

3.5.6 Fifth Painlevé Equation

► **Form of the fifth Painlevé equation. Relations between solutions.**

The *fifth Painlevé equation* has the form

$$y''_{zz} = \frac{3y - 1}{2y(y-1)}(y'_z)^2 - \frac{y'_z}{z} + \frac{(y-1)^2}{z^2}\left(\alpha y + \frac{\beta}{y}\right) + \gamma\frac{y}{z} + \frac{\delta y(y+1)}{y-1}. \qquad (3.5.6.1)$$

If we pass on to the new independent variable $z = e^{\zeta}$, the solutions are single-valued functions of ζ.

Solutions of the fifth Painlevé equation (3.5.6.1) corresponding to different values of parameters are related by:

$$\begin{aligned} y(z, \alpha, \beta, \gamma, \delta) &= y(-z, \alpha, \beta, -\gamma, \delta), \\ y(z, \alpha, \beta, \gamma, \delta) &= \frac{1}{y(z, -\beta, -\alpha, -\gamma, \delta)}. \end{aligned}$$

► **Rational particular solutions.**

Let $y = y(z, \alpha, \beta, \gamma, \delta)$ be a solution of equation (3.5.6.1). Then special rational solutions of the fourth Painlevé equation are

$$\begin{aligned} y(z, \tfrac{1}{2}, -\tfrac{1}{2}\mu^2, 2k - k\mu, -\tfrac{1}{2}k^2) = kz + \mu, \\ y(z, \tfrac{1}{2}, k^2\mu, 2k\mu, \mu) = k/(k+z), \\ y(z, \tfrac{1}{8}, -\tfrac{1}{8}, -k\mu, \mu) = (k+z)/(k-z), \end{aligned}$$

where k and μ are arbitrary constants.

If $\delta \neq 0$ in (3.5.6.1), then set $\delta = 1/2$, without loss of generality. In this case the fifth Painlevé equation has a rational solution iff one of the following holds with $m, n \in \mathbb{Z}$ and $\epsilon \pm 1$:

(a) $\alpha = \frac{1}{2}(m+\epsilon\gamma)^2$ and $\beta = -\frac{1}{2}n^2$, where $n>0$, $n+m$ is odd, and $\alpha \neq 0$ when $|m|<n$.

(b) $\alpha = \frac{1}{2}n^2$ and $\beta = -\frac{1}{2}(m+\epsilon\gamma)^2$, where $n>0$, $n+m$ is odd, and $\beta \neq 0$ when $|m|<n$.

(c) $\alpha = \frac{1}{2}a^2$, $\beta = -\frac{1}{2}(a+n)^2$, and $\gamma = m$, where $n+m$ is even, and a is an arbitrary constant.

(d) $\alpha = \frac{1}{2}(b+n)^2$, $\beta = -\frac{1}{2}b^2$, and $\gamma = m$, where $n+m$ is even, and b is an arbitrary constant.

(e) $\alpha = \frac{1}{8}(2m+1)^2$, $\beta = -\frac{1}{8}(2n+1)^2$, and $\gamma \notin= \mathbb{Z}$. These rational solutions have the form

$$y = \lambda z + \mu + \frac{P_{n-1}(z)}{Q_n(z)},$$

where $P_{n-1}(z)$ and $Q_n(z)$ are polynomials of degrees $n-1$ and n, respectively, with no common zeros.

► Elementary nonrational particular solutions.

Elementary nonrational solutions of the fifth Painlevé equation are

$$y(z,\mu,-\tfrac{1}{8},-k^2\mu,0) = 1 + kz^{1/2},$$
$$y(z,0,0,\mu,-\tfrac{1}{2}\mu^2) = k\exp(\mu z),$$

where k and μ are arbitrary constants.

Equation (3.5.6.1), with $\delta = 0$, has algebraic solutions if either

$$\alpha = \tfrac{1}{2}\mu^2, \quad \beta = -\tfrac{1}{8}(2n-1)^2, \quad \gamma = -1,$$

or

$$\alpha = \tfrac{1}{8}(2n-1)^2, \quad \beta = -\tfrac{1}{2}\mu^2, \quad \gamma = 1,$$

with $n \in \mathbb{Z}$. These are rational solutions in $\zeta = z^{1/2}$ of the form

$$y = P_{n^2-n+1}(\zeta)/Q_{n^2-n}(\zeta),$$

where $P_{n^2-n+1}(\zeta)$ and $Q_{n^2-n}(\zeta)$ are polynomials of degrees n^2-n+1 and n^2-n, respectively, with no common zeros.

► Cases when the fifth Painlevé equation are solvable by quadrature.

1°. Equation (3.5.6.1), with $\gamma = \delta = 0$, has a first integral

$$z^2(y'_z)^2 = (y-1)^2(2\alpha y^2 + Cy - 2\beta),$$

which is solvable by quadrature (C is an arbitrary constant).

2°. On setting $z = e^t$ in (3.5.6.1), we obtain

$$y''_{tt} = \frac{3y-1}{2y(y-1)}(y'_t)^2 + (y-1)^2\Big(\alpha y + \frac{\beta}{y}\Big) + \gamma y e^t + \frac{\delta y(y+1)}{y-1}e^{2t}. \qquad (3.5.6.2)$$

If $\gamma = \delta = 0$, equation (3.5.6.2) is reduced, by integration, to a first-order autonomous equation:

$$y'_t = (y-1)\sqrt{2\alpha y^2 + Cy - 2\beta},$$

which is readily integrable by quadrature.

► **Solutions in terms of Whittaker functions.**

If $\delta \neq 0$ in (3.5.6.1), then set $\delta = 1/2$, without loss of generality. Then the fifth Painlevé equation has solutions expressible in terms of Whittaker functions, only in the following three cases:

$$(a) \quad a + b + \epsilon_3\gamma = 2n + 1, \tag{3.5.6.3}$$

$$(b) \quad a = n, \tag{3.5.6.4}$$

$$(c) \quad b = n, \tag{3.5.6.5}$$

where $n \in \mathbb{Z}$, $a = \epsilon_1\sqrt{2\alpha}$, and $b = \epsilon_2\sqrt{-2\beta}$, with $\epsilon_j = \pm 1$, $j = 1,\ 2,\ 3$, independently.

In the case when $n = 0$ in (3.5.6.3), any solution of the Riccati equation

$$zy'_z = ay^2 + (b - a + \epsilon_3 z)y - b \tag{3.5.6.6}$$

is simultaneously a solution of the fifth Painlevé equation (3.5.6.1). If $a \neq 0$, then equation (3.5.6.6) has the solution

$$y = -z\phi'_z(z)/\phi(z),$$

where

$$\phi(z) = \zeta^{-k}\exp\left(\tfrac{1}{2}\zeta\right)\left[C_1 M_{k,\mu}(\zeta) + C_2 W_{k,\mu}(\zeta)\right],$$

with $\zeta = \epsilon_3 z$, $k = \frac{1}{2}(a - b + 1)$, $\mu = \frac{1}{2}(a + b)$; C_1 and C_2 are arbitrary constants, and $M_{k,\mu}(\zeta)$ and $W_{k,\mu}(\zeta)$ are Whittaker functions.

3.5.7 Sixth Painlevé Equation

► **Form of the sixth Painlevé equation. Relations between solutions.**

The *sixth Painlevé equation* has the form

$$y''_{zz} = \frac{1}{2}\left(\frac{1}{y} + \frac{1}{y-1} + \frac{1}{y-z}\right)(y'_z)^2 - \left(\frac{1}{z} + \frac{1}{z-1} + \frac{1}{y-z}\right)y'_z + \frac{y(y-1)(y-z)}{z^2(z-1)^2}\left[\alpha + \beta\frac{z}{y^2} + \gamma\frac{z-1}{(y-1)^2} + \delta\frac{z(z-1)}{(y-z)^2}\right]. \tag{3.5.7.1}$$

In equation (3.5.7.1), the points $z = 0$, $z = 1$, and $z = \infty$ are fixed logarithmic branch points.

Solutions of the sixth Painlevé equation (3.5.7.1) corresponding to different values of parameters are related by:

$$y(z,\ -\beta,\ -\alpha,\ \gamma,\ \delta) = \frac{1}{y\left(\dfrac{1}{z},\ \alpha,\ \beta,\ \gamma,\ \delta\right)},$$

$$y(z,\ -\beta,\ -\gamma,\ \alpha,\ \delta) = 1 - \frac{1}{y\left(\dfrac{1}{1-z},\ \alpha,\ \beta,\ \gamma,\ \delta\right)},$$

$$y\left(z,\ -\beta,\ -\alpha,\ -\delta + \frac{1}{2},\ -\gamma + \frac{1}{2}\right) = \frac{z}{y(z,\ \alpha,\ \beta,\ \gamma,\ \delta)}.$$

The successive application of these relations yields 24 equations of the form (3.5.7.1) with different values of parameters related by known transformations.

► **Rational particular solutions.**

Let $y = y(z, \alpha, \beta, \gamma, \delta)$ be a solution of equation (3.5.7.1). Then special rational solutions of the sixth Painlevé equation are

$$\begin{aligned} y(z, \mu, -k^2\mu, \tfrac{1}{2}, -\tfrac{1}{2} - \mu(k-1)^2) &= kz, \\ y(z, 0, 0, 2, 0) &= kz^2, \\ y(z, 0, 0, \tfrac{1}{2}, -\tfrac{3}{2}) &= k/z, \\ y(z, 0, 0, 2, -4) &= k/z^2, \\ y(z, \tfrac{1}{2}(k+\mu)^2, -\tfrac{1}{2}, \tfrac{1}{2}(\mu-1)^2, \tfrac{1}{2}k(2-k)) &= z/(k+\mu z), \end{aligned}$$

where k and μ are arbitrary constants.

In the general case, the sixth Painlevé equation has rational solutions if

$$\begin{gathered} a+b+c+d = 2n+1, \qquad n \in \mathbb{Z}, \\ a = \epsilon_1\sqrt{2\alpha}, \quad b = \epsilon_2\sqrt{-2\beta}, \quad c = \epsilon_3\sqrt{2\gamma}, \quad d = \epsilon_4\sqrt{1-2\delta}, \end{gathered}$$

where $\epsilon_j = \pm 1$, $j = 1, 2, 3, 4$, independently, and at least one of numbers a, b, c or d is an integer.

► **Solutions in terms of the elliptic function.**

1°. If $\alpha = \beta = \gamma = \delta = 0$, the general solution of equation (19) has the form:

$$y = E(C_1\omega_1 + C_2\omega_2,\ z),$$

where $E(u, z)$ is the elliptic function, defined by the integral

$$u = \int_0^E \frac{dy}{\sqrt{y(y-1)(y-z)}}, \tag{3.5.7.2}$$

with periods $2\omega_1$ and $2\omega_2$, which are functions of z.

2°. If $\alpha = \beta = \gamma = 0$, $\delta = \frac{1}{2}$, the general solution of equation (3.5.7.1) has the form:

$$y = E(w + C_1\omega_1 + C_2\omega_2,\ z),$$

where $w \not\equiv 0$ is any particular solution of the linear equation

$$z(z-1)w''_{zz} + (2z-1)w'_z + \tfrac{1}{4}w = 0$$

and $E(u, z)$ is the elliptic function defined by formula (3.5.7.2).

► **Solutions in terms of hypergeometric functions.**

Equation (3.5.7.1) has solutions expressible in terms of hypergeometric functions iff

$$\begin{gathered} a+b+c+d = 2n+1, \qquad n \in \mathbb{Z}, \\ a = \epsilon_1\sqrt{2\alpha}, \quad b = \epsilon_2\sqrt{-2\beta}, \quad c = \epsilon_3\sqrt{2\gamma}, \quad d = \epsilon_4\sqrt{1-2\delta}, \end{gathered} \tag{3.5.7.3}$$

with $\epsilon_j = \pm 1$, $j = 1,\ 2,\ 3$, independently.

If $n = 1$ in (3.5.7.3), then every solution of the Riccati equation

$$z(z-1)y'_z = ay^2 + [(b+c)z - a - c]y - bz, \tag{3.5.7.4}$$

is simultaneously a solution of equation (3.5.7.1). If $a \neq 0$, then (3.5.7.4) has the solution

$$y = \frac{\zeta - 1}{a}\frac{\phi'_\zeta(\zeta)}{\phi(\zeta)}, \qquad \zeta = \frac{1}{1-z},$$

where

$$\phi(\zeta) = C_1 F(b, -a, b+c; \zeta) + C_2 \zeta^{1-b-c} F(1-a-b-c, 1-c, 2-b-c; \zeta),$$

C_1 and C_2 are arbitrary constants, and $F(\alpha, \beta, \gamma; \zeta)$ is the hypergeometric function.

◆ *For more details about Painlevé equations (including of some illustrative figures of Painlevé transcendental functions), see the list of references given below.*

⊙ *Literature for Section 3.5*: P. Painlevé (1900), B. Gambier (1910), V. V. Golubev (1950), A. S. Fokas and M. J. Ablowitz (1982), A. R. Its and V. Yu. Novokshenov (1986), V. I. Gromak and N. A. Lukashevich (1990), R. Conte (1999), A. R. Chowdhury (2000), V. F. Zaitsev and A. D. Polyanin (2001), V. I. Gromak (2002), P. A. Clarkson (2003, 2006), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007), F. W. J. Olver, D. W. Lozier, R. F. Boisvert, and C. W. Clark (2010).

3.6 Perturbation Methods of Mechanics and Physics

3.6.1 Preliminary Remarks. Summary Table of Basic Methods

Perturbation methods are widely used in nonlinear mechanics and theoretical physics for solving problems that are described by differential equations with a small parameter ε. The primary purpose of these methods is to obtain an approximate solution that would be equally suitable at all (small, intermediate, and large) values of the independent variable as $\varepsilon \to 0$.

Equations with a small parameter can be classified according to the following:

(i) the order of the equation remains the same at $\varepsilon = 0$;

(ii) the order of the equation reduces at $\varepsilon = 0$.

For the first type of equations, solutions of related problems* are sufficiently smooth (little varying as ε decreases). The second type of equation is said to be degenerate at $\varepsilon = 0$, or singularly perturbed. In related problems, thin boundary layers usually arise whose thickness is significantly dependent on ε; such boundary layers are characterized by high gradients of the unknown.

All perturbation methods have a limited domain of applicability; the possibility of using one or another method depends on the type of equations or problems involved. The most commonly used methods are summarized in Table 3.2 (the method of regular series expansions is set out in Section 3.6.2). In subsequent paragraphs, additional remarks and specific examples are given for some of the methods. In practice, one usually confines oneself to a few leading terms of the asymptotic expansion.

In many problems of nonlinear mechanics and theoretical physics, the independent variable is dimensionless time t. Therefore, in this subsection we use the conventional t $(0 \le t < \infty)$ instead of x.

*Further on, we assume that the initial and/or boundary conditions are independent of the parameter ε.

TABLE 3.2
Perturbation methods of nonlinear mechanics and theoretical physics
(the third column gives n leading asymptotic terms with respect to the small parameter ε).

Method name	Examples of problems solved by the method	Form of the solution sought	Additional conditions and remarks
Method of scaled parameters $(0\le t<\infty)$	One looks for periodic solutions of the equation $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$; see also Section 3.6.3	$y(t)=\sum_{k=0}^{n-1}\varepsilon^k y_k(z)$, $t=z\left(1+\sum_{k=1}^{n-1}\varepsilon^k\omega_k\right)$	Unknowns: y_k and ω_k; $y_{k+1}/y_k=O(1)$; secular terms are eliminated through selection of the constants ω_k
Method of strained coordinates $(0\le t<\infty)$	Cauchy problem: $y'_t=f(t,y,\varepsilon)$; $y(t_0)=y_0$ (f is of a special form); see also the problem in the method of scaled parameters	$y(t)=\sum_{k=0}^{n-1}\varepsilon^k y_k(z)$, $t=z+\sum_{k=1}^{n-1}\varepsilon^k\varphi_k(z)$	Unknowns: y_k and φ_k; $y_{k+1}/y_k=O(1)$, $\varphi_{k+1}/\varphi_k=O(1)$
Averaging method $(0\le t<\infty)$	Cauchy problem: $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$, $y(0)=y_0$, $y'_t(0)=y_1$; for more general problems see Section 3.6.4	$y=a(t)\cos\varphi(t)$, the amplitude a and phase φ are governed by the equations $\frac{da}{dt}=-\frac{\varepsilon}{\omega_0}f_s(a)$, $\frac{d\varphi}{dt}=\omega_0-\frac{\varepsilon}{a\omega_0}f_c(a)$	Unknowns: a and φ; $f_s=\frac{1}{2\pi}\int_0^{2\pi}\sin\varphi F\,d\varphi$, $f_c=\frac{1}{2\pi}\int_0^{2\pi}\cos\varphi F\,d\varphi$, $F=f(a\cos\varphi,-a\omega_0\sin\varphi)$
Krylov–Bogolyubov–Mitropolskii method $(0\le t<\infty)$	One looks for periodic solutions of the equation $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$; Cauchy problem for this and other equations	$y=a\cos\varphi+\sum_{k=1}^{n-1}\varepsilon^k y_k(a,\varphi)$, a and φ are determined by the equations $\frac{da}{dt}=\sum_{k=1}^{n}\varepsilon^k A_k(a)$, $\frac{d\varphi}{dt}=\omega_0+\sum_{k=1}^{n}\varepsilon^k\Phi_k(a)$	Unknowns: y_k, A_k, Φ_k; y_k are 2π-periodic functions of φ; the y_k are assumed not to contain $\cos\varphi$
Method of two-scale expansions $(0\le t<\infty)$	Cauchy problem: $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$, $y(0)=y_0$, $y'_t(0)=y_1$; for boundary value problems see Section 3.6.5	$y=\sum_{k=0}^{n-1}\varepsilon^k y_k(\xi,\eta)$, where $\xi=\varepsilon t$, $\eta=t\left(1+\sum_{k=2}^{n-1}\varepsilon^k\omega_k\right)$, $\frac{d}{dt}=\varepsilon\frac{\partial}{\partial\xi}+\left(1+\varepsilon^2\omega_2+\cdots\right)\frac{\partial}{\partial\eta}$	Unknowns: y_k and ω_k; $y_{k+1}/y_k=O(1)$; secular terms are eliminated through selection of ω_k
Multiple scales method $(0\le t<\infty)$	One looks for periodic solutions of the equation $y''_{tt}+\omega_0^2 y=\varepsilon f(y,y'_t)$; Cauchy problem for this and other equations	$y=\sum_{k=0}^{n-1}\varepsilon^k y_k$, where $y_k=y_k(T_0,T_1,\ldots,T_n)$, $T_k=\varepsilon^k t$ $\frac{d}{dt}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+\cdots+\varepsilon^n\frac{\partial}{\partial T_n}$	Unknowns: y_k; $y_{k+1}/y_k=O(1)$; for $n=1$, this method is equivalent to the averaging method
Method of matched asymptotic expansions $(0\le x\le b)$	Boundary value problem: $\varepsilon y''_{xx}+f(x,y)y'_x=g(x,y)$, $y(0)=y_0$, $y(b)=y_b$ (f assumed positive); for other problems see Section 3.6.6	Outer expansion: $y=\sum_{k=0}^{n-1}\sigma_k(\varepsilon)y_k(x)$, $O(\varepsilon)\le x\le b$; inner expansion ($z=x/\varepsilon$): $\widetilde y=\sum_{k=0}^{n-1}\widetilde\sigma_k(\varepsilon)\widetilde y_k(z)$, $0\le x\le O(\varepsilon)$	Unknowns: y_k, $\widetilde y_k$, σ_k, $\widetilde\sigma_k$; $y_{k+1}/y_k=O(1)$, $\widetilde y_{k+1}/\widetilde y_k=O(1)$; the procedure of matching expansions is used: $y(x\to0)=\widetilde y(z\to\infty)$
Method of composite expansions $(0\le x\le b)$	Boundary value problem: $\varepsilon y''_{xx}+f(x,y)y'_x=g(x,y)$, $y(0)=y_0$, $y(b)=y_b$ (f assumed positive); boundary value problems for other equations	$y=Y(x,\varepsilon)+\widetilde Y(z,\varepsilon)$, $Y=\sum_{k=0}^{n-1}\sigma_k(\varepsilon)Y_k(x)$, $\widetilde Y=\sum_{k=0}^{n-1}\widetilde\sigma_k(\varepsilon)\widetilde Y_k(z)$, $z=\frac{x}{\varepsilon}$; here, $\widetilde Y_k\to0$ as $z\to\infty$	Unknowns: Y_k, $\widetilde Y_k$, σ_k, $\widetilde\sigma_k$; $Y(b,\varepsilon)=y_b$, $Y(0,\varepsilon)+\widetilde Y(0,\varepsilon)=y_0$; two forms for the equation (in terms of x and z) are used to obtain solutions

3.6.2 Method of Regular (Direct) Expansion in Powers of the Small Parameter

We consider an equation of general form with a parameter ε:

$$y''_{tt} + f(t, y, y'_t, \varepsilon) = 0. \tag{3.6.2.1}$$

We assume that the function f can be represented as a series in powers of ε:

$$f(t, y, y'_t, \varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n f_n(t, y, y'_t). \tag{3.6.2.2}$$

Solutions of the Cauchy problem and various boundary value problems for equation (3.6.2.1) with $\varepsilon \to 0$ are sought in the form of a power series expansion:

$$y = \sum_{n=0}^{\infty} \varepsilon^n y_n(t). \tag{3.6.2.3}$$

One should substitute (3.6.2.3) into equation (3.6.2.1) taking into account (3.6.2.2). Then the functions f_n are expanded into a power series in the small parameter and the coefficients of like powers of ε are collected and equated to zero to obtain a system of equations for y_n:

$$y''_0 + f_0(t, y_0, y'_0) = 0, \tag{3.6.2.4}$$

$$y''_1 + F(t, y_0, y'_0)y'_1 + G(t, y_0, y'_0)y_1 + f_1(t, y_0, y'_0) = 0, \tag{3.6.2.5}$$

$$F = \frac{\partial f_0}{\partial y'}, \qquad G = \frac{\partial f_0}{\partial y}.$$

Here only the first two equations are written out. The prime denotes differentiation with respect to t. To obtain the initial (or boundary) conditions for y_n, the expansion (3.6.2.3) is taken into account.

The success in the application of this method is primarily determined by the possibility of constructing a solution of equation (3.6.2.4) for the leading term y_0. It is significant to note that the other terms y_n with $n \geq 1$ are governed by linear equations with homogeneous initial conditions.

Example 3.26. The *Duffing equation*

$$y''_{tt} + y + \varepsilon y^3 = 0 \tag{3.6.2.6}$$

with initial conditions

$$y(0) = a, \quad y'_t(0) = 0$$

describes the motion of a cubic oscillator, i.e., oscillations of a point mass on a nonlinear spring. Here, y is the deviation of the point mass from the equilibrium and t is dimensionless time.

For $\varepsilon \to 0$, an approximate solution of the problem is sought in the form of the asymptotic expansion (3.6.2.3). We substitute (3.6.2.3) into equation (3.6.2.6) and initial conditions and expand in powers of ε. On equating the coefficients of like powers of the small parameter to zero, we obtain the following problems for y_0 and y_1:

$$y''_0 + y_0 = 0, \qquad y_0 = a, \quad y'_0 = 0;$$

$$y''_1 + y_1 = -y_0^3, \qquad y_1 = 0, \quad y'_1 = 0.$$

The solution of the problem for y_0 is given by

$$y_0 = a\cos t.$$

Substituting this expression into the equation for y_1 and taking into account the identity $\cos^3 t = \frac{1}{4}\cos 3t + \frac{3}{4}\cos t$, we obtain

$$y_1'' + y_1 = -\tfrac{1}{4}a^3(\cos 3t + 3\cos t), \qquad y_1 = 0, \quad y_1' = 0.$$

Integrating yields

$$y_1 = -\tfrac{3}{8}a^3 t\sin t + \tfrac{1}{32}a^3(\cos 3t - 3\cos t).$$

Thus the two-term solution of the original problem is given by

$$y = a\cos t + \varepsilon a^3\bigl[-\tfrac{3}{8}t\sin t + \tfrac{1}{32}(\cos 3t - 3\cos t)\bigr] + O(\varepsilon^2).$$

Remark 3.17. The term $t\sin t$ causes $y_1/y_0 \to \infty$ as $t \to \infty$. For this reason, the solution obtained is unsuitable at large times. It can only be used for $\varepsilon t \ll 1$; this results from the condition of applicability of the expansion, $y_0 \gg \varepsilon y_1$.

This circumstance is typical of the method of regular series expansions with respect to the small parameter; in other words, the expansion becomes unsuitable at large values of the independent variable. This method is also inapplicable if the expansion (3.6.2.3) begins with negative powers of ε. Methods that allow avoiding the above difficulties are discussed below in Sections 3.6.3 through 3.6.5.

Remark 3.18. Growing terms as $t \to \infty$, like $t\sin t$, that narrow down the domain of applicability of asymptotic expansions are called *secular*.

3.6.3 Method of Scaled Parameters (Lindstedt–Poincaré Method)

We illustrate the characteristic features of the method of scaled parameters with a specific example (the transformation of the independent variable we use here as well as the form of the expansion are specified in the first row of Table 3.2).

Example 3.27. Consider the Duffing equation (3.6.2.6) again. On performing the change of variable

$$t = z(1 + \varepsilon\omega_1 + \cdots),$$

we have

$$y''_{zz} + (1 + \varepsilon\omega_1 + \cdots)^2(y + \varepsilon y^3) = 0. \tag{3.6.3.1}$$

The solution is sought in the series form

$$y = y_0(z) + \varepsilon y_1(z) + \cdots.$$

Substituting it into equation (3.6.3.1) and matching the coefficients of like powers of ε, we arrive at the following system of equations for two leading terms of the series:

$$y_0'' + y_0 = 0, \tag{3.6.3.2}$$

$$y_1'' + y_1 = -y_0^3 - 2\omega_1 y_0, \tag{3.6.3.3}$$

where the prime denotes differentiation with respect to z.

The general solution of equation (3.6.3.2) is given by

$$y_0 = a\cos(z + b), \tag{3.6.3.4}$$

where a and b are constants of integration. Taking into account (3.6.3.4) and rearranging terms, we reduce equation (3.6.3.3) to

$$y_1'' + y_1 = -\tfrac{1}{4}a^3\cos\bigl[3(z + b)\bigr] - 2a\bigl(\tfrac{3}{8}a^2 + \omega_1\bigr)\cos(z + b). \tag{3.6.3.5}$$

For $\omega_1 \neq -\frac{3}{8}a^2$, the particular solution of equation (3.6.3.5) contains a secular term proportional to $z\cos(z+b)$. In this case, the condition of applicability of the expansion $y_1/y_0 = O(1)$ (see the first row and the last column of Table 3.2) cannot be satisfied at sufficiently large z. For this condition to be met, one should set

$$\omega_1 = -\tfrac{3}{8}a^2. \tag{3.6.3.6}$$

In this case, the solution of equation (3.6.3.5) is given by

$$y_1 = \tfrac{1}{32}a^3\cos\bigl[3(z+b)\bigr]. \tag{3.6.3.7}$$

Subsequent terms of the expansion can be found likewise.

With (3.6.3.4), (3.6.3.6), and (3.6.3.7), we obtain a solution of the Duffing equation in the form

$$y = a\cos(\omega t+b) + \tfrac{1}{32}\varepsilon a^3\cos\bigl[3(\omega t+b)\bigr] + O(\varepsilon^2),$$
$$\omega = \bigl[1-\tfrac{3}{8}\varepsilon a^2 + O(\varepsilon^2)\bigr]^{-1} = 1+\tfrac{3}{8}\varepsilon a^2 + O(\varepsilon^2).$$

3.6.4 Averaging Method (Van der Pol–Krylov–Bogolyubov Scheme)

► Averaging method for equations of a special form.

1°. The averaging method involved two stages. First, the second-order nonlinear equation

$$y''_{tt} + \omega_0^2 y = \varepsilon f(y, y'_t) \tag{3.6.4.1}$$

is reduced with the transformation

$$y = a\cos\varphi, \quad y'_t = -\omega_0 a\sin\varphi, \qquad \text{where} \quad a = a(t), \quad \varphi = \varphi(t),$$

to an equivalent system of two first-order differential equations:

$$\begin{aligned} a'_t &= -\frac{\varepsilon}{\omega_0} f(a\cos\varphi, -\omega_0 a\sin\varphi)\sin\varphi, \\ \varphi'_t &= \omega_0 - \frac{\varepsilon}{\omega_0 a} f(a\cos\varphi, -\omega_0 a\sin\varphi)\cos\varphi. \end{aligned} \tag{3.6.4.2}$$

The right-hand sides of equations (3.6.4.2) are periodic in φ, with the amplitude a being a slow function of time t. The amplitude and the oscillation character are changing little during the time the phase φ changes by 2π.

At the second stage, the right-hand sides of equations (3.6.4.2) are being averaged with respect to φ. This procedure results in an approximate system of equations:

$$\begin{aligned} a'_t &= -\frac{\varepsilon}{\omega_0} f_{\rm s}(a), \\ \varphi'_t &= \omega_0 - \frac{\varepsilon}{\omega_0 a} f_{\rm c}(a), \end{aligned} \tag{3.6.4.3}$$

where

$$f_{\rm s}(a) = \frac{1}{2\pi}\int_0^{2\pi} \sin\varphi\, f(a\cos\varphi, -\omega_0 a\sin\varphi)\,d\varphi,$$
$$f_{\rm c}(a) = \frac{1}{2\pi}\int_0^{2\pi} \cos\varphi\, f(a\cos\varphi, -\omega_0 a\sin\varphi)\,d\varphi.$$

System (3.6.4.3) is substantially simpler than the original system (3.6.4.2)—the first equation in (3.6.4.3), for the oscillation amplitude a, is a separable equation and, hence, can readily be integrated; then the second equation in (3.6.4.3), which is linear in φ, can also be integrated.

Note that the Krylov–Bogolyubov–Mitropolskii method (the fourth row in Table 3.2) generalizes the above approach and allows obtaining subsequent asymptotic terms as $\varepsilon \to 0$.

► **General scheme of the averaging method.**

Below we outline the general scheme of the averaging method. We consider the second-order nonlinear equation with a small parameter:

$$y''_{tt} + g(t, y, y'_t) = \varepsilon f(t, y, y'_t). \tag{3.6.4.4}$$

Equation (3.6.4.4) should first be transformed to the equivalent system of equations

$$\begin{aligned} y'_t &= u, \\ u'_t &= -g(t, y, u) + \varepsilon f(t, y, u). \end{aligned} \tag{3.6.4.5}$$

Suppose the general solution of the "truncated" system (3.6.4.5), with $\varepsilon = 0$, is known:

$$y_0 = \varphi(t, C_1, C_2), \qquad u_0 = \psi(t, C_1, C_2), \tag{3.6.4.6}$$

where C_1 and C_2 are constants of integration. Taking advantage of the method of variation of constants, we pass from the variables y, u in (3.6.4.5) to Lagrange's variables x_1, x_2 according to the formulas

$$y = \varphi(t, x_1, x_2), \qquad u = \psi(t, x_1, x_2), \tag{3.6.4.7}$$

where φ and ψ are the same functions that define the general solution of the "truncated" system (3.6.4.6). Transformation (3.6.4.7) enables the reduction of system (3.6.4.5) to the *standard form*

$$\begin{aligned} x'_1 &= \varepsilon F_1(t, x_1, x_2), \\ x'_2 &= \varepsilon F_2(t, x_1, x_2). \end{aligned} \tag{3.6.4.8}$$

Here the prime denotes differentiation with respect to t and

$$F_1 = \frac{\varphi_2 f(t, \varphi, \psi)}{\varphi_2\psi_1 - \varphi_1\psi_2}, \quad F_2 = -\frac{\varphi_1 f(t, \varphi, \psi)}{\varphi_2\psi_1 - \varphi_1\psi_2}; \quad \varphi_k = \frac{\partial\varphi}{\partial x_k}, \quad \psi_k = \frac{\partial\psi}{\partial x_k},$$
$$\varphi = \varphi(t, x_1, x_2), \quad \psi = \psi(t, x_1, x_2).$$

It is noteworthy that system (3.6.4.8) is equivalent to the original equation (3.6.4.4). The unknowns x_1 and x_2 are slow functions of time.

As a result of averaging, system (3.6.4.8) is replaced by a simpler, approximate autonomous system of equations:

$$\begin{aligned} x'_1 &= \varepsilon \mathcal{F}_1(x_1, x_2), \\ x'_2 &= \varepsilon \mathcal{F}_2(x_1, x_2), \end{aligned} \tag{3.6.4.9}$$

where

$$\mathcal{F}_k(x_1, x_2) = \frac{1}{T}\int_0^T F_k(t, x_1, x_2)\, dt \qquad \text{if } F_k \text{ is a } T\text{-periodic function of } t;$$
$$\mathcal{F}_k(x_1, x_2) = \lim_{T\to\infty} \frac{1}{T}\int_0^T F_k(t, x_1, x_2)\, dt \qquad \text{if } F_k \text{ is not periodic in } t.$$

Remark 3.19. The averaging method is applicable to equations (3.6.4.1) and (3.6.4.4) with non-smooth right-hand sides.

Remark 3.20. The averaging method has rigorous mathematical substantiation. There is also a procedure that allows finding subsequent asymptotic terms. For this procedure, e.g., see the books by Bogolyubov and Mitropolskii (1974), Zhuravlev and Klimov (1988), and Arnold, Kozlov, and Neishtadt (1993).

3.6.5 Method of Two-Scale Expansions (Cole–Kevorkian Scheme)

► **Method of two-scale expansions for a specific example (Van der Pol equation).**

We illustrate the characteristic features of the method of two-scale expansions with a specific example. Thereafter we outline possible generalizations and modifications of the method.

Example 3.28. Consider the *Van der Pol equation*

$$y''_{tt} + y = \varepsilon(1 - y^2)y'_t. \qquad (3.6.5.1)$$

The solution is sought in the form (see the fifth row in Table 3.2):

$$\begin{aligned} &y = y_0(\xi,\eta) + \varepsilon y_1(\xi,\eta) + \varepsilon^2 y_2(\xi,\eta) + \cdots, \\ &\xi = \varepsilon t, \quad \eta = \big(1 + \varepsilon^2\omega_2 + \cdots\big)t. \end{aligned} \qquad (3.6.5.2)$$

On substituting (3.6.5.2) into (3.6.5.1) and on matching the coefficients of like powers of ε, we obtain the following system for two leading terms:

$$\frac{\partial^2 y_0}{\partial \eta^2} + y_0 = 0, \qquad (3.6.5.3)$$

$$\frac{\partial^2 y_1}{\partial \eta^2} + y_1 = -2\frac{\partial^2 y_0}{\partial\xi\partial\eta} + (1 - y_0^2)\frac{\partial y_0}{\partial \eta}. \qquad (3.6.5.4)$$

The general solution of equation (3.6.5.3) is given by

$$y_0 = A(\xi)\cos\eta + B(\xi)\sin\eta. \qquad (3.6.5.5)$$

The dependence of A and B on the slow variable ξ is not being established at this stage.

We substitute (3.6.5.5) into the right-hand side of equation (3.6.5.4) and perform elementary manipulations to obtain

$$\begin{aligned} \frac{\partial^2 y_1}{\partial \eta^2} + y_1 = {} & \big[-2B'_\xi + \tfrac{1}{4}B(4 - A^2 - B^2)\big]\cos\eta + \big[2A'_\xi - \tfrac{1}{4}A(4 - A^2 - B^2)\big]\sin\eta \\ & + \tfrac{1}{4}(B^3 - 3A^2B)\cos 3\eta + \tfrac{1}{4}(A^3 - 3AB^2)\sin 3\eta. \end{aligned} \qquad (3.6.5.6)$$

The solution of this equation must not contain unbounded terms as $\eta \to \infty$; otherwise the necessary condition $y_1/y_0 = O(1)$ is not satisfied. Therefore the coefficients of $\cos\eta$ and $\sin\eta$ must be set equal to zero:

$$\begin{aligned} -2B'_\xi + \tfrac{1}{4}B(4 - A^2 - B^2) = 0, \\ 2A'_\xi - \tfrac{1}{4}A(4 - A^2 - B^2) = 0. \end{aligned} \qquad (3.6.5.7)$$

Equations (3.6.5.7) serve to determine $A = A(\xi)$ and $B = B(\xi)$. We multiply the first equation in (3.6.5.7) by $-B$ and the second by A and add them together to obtain

$$r'_\xi - \tfrac{1}{8}r(4 - r^2) = 0, \qquad \text{where} \quad r^2 = A^2 + B^2. \qquad (3.6.5.8)$$

The integration by separation of variables yields

$$r^2 = \frac{4r_0^2}{r_0^2 + (4 - r_0^2)e^{-\xi}}, \qquad (3.6.5.9)$$

where r_0 is the initial oscillation amplitude.

On expressing A and B in terms of the amplitude r and phase φ, we have $A = r\cos\varphi$ and $B = -r\sin\varphi$. Substituting these expressions into either of the two equations in (3.6.5.7) and using (3.6.5.8), we find that $\varphi'_\xi = 0$ or $\varphi = \varphi_0 = \text{const}$. Therefore the leading asymptotic term can be represented as

$$y_0 = r(\xi)\cos(\eta + \varphi_0),$$

where $\xi = \varepsilon t$ and $\eta = t$, and the function $r(\xi)$ is determined by (3.6.5.9).

► **General scheme of the method of two-scale expansions.**

The method of two-scale expansions can also be used for solving boundary value problems where the small parameter appears together with the highest derivative as a factor (such problems for $0 \le x \le a$ are indicated in the seventh row of Table 3.2 and in Section 3.6.6). In the case where a boundary layer arises near the point $x = 0$ (and its thickness has an order of magnitude of ε), the solution is sought in the form

$$y = y_0(\xi, \eta) + \varepsilon y_1(\xi, \eta) + \varepsilon^2 y_2(\xi, \eta) + \cdots,$$
$$\xi = x, \quad \eta = \varepsilon^{-1}\big[g_0(x) + \varepsilon g_1(x) + \varepsilon^2 g_2(x) + \cdots\big],$$

where the functions $y_k = y_k(\xi, \eta)$ and $g_k = g_k(x)$ are to be determined. The derivative with respect to x is calculated in accordance with the rule

$$\frac{d}{dx} = \frac{\partial}{\partial \xi} + \eta'_x \frac{\partial}{\partial \eta} = \frac{\partial}{\partial \xi} + \frac{1}{\varepsilon}\big(g'_0 + \varepsilon g'_1 + \varepsilon^2 g'_2 + \cdots\big)\frac{\partial}{\partial \eta}.$$

Additional conditions are imposed on the asymptotic terms in the domain under consideration; namely, $y_{k+1}/y_k = O(1)$ and $g_{k+1}/g_k = O(1)$ for $k = 0, 1, \ldots$, and $g_0(x) \to x$ as $x \to 0$.

Remark 3.21. The two-scale method is also used to solve problems that arise in mechanics and physics and are described by partial differential equations.

3.6.6 Method of Matched Asymptotic Expansions

► **Method of matched asymptotic expansions for a specific example.**

We illustrate the characteristic features of the method of matched asymptotic expansions with a specific example (the form of the expansions is specified in the seventh row of Table 3.2). Thereafter we outline possible generalizations and modifications of the method.

Example 3.29. Consider the linear boundary value problem

$$\varepsilon y''_{xx} + y'_x + f(x)y = 0, \qquad (3.6.6.1)$$
$$y(0) = a, \quad y(1) = b, \qquad (3.6.6.2)$$

where $0 < f(0) < \infty$.

At $\varepsilon = 0$ equation (3.6.6.1) degenerates; the solution of the resulting first-order equation

$$y'_x + f(x)y = 0 \qquad (3.6.6.3)$$

cannot meet the two boundary conditions (3.6.6.2) simultaneously. It can be shown that the condition at $x = 0$ has to be omitted in this case (a boundary layer arises near this point).

The leading asymptotic term of the outer expansion, $y = y_0(x) + O(\varepsilon)$, is determined by equation (3.6.6.3). The solution of (3.6.6.3) that satisfies the second boundary condition in (3.6.6.2) is given by

$$y_0(x) = b \exp\left[\int_x^1 f(\xi)\, d\xi\right]. \qquad (3.6.6.4)$$

We seek the leading term of the inner expansion, in the boundary layer adjacent to the left boundary, in the following form (see the seventh row and third column in Table 3.2):

$$\widetilde{y} = \widetilde{y}_0(z) + O(\varepsilon), \qquad z = x/\varepsilon, \qquad (3.6.6.5)$$

where z is the extended variable. Substituting (3.6.6.5) into (3.6.6.1) and extracting the coefficient of ε^{-1}, we obtain

$$\widetilde{y}_0'' + \widetilde{y}_0' = 0, \tag{3.6.6.6}$$

where the prime denotes differentiation with respect to z. The solution of equation (3.6.6.6) that satisfies the first boundary condition in (3.6.6.2) is given by

$$\widetilde{y}_0 = a - C + Ce^{-z}. \tag{3.6.6.7}$$

The constant of integration C is determined from the condition of matching the leading terms of the outer and inner expansions:

$$y_0(x \to 0) = \widetilde{y}_0(z \to \infty). \tag{3.6.6.8}$$

Substituting (3.6.6.4) and (3.6.6.7) into condition (3.6.6.8) yields

$$C = a - be^{\langle f \rangle}, \qquad \text{where} \quad \langle f \rangle = \int_0^1 f(x)\,dx. \tag{3.6.6.9}$$

Taking into account relations (3.6.6.4), (3.6.6.5), (3.6.6.7), and (3.6.6.9), we represent the approximate solution in the form

$$y = \begin{cases} be^{\langle f \rangle} + \bigl(a - be^{\langle f \rangle}\bigr)e^{-x/\varepsilon} & \text{for } \ 0 \le x \le O(\varepsilon), \\ b\exp\Bigl[\displaystyle\int_x^1 f(\xi)\,d\xi\Bigr] & \text{for } \ O(\varepsilon) \le x \le 1. \end{cases} \tag{3.6.6.10}$$

It is apparent that inside the thin boundary layer, whose thickness is proportional to ε, the solution rapidly changes by a finite value, $\Delta = be^{\langle f \rangle} - a$.

To determine the function y on the entire interval $x \in [0,1]$ using formula (3.6.6.10), one has to "switch" at some intermediate point $x = x_0$ from one part of the solution to the other. Such switching is not convenient and, in practice, one often resorts to a *composite solution* instead of using the double formula (3.6.6.10). In similar cases, a composite solution is defined as

$$y = y_0(x) + \widetilde{y}_0(z) - A, \qquad A = \lim_{x \to 0} y_0(x) = \lim_{z \to \infty} \widetilde{y}_0(z).$$

In the problem under consideration, we have $A = be^{\langle f \rangle}$ and hence the composite solution becomes

$$y = \bigl(a - be^{\langle f \rangle}\bigr)e^{-x/\varepsilon} + b\exp\Bigl[\int_x^1 f(\xi)\,d\xi\Bigr].$$

For $\varepsilon \ll x \le 1$, this solution transforms to the outer solution $y_0(x)$ and for $0 \le x \ll \varepsilon$, to the inner solution, thus providing an approximate representation of the unknown over the entire domain.

► General scheme of the method of matched asymptotic expansions. Some remarks.

We now consider an equation of the general form

$$\varepsilon y''_{xx} = F(x, y, y'_x) \tag{3.6.6.11}$$

subject to boundary conditions (3.6.6.2).

For the leading term of the outer expansion $y = y_0(x) + \cdots$, we have the equation

$$F(x, y_0, y_0') = 0.$$

In the general case, when using the method of matched asymptotic expansions, the position of the boundary layer and the form of the inner (extended) variable have to be determined in the course of the solution of the problem.

First we assume that the boundary layer is located near the left boundary. In (3.6.6.11), we make a change of variable $z = x/\delta(\varepsilon)$ and rewrite the equation as

$$y''_{zz} = \frac{\delta^2}{\varepsilon} F\Big(\delta z, y, \frac{1}{\delta} y'_z\Big). \tag{3.6.6.12}$$

The function $\delta = \delta(\varepsilon)$ is selected so that the right-hand side of equation (3.6.6.12) has a nonzero limit value as $\varepsilon \to 0$, provided that z, y, and y'_z are of the order of 1.

Example 3.30. For $F(x, y, y'_x) = -kx^\lambda y'_x + y$, where $0 \le \lambda < 1$, the substitution $z = x/\delta(\varepsilon)$ brings equation (3.6.6.11) to

$$y''_{zz} = -\frac{\delta^{1+\lambda}}{\varepsilon} kz^\lambda y'_z + \frac{\delta^2}{\varepsilon} y.$$

In order that the right-hand side of this equation has a nonzero limit value as $\varepsilon \to 0$, one has to set $\delta^{1+\lambda}/\varepsilon = 1$ or $\delta^{1+\lambda}/\varepsilon = \text{const}$, where const is any positive number. It follows that $\delta = \varepsilon^{\frac{1}{1+\lambda}}$.

The leading asymptotic term of the inner expansion in the boundary layer, $y = \widetilde{y}_0(z) + \cdots$, is determined by the equation $\widetilde{y}''_0 + kz^\lambda \widetilde{y}'_0 = 0$, where the prime denotes differentiation with respect to z.

If the position of the boundary layer is selected incorrectly, the outer and inner expansions cannot be matched. In this situation, one should consider the case where an arbitrary boundary layer is located on the right (this case is reduced to the previous one with the change of variable $x = 1 - z$). In Example 3.30 above, the boundary layer is on the left if $k > 0$ and on the right if $k < 0$.

There is a procedure for matching subsequent asymptotic terms of the expansion (see the seventh row and last column in Table 3.2). In its general form, this procedure can be represented as

inner expansion of the outer expansion (y*-expansion for* $x \to 0$)

$\quad = $ *outer expansion of the inner expansion* ($\widetilde{y}$*-expansion for* $z \to \infty$).

Remark 3.22. The method of matched asymptotic expansions can also be applied to construct periodic solutions of singularly perturbed equations (e.g., in the problem of relaxation oscillations of the Van der Pol oscillator).

Remark 3.23. Two boundary layers can arise in some problems (e.g., in cases where the right-hand side of equation (3.6.6.11) does not explicitly depend on y'_x).

Remark 3.24. The method of matched asymptotic expansions is also used for solving equations (in semi-infinite domains) that do not degenerate at $\varepsilon = 0$. In such cases, there are no boundary layers; the original variable is used in the inner domain, and an extended coordinate is introduced in the outer domain.

Remark 3.25. The method of matched asymptotic expansions is successfully applied for the solution of various problems in mathematical physics that are described by partial differential equations; in particular, it plays an important role in the theory of heat and mass transfer and in hydrodynamics.

⊙ *Literature for Section 3.6*: M. Van Dyke (1964), G. D. Cole (1968), G. E. O. Giacaglia (1972), A. H. Nayfeh (1973, 1981), N. N. Bogolyubov and Yu. A. Mitropolskii (1974), J. Kevorkian and J. D. Cole (1981, 1996), P. A. Lagerstrom (1988), V. Ph. Zhuravlev and D. M. Klimov (1988), J. A. Murdock (1991), V. I. Arnold, V. V. Kozlov, and A. I. Neishtadt (1993), V. F. Zaitsev and A. D. Polyanin (2001), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

3.7 Galerkin Method and Its Modifications (Projection Methods)

3.7.1 Approximate Solution for a Boundary Value Problem

► Approximate solution is linear with respect to unknown coefficients.

Consider a boundary value problem for the equation

$$\mathfrak{F}[y] - f(x) = 0 \tag{3.7.1.1}$$

with linear homogeneous boundary conditions* at the points $x = x_1$ and $x = x_2$ ($x_1 \le x \le x_2$). Here, $\mathfrak{F}$ is a linear or nonlinear differential operator of the second order (or a higher order operator); $y = y(x)$ is the unknown function and $f = f(x)$ is a given function. It is assumed that $\mathfrak{F}[0] = 0$.

Let us choose a sequence of linearly independent functions (called *basis functions*)

$$\varphi = \varphi_n(x) \qquad (n = 1,\, 2,\, \dots,\, N) \tag{3.7.1.2}$$

satisfying the same boundary conditions as $y = y(x)$. According to all methods that will be considered below, an approximate solution of equation (3.7.1.1) is sought as a linear combination

$$y_N = \sum_{n=1}^{N} A_n \varphi_n(x), \tag{3.7.1.3}$$

with the unknown coefficients A_n to be found in the process of solving the problem.

The finite sum (3.7.1.3) is called an *approximation function*. The remainder term R_N obtained after the finite sum has been substituted into the left-hand side of equation (3.7.1.1),

$$R_N = \mathfrak{F}[y_N] - f(x). \tag{3.7.1.4}$$

If the remainder R_N is identically equal to zero, then the function y_N is the exact solution of equation (3.7.1.1). In general, $R_N \not\equiv 0$.

► General form of an approximate solution.

Instead of the approximation function (3.7.1.3), which is linear in the unknown coefficients A_n, one can look for a more general form of the approximate solution:

$$y_N = \Phi(x, A_1, \dots, A_N), \tag{3.7.1.5}$$

where $\Phi(x, A_1, \dots, A_N)$ is a given function (based on experimental data or theoretical considerations suggested by specific features of the problem) satisfying the boundary conditions for any values of the coefficients $A_1, \dots, A_N$.

*For second-order ODEs, nonhomogeneous boundary conditions can be reduced to homogeneous ones by the change of variable $z = A_2x^2 + A_1x + A_0 + y$ (the constants A_2, A_1, and A_0 are selected using the method of undetermined coefficients).

3.7.2 Galerkin Method. General Scheme

In order to find the coefficients A_n in (3.7.1.3), consider another sequence of linearly independent functions

$$\psi = \psi_k(x) \qquad (k = 1,\, 2,\, \dots,\, N). \tag{3.7.2.1}$$

Let us multiply both sides of (3.7.1.4) by ψ_k and integrate the resulting relation over the region $V = \{x_1 \le x \le x_2\}$, in which we seek the solution of equation (3.7.1.1). Next, we equate the corresponding integrals to zero (for the exact solutions, these integrals are equal to zero). Thus, we obtain the following system of linear algebraic equations for the unknown coefficients A_n:

$$\int_{x_1}^{x_2} \psi_k R_N \, dx = 0 \qquad (k = 1,\, 2,\, \dots,\, N). \tag{3.7.2.2}$$

Relations (3.7.2.2) mean that the approximation function (3.7.1.3) satisfies equation (3.7.1.1) "on the average" (i.e., in the integral sense) with weights ψ_k. Introducing the scalar product $\langle g, h\rangle = \int_{x_1}^{x_2} gh\, dx$ of arbitrary functions g and h, we can consider equations (3.7.2.2) as the condition of orthogonality of the remainder R_N to all weight functions ψ_k.

The Galerkin method can be applied not only to boundary value problems, but also to eigenvalue problems (in the latter case, one takes $f = \lambda y$ and seeks eigenfunctions y_n, together with eigenvalues λ_n).

Mathematical justification of the Galerkin method for specific boundary value problems can be found in the literature listed at the end of Section 3.7. Below we describe some other methods that are in fact special cases of the Galerkin method.

Remark 3.26. Most often, one takes suitable sequences of polynomials or trigonometric functions as $\varphi_n(x)$ in the approximation function (3.7.1.3).

3.7.3 Bubnov–Galerkin, Moment, and Least Squares Methods

► **Bubnov–Galerkin method.**

The sequences of functions (3.7.1.2) and (3.7.2.1) in the Galerkin method can be chosen arbitrarily. In the case of equal functions,

$$\varphi_k(x) = \psi_k(x) \qquad (k = 1,\, 2,\, \dots,\, N), \tag{3.7.3.1}$$

the method is often called the *Bubnov–Galerkin method*.

► **Moment method.**

2°. The *moment method* is the Galerkin method with the weight functions (3.7.2.1) being powers of x,

$$\psi_k = x^k. \tag{3.7.3.2}$$

► **Least squares method.**

Sometimes, the functions ψ_k are expressed in terms of φ_k by the relations

$$\psi_k = \mathfrak{F}[\varphi_k] \qquad (k = 1,\, 2,\, \dots),$$

where $\mathfrak{F}$ is the differential operator of equation (3.7.1.1). This version of the Galerkin method is called the *least squares method*.

3.7.4 Collocation Method

In the collocation method, one chooses a sequence of points x_k, $k = 1, \dots, N$, and imposes the condition that the remainder (3.7.1.4) be zero at these points,

$$R_N = 0 \quad \text{at} \quad x = x_k \qquad (k = 1, \dots, N). \tag{3.7.4.1}$$

When solving a specific problem, the points x_k, at which the remainder R_N is set equal to zero, are regarded as most significant. The number of collocation points N is taken equal to the number of the terms of the series (3.7.1.3). This enables one to obtain a complete system of algebraic equations for the unknown coefficients A_n (for linear boundary value problems, this algebraic system is linear).

Note that the collocation method is a special case of the Galerkin method with the sequence (3.7.2.1) consisting of the Dirac delta functions:

$$\psi_k = \delta(x - x_k).$$

In the collocation method, there is no need to calculate integrals, and this essentially simplifies the procedure of solving nonlinear problems (although usually this method yields less accurate results than other modifications of the Galerkin method).

Example 3.31. Consider the boundary value problem for the linear variable-coefficient second-order ordinary differential equation

$$y''_{xx} + g(x)y - f(x) = 0 \tag{3.7.4.2}$$

subject to the boundary conditions of the first kind

$$y(-1) = y(1) = 0. \tag{3.7.4.3}$$

Assume that the coefficients of equation (3.7.4.2) are smooth even functions, so that $f(x) = f(-x)$ and $g(x) = g(-x)$. We use the collocation method for the approximate solution of problem (3.7.4.2)–(3.7.4.3).

1°. Take the polynomials

$$y_n(x) = x^{2n-2}(1 - x^2), \qquad n = 1,\, 2,\, \dots\, N,$$

as the basis functions; they satisfy the boundary conditions (3.7.4.3), $y_n(\pm 1) = 0$.

Let us consider three collocation points

$$x_1 = -\sigma, \quad x_2 = 0, \quad x_3 = \sigma \qquad (0 < \sigma < 1) \tag{3.7.4.4}$$

and confine ourselves to two basis functions ($N = 2$), so that the approximation function is taken in the form

$$y(x) = A_1(1 - x^2) + A_2 x^2 (1 - x^2). \tag{3.7.4.5}$$

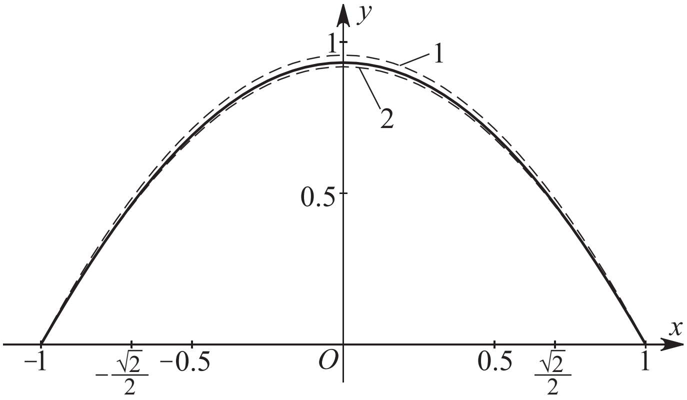

Figure 3.4: Comparison of the numerical solution of problem (3.7.4.2), (3.7.4.3), (3.7.4.7) with the approximate analytical solution (3.7.4.5), (3.7.4.8) obtained with the collocation method.

Substituting (3.7.4.5) in the left-hand side of equation (3.7.4.2) yields the remainder

$$R(x) = A_1\bigl[-2 + (1 - x^2)g(x)\bigr] + A_2\bigl[2 - 12x^2 + x^2(1 - x^2)g(x)\bigr] - f(x).$$

It must vanish at the collocation points (3.7.4.4). Taking into account the properties $f(\sigma) = f(-\sigma)$ and $g(\sigma) = g(-\sigma)$, we obtain two linear algebraic equations for the coefficients A_1 and A_2:

$$\begin{aligned} A_1\bigl[-2 + g(0)\bigr] + 2A_2 - f(0) &= 0 \qquad (\text{at } x = 0), \\ A_1\bigl[-2 + (1 - \sigma^2)g(\sigma)\bigr] + A_2\bigl[2 - 12\sigma^2 + \sigma^2(1 - \sigma^2)g(\sigma)\bigr] - f(\sigma) &= 0 \qquad (\text{at } x = \pm\sigma). \end{aligned} \tag{3.7.4.6}$$

2°. To be specific, let us take the following functions entering equation (3.7.4.2):

$$f(x) = -1, \quad g(x) = 1 + x^2. \tag{3.7.4.7}$$

On solving the corresponding system of algebraic equations (3.7.4.6), we find the coefficients

$$A_1 = \frac{\sigma^4 + 11}{\sigma^4 + 2\sigma^2 + 11}, \qquad A_2 = -\frac{\sigma^2}{\sigma^4 + 2\sigma^2 + 11}. \tag{3.7.4.8}$$

In Fig. 3.4, the solid line depicts the numerical solution to problem (3.7.4.2)–(3.7.4.3), with the functions (3.7.4.7), obtained by the shooting method (see Section 3.8.5). The dashed lines 1 and 2 show the approximate solutions obtained by the collocation method using the formulas (3.7.4.5), (3.7.4.8) with $\sigma = \frac{1}{2}$ (equidistant points) and $\sigma = \frac{\sqrt{2}}{2}$ (Chebyshev points, see Section 4.5), respectively. It is evident that both cases provide good coincidence of the approximate and numerical solutions; the use of Chebyshev points gives a more accurate result.

Remark 3.27. The theorem of convergence of the collocation method for linear boundary value problems is given in Section 4.5, where nth-order differential equations are considered.

3.7.5 Method of Partitioning the Domain

The domain $V = \{x_1 \le x \le x_2\}$ is split into N subdomains: $V_k = \{x_{k1} \le x \le x_{k2}\}$, $k = 1, \ldots, N$. In this method, the weight functions are chosen as follows:

$$\psi_k(x) = \begin{cases} 1 & \text{for } x \in V_k, \\ 0 & \text{for } x \notin V_k. \end{cases}$$

The subdomains V_k are chosen according to the specific properties of the problem under consideration and can generally be arbitrary (the union of all subdomains V_k may differ from the domain V, and some V_k and V_m may overlap).

3.7.6 Least Squared Error Method

Sometimes, in order to find the coefficients A_n of the approximation function (3.7.1.3), one uses the least squared error method based on the minimization of the functional:

$$\Phi = \int_{x_1}^{x_2} R_N^2 \, dx \to \min . \qquad (3.7.6.1)$$

For given functions φ_n in (3.7.1.3), the integral Φ is a function with respect to the coefficients A_n. The corresponding necessary conditions of minimum in (3.7.6.1) have the form

$$\frac{\partial \Phi}{\partial A_n} = 0 \qquad (n = 1, \dots, N).$$

This is a system of algebraic (transcendental) equations for the coefficients A_n.

⊙ *Literature for Section 3.7*: L. V. Kantorovich and V. I. Krylov (1962), M. A. Krasnosel’skii, G. M. Vainikko, et al. (1969), B. A. Finlayson (1972), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

3.8 Iteration and Numerical Methods

3.8.1 Method of Successive Approximations (Cauchy Problem)

The method of successive approximations is implemented in two steps. First, the Cauchy problem

$$y''_{xx} = f(x, y, y'_x) \qquad \text{(equation)}, \qquad (3.8.1.1)$$
$$y(x_0) = y_0, \quad y'_x(x_0) = y'_0 \qquad \text{(initial conditions)} \qquad (3.8.1.2)$$

is reduced to an equivalent system of integral equations by the introduction of the new variable $u(x) = y'_x$. These integral equations have the form

$$u(x) = y'_0 + \int_{x_0}^{x} f\big(t, y(t), u(t)\big)\, dt, \qquad y(x) = y_0 + \int_{x_0}^{x} u(t)\, dt. \qquad (3.8.1.3)$$

Then the solution of system (3.8.1.3) is sought by means of successive approximations defined by the following recurrence formulas:

$$u_{n+1}(x) = y'_0 + \int_{x_0}^{x} f\big(t, y_n(t), u_n(t)\big)\, dt, \quad y_{n+1}(x) = y_0 + \int_{x_0}^{x} u_n(t)\, dt; \qquad n = 0,\ 1,\ 2,\ \dots$$

As the initial approximation, one can take $y_0(x) = y_0$ and $u_0(x) = y'_0$.

Remark 3.28. If the right-hand side of equation (3.8.1.1) is independent of the derivative, i.e., $f(x, y, y'_x) = f(x, y)$, the equation can simply be differentiated twice taking into account the initial conditions without reducing it to system (3.8.1.3). In doing so, we arrive at the integral equation

$$y = y_0 + y'_0 x + \int_0^x (x - t) f\big(t, y(t)\big)\, dt.$$

The solution of problem (3.8.1.1)–(3.8.1.1) (3.8.1.3) is sought using successive approximations defined by the recurrence formulas

$$y_{n+1}(x) = y_0 + y_0' x + \int_0^x (x-t) f\big(t, y_n(t)\big)\, dt; \qquad n = 0,\, 1,\, 2,\, \dots$$

As the initial approximation, one can take $y_0(x) = y_0 + y_0' x$.

3.8.2 Runge–Kutta Method (Cauchy Problem)

For the numerical integration of the Cauchy problem (3.8.1.1)–(3.8.1.2), one often uses the Runge–Kutta method of the fourth-order approximation.

Let the mesh increment h be sufficiently small. We introduce the following notation:

$$x_k = x_0 + kh, \quad y_k = y(x_k), \quad y_k' = y_x'(x_k), \quad f_k = f(x_k, y_k, y_k'); \qquad k = 0,\, 1,\, 2,\, \dots$$

The desired values y_k and y_k' are successively found by the formulas

$$\begin{aligned} y_{k+1} &= y_k + h y_k' + \tfrac{1}{6} h^2 (\varphi_1 + \varphi_2 + \varphi_3), \\ y_{k+1}' &= y_k' + \tfrac{1}{6} h (\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4), \end{aligned}$$

where

$$\begin{aligned} \varphi_1 &= f\big(x_k,\, y_k,\, y_k'\big), \\ \varphi_2 &= f\big(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h y_k',\, y_k' + \tfrac{1}{2}h\varphi_1\big), \\ \varphi_3 &= f\big(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h y_k' + \tfrac{1}{4}h^2\varphi_1,\, y_k' + \tfrac{1}{2}h\varphi_2\big), \\ \varphi_4 &= f\big(x_k + h,\, y_k + h y_k' + \tfrac{1}{2}h^2\varphi_2,\, y_k' + h\varphi_3\big). \end{aligned}$$

In practice, the step Δx is determined in the same way as for first-order equations (see Remark 1.32 in Section 1.13.1).

3.8.3 Reduction to a System of Equations (Cauchy Problem)

The Cauchy problem (3.8.1.1)–(3.8.1.2) for a single second-order equation can be reduced with the new variable $z = y_x'$ to the Cauchy problem for a system of two first-order equations:

$$\begin{aligned} &y_x' = z, \quad z_x' = f(x, y, z) \quad &&\text{(equations)}, \\ &y(x_0) = y_0, \quad z(x_0) = y_0' \quad &&\text{(initial conditions)}. \end{aligned}$$

This problem can be numerically integrated using the methods described in Section 7.4.

3.8.4 Predictor–Corrector Methods (Cauchy Problem)

► **Second-order equation of the general form.**

We look at equation (3.8.1.1).

(i) *Predictor step.* With the values at x_{k-3}, x_{k-2}, x_{k-1}, and x_k, one uses the formula

$$\widetilde{y}_{k+1}' = y_{k-3}' + \tfrac{4}{3} h (2f_k - f_{k-1} + 2f_{k-2})$$

to compute an initial guess value of the derivative at x_{k+1}.

(ii) *Corrector step.* One improves the initial guess by computing the value of y and its derivative at x_{k+1} using the formulas

$$y_{k+1} = y_{k-1} + \tfrac{1}{3}h(\widetilde{y}'_{k+1} + 4y'_k + y'_{k-1}),$$
$$y'_{k+1} = y'_{k-1} + \tfrac{1}{3}h(\widetilde{f}_{k+1} + 4f_k + f_{k-1}),$$

where $\widetilde{f}_{k+1} = f(x_{k+1}, y_{k+1}, \widetilde{y}'_{k+1})$.

► **Second-order equation of a special form.**

If the right-hand side of equation (3.8.1.1) is independent of the derivative, i.e., $f = f(x, y)$, one can use the predictor formula

$$\widetilde{y}_{k+1} = 2y_{k-1} - y_{k-3} + \tfrac{4}{3}h^2(f_k + f_{k-1} + f_{k-2})$$

and then use *Stoermer's rule* as the corrector:

$$y_{k+1} = 2y_k - y_{k-1} + \tfrac{1}{12}h^2(\widetilde{f}_{k+1} + 10f_k + f_{k-1}).$$

3.8.5 Shooting Method (Boundary Value Problems)

The key idea of the shooting method is to reduce the solution of the original boundary value problem for a given differential equation to multiple solutions of auxiliary Cauchy problems for the same differential of equation.

► **Boundary problems with first, second, third and mixed boundary conditions.**

1°. Suppose that one deals with a boundary value problem, in the domain $x_1 \le x \le x_2$, for equation (3.8.1.1) subject to the simple boundary conditions of the first kind

$$y(x_1) = a, \quad y(x_2) = b, \qquad (3.8.5.1)$$

where a and b are given numbers.

Let us look at an auxiliary Cauchy problem for equation (3.8.1.1) with the initial conditions

$$y(x_1) = a, \quad y'_x(x_1) = \lambda. \qquad (3.8.5.2)$$

For any λ, the solution to this Cauchy problem satisfies the first boundary condition in (3.8.5.1) at the point $x = x_1$ (the solution can be obtained by the Runge–Kutta method or any other suitable numerical method). The original problem will be solved if we find a value $\lambda = \lambda_*$ such that the solution $y = y(x, \lambda_*)$ coincides at the point $x = x_2$ with the value required by the second boundary condition in (3.8.5.1):

$$y(x_2, \lambda_*) = b.$$

First, we set an arbitrary number $\lambda = \lambda_1$ (e.g., $\lambda_1 = 0$) and solve the Cauchy problem (3.8.1.1), (3.8.5.2) numerically. The solution results in the number

$$\Delta_1 = y(x_2, \lambda_1) - b. \qquad (3.8.5.3)$$

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Then, we choose a different value $\lambda = \lambda_2$ and compute

$$\Delta_2 = y(x_2, \lambda_2) - b. \tag{3.8.5.4}$$

Suppose that λ_2 has been chosen so that Δ_1 and Δ_2 have different signs (perhaps, a few tries will be required to choose a suitable λ_2). By virtue of the continuity of the solution in λ, the desired value λ_* will lie between λ_1 and λ_2. Then, we set, for example, $\lambda_3 = \frac{1}{2}(\lambda_1 + \lambda_2)$ and solve the Cauchy problem to obtain Δ_3. Out of the two previous values λ_j ($j = 1,\ 2$), we keep the one for which Δ_j and Δ_3 have different signs. The desired λ_* will be between the λ_j and λ_3. Further, by setting $\lambda_4 = \frac{1}{2}(\lambda_j + \lambda_3)$, we find Δ_4 and so on. The process is repeated until we find λ_* with a required accuracy.

Remark 3.29. The above algorithm can be improved by using, instead of bisections, the following formulas:

$$\lambda_3 = \frac{|\Delta_2|\lambda_1 + |\Delta_1|\lambda_2}{|\Delta_2| + |\Delta_1|}, \quad \lambda_4 = \frac{|\Delta_3|\lambda_j + |\Delta_j|\lambda_3}{|\Delta_3| + |\Delta_j|}, \quad \dots$$

2°. Table 3.3 lists the initial conditions that should be used in the auxiliary Cauchy problem to numerically solve boundary value problems for the second-order equation (3.8.1.1) with different linear boundary conditions at the left endpoint. The parameter λ in the Cauchy problem is selected so as to satisfy the boundary condition at the right endpoint.

TABLE 3.3
Initial conditions in the auxiliary Cauchy problem used to solve boundary value problems by the shooting method ($x_1 \le x \le x_2$)

No	Problem	Boundary condition at the left end	Initial conditions
1	First boundary value problem	$y(x_1) = a$	$y(x_1) = a,\ y'_x(x_1) = \lambda$
2	Second boundary value problem	$y'_x(x_1) = a$	$y(x_1) = \lambda,\ y'_x(x_1) = a$
3	Third boundary value problem	$y'_x(x_1) - ky(x_1) = a$	$y(x_1) = \lambda,\ y'_x(x_1) = a + k\lambda$

Importantly, nonlinear boundary value problems can have one solution, several solutions, or no solutions at all (see Examples 3.14 and 3.17, which illustrate all these scenarios based on exact analyses of two one-parameter problems from combustion theory). Therefore, special care is required when treating nonlinear problems; after finding a suitable $\lambda = \lambda_1$, one should look for other possible allowable values in a wider range of λ. If one fails to find a suitable λ_1, one should consider the possibility that the problem may simply have no solution.

► Problems with more complex linear or nonlinear boundary conditions.

In a similar way, one constructs the solution of the boundary value problem with nonlinear boundary conditions of the form

$$y'_x = \varphi(y) \quad \text{at} \quad x = x_1, \tag{3.8.5.5}$$
$$\psi(y, y'_x) = 0 \quad \text{at} \quad x = x_2. \tag{3.8.5.6}$$

The first boundary condition is a generalization of a linear nonhomogeneous boundary condition of the third kind. This condition can arise, for example, in mass transfer problems with a heterogeneous reaction, where $g(y)$ defines the rate of the chemical reaction. The second boundary condition is quite general.

Consider an auxiliary Cauchy problem for equation (3.8.1.1) with the initial conditions

$$y(x_1) = \lambda, \qquad y'_x(x_1) = \varphi(\lambda). \tag{3.8.5.7}$$

For any λ, the solution to this Cauchy problem will satisfy the first boundary condition (3.8.5.5).

We set an arbitrary value $\lambda = \lambda_1$ and solve the Cauchy problem (3.8.1.1), (3.8.5.5) numerically to obtain the number

$$\Delta_1 = \psi(y, y'_x)|_{\lambda=\lambda_1,\, x=x_2}. \tag{3.8.5.8}$$

Then we set a different value $\lambda = \lambda_2$ and compute

$$\Delta_2 = \psi(y, y'_x)|_{\lambda=\lambda_2,\, x=x_2}. \tag{3.8.5.9}$$

We assume that λ_2 is chosen so that Δ_1 and Δ_2 have different signs. The desired value $\lambda = \lambda_*$, for which the boundary condition (3.8.5.6) is satisfied exactly, will lie between λ_1 and λ_2. The subsequent procedure of numerical solution coincides with that outlined above for equation (3.8.1.1) with the simple linear boundary conditions of the first kind (3.8.5.1).

Remark 3.30. In a similar way, one can solve the boundary value problem described by equation (3.8.1.1), boundary condition (3.8.5.5), and the nonlocal linear condition

$$\int_{x_1}^{x_2} h(x)y(x)\,dx = c, \tag{3.8.5.10}$$

where $h(x)$ is a given function and c is a given number. To this end, one solves the Cauchy problem (3.8.1.1), (3.8.5.5) numerically with two different values $\lambda = \lambda_1$ and $\lambda = \lambda_2$ such that

$$\Delta_1 = \int_{x_1}^{x_2} h(x)y(x,\lambda_1)\,dx - c \quad \text{and} \quad \Delta_2 = \int_{x_1}^{x_2} h(x)y(x,\lambda_2)\,dx - c$$

have different signs. The subsequent procedure of numerical solution completely coincides with the one outlined above for equation (3.8.1.1) with the boundary conditions of the first kind (3.8.5.1).

Remark 3.31. One should bear in mind that the boundary value problem (3.8.1.1), (3.8.5.5), (3.8.5.6) can have two or more solutions, corresponding to different values λ_{*i}.

In a similar way, one can solve the boundary value problem described by equation (3.8.1.1), boundary condition (3.8.5.5), and the general nonlocal nonlinear condition

$$\int_{x_1}^{x_2} \Phi(x, y(x))\,dx = c,$$

where $\Phi(x, y)$ is a given function. In particular, this condition with the quadratic function $\Phi(x, y) = y^2$, independent of x, represents a normalization condition (which arises, for example, in quantum mechanics).

3.8.6 Numerical Methods for Problems with Equations Defined Implicitly or Parametrically

► Numerical solution of the Cauchy problem for parametrically defined equations.

In this paragraph, we outline the ideas of two numerical methods for solving the Cauchy problem for the second-order equation represented in parametric form using two relations (see Section 3.2.8)

$$y'_x = F(x,y,t), \qquad y''_{xx} = G(x,y,t) \tag{3.8.6.1}$$

with the initial conditions (3.8.1.2).

First method. We start directly from equations (3.8.6.1). Consider two auxiliary Cauchy problems

$$y'_x = F(x,y,t), \qquad y(x_0) = y_0 \qquad \text{(first problem);} \tag{3.8.6.2}$$

$$y''_{xx} = G(x,y,t), \qquad y(x_0) = y_0, \;\; y'_x(x_0) = y'_0 \qquad \text{(second problem).} \tag{3.8.6.3}$$

Let $y_F = y_F(x,t)$ and $y_G = y_G(x,t)$ be their respective solutions. Introduce the difference of the two solutions

$$\Delta(x,t) = y_G(x,t) - y_F(x,t). \tag{3.8.6.4}$$

Now we fix a value of the parameter, $t = t_k$, and find numerical solutions $y_F(x,t_k)$ and $y_G(x,t_k)$ using, for example, the Runge–Kutta method. Further, by varying x, we find an x_k at which the right-hand side of equation (3.8.6.4) vanishes: $\Delta(x_k,t_k) = 0$. To this x_k there corresponds the value of the desired function $y_k = y_F(x_k,t_k) = y_G(x_k,t_k)$. Thus, to each t_k there corresponds a point (x_k,y_k) in the (x,y) plane at which the curves $y_F = y_F(x,t_k)$ and $y_G = y_G(x,t_k)$ intersect. On taking another value of the parameter, $t = t_{k+1}$, we find a new point (x_{k+1},y_{k+1}). The combination of discrete points (x_k,y_k) with $k = 0,\,1,\,2,\,\ldots$ defines an approximation to the solution $y = y(x)$ of the original problem (3.8.6.1), (3.8.1.2).

The initial value $t = t_0$ is determined from the algebraic (or transcendental) equation

$$y'_0 = F(x_0,y_0,t_0), \tag{3.8.6.5}$$

where x_0, y_0, and y'_0 are the values appearing in the initial conditions (3.8.6.2)–(3.8.6.2), obtained from (3.8.1.2)).

Second method. With the method outlined in Section 3.2.8, we reduce the parametric equation (3.8.6.1) to a standard system of first-order differential equations for $x = x(t)$ and $y = y(t)$ (see equations (3.2.8.4) and (3.2.8.5)):

$$x'_t = \frac{F_t}{G - F_x - FF_y}, \qquad y'_t = \frac{FF_t}{G - F_x - FF_y}. \tag{3.8.6.6}$$

Suppose that $G - F_x - FF_y \neq 0$. Then system (3.8.6.6) subject to the initial conditions

$$x(t_0) = x_0, \qquad y(t_0) = y_0, \tag{3.8.6.7}$$

where t_0 is found from the algebraic (or transcendental) of equation (3.8.6.5), is solved numerically using, for example, the Runge–Kutta method (see Section 7.4.1 for relevant formulas). This solution will also solve the original parametric problem (3.8.6.1), (3.8.1.2).

Remark 3.32. In general, the algebraic (or transcendental) equation (3.8.6.5) can have several different roots, in which case the original problem (3.8.6.1), (3.8.1.2) will have the same number of different solutions.

► **First boundary value problem. Numerical solution procedure.**

Let us look at the first boundary value problem for the parametric second-order ODE (3.8.6.1) in the range $x_1 \le x \le x_2$ with the boundary conditions

$$y(x_1) = y_1, \quad y(x_2) = y_2. \tag{3.8.6.8}$$

Below we present the main idea of a numerical procedure to solve this kind of problem.

Consider two auxiliary Cauchy problems for the equation (see the second equation in Eq. (3.8.6.1))

$$y''_{xx} = G(x, y, t) \tag{3.8.6.9}$$

subject to the initial conditions

$$y(x_1) = y_1, \quad y'_x(x_1) = F(x_1, y_1, t) \qquad \text{(problem 1);} \tag{3.8.6.10}$$
$$y(x_2) = y_2, \quad y'_x(x_2) = F(x_2, y_2, t) \qquad \text{(problem 2).} \tag{3.8.6.11}$$

By choosing a specific value of the parameter, $t = t_k$, we solve the auxiliary Cauchy problems numerically (e.g., by the Runge–Kutta method) to obtain $y^1 = y^1(x, t_k)$ and $y^2 = y^2(x, t_k)$, respectively (the superscripts indicate the problem number). To any t_k there corresponds a point (x_k, y_k) in the (x, y) plane at which the curves corresponding to the solutions $y^1 = y^1(x, t_k)$ and $y^2 = y^2(x, t_k)$ intersect. By choosing a different value, t_{k+1}, we find another point, (x_{k+1}, y_{k+1}). The discrete set of points (x_k, y_k) with $k = 0,\ 1,\ 2,\ \dots$ defines an approximation to the solution $y = y(x)$ of the original boundary value problem (3.8.6.1), (3.8.6.8).

► **Numerical integration of equations defined implicitly.**

Let us look at the Cauchy problem for the implicit equation

$$y'_x = F(x, y, y''_{xx}) \tag{3.8.6.12}$$

subject to the initial condition (3.8.1.2).

The substitution $y''_{xx} = t$ reduces equation (3.8.6.12) to the parametric equation

$$y'_x = F(x, y, t), \qquad y''_{xx} = t \tag{3.8.6.13}$$

with the initial conditions (3.8.1.2).

Problem (3.8.6.13), (3.8.1.2) is a special case of problem (3.8.6.1), (3.8.1.2) in which $G(x, y, t) = t$, and hence, it can be solved with the numerical methods described previously.

► **Differential-algebraic equations.**

Parametrically defined nonlinear differential equations of the form (3.8.6.1) are a special class of coupled (DAEs for short). Numerical methods for DAEs other than those discussed above can be found in the books by Hairer, Lubich, and Roche (1989), Schiesser (1994), Hairer and Wanner (1996), Brenan, Campbell, and Petzold (1996), Ascher and Petzold (1998), and Rabier and Rheinboldt (2002).

3.8.7 Numerical Solution Blow-Up Problems*

► **Preliminary remarks. Blow-up solutions with a power-law singularity.**

Below, we will be concerned with *blow-up problems*, whose solution tends to infinity as the independent variable approaches a finite value $x = x_*$, which is unknown in advance. The important question arises as to how one can determine the singular point x_* with numerical methods.

Example 3.32. Consider the model Cauchy problem for the nonlinear second-order ODE

$$y''_{xx} = 2y^3 \quad (x > 0), \qquad y(0) = 1, \quad y'_x(0) = 1. \tag{3.8.7.1}$$

Its exact solution is given by

$$y = \frac{1}{1-x} \tag{3.8.7.2}$$

and has a power-law singularity (a pole) at $x_* = 1$. For $x > x_*$, there is no solution.

If one solves problem (3.8.7.1) using, for example, explicit Runge–Kutta methods of different order of accuracy, one obtains a numerical solution which is positive, monotonically increases, and exists for arbitrarily large x_k. From the form of the solution, one cannot conclude that the exact solution has a pole (it appears that the exact solution rapidly increases and exists for any $x > 0$). Note that the standard explicit schemes do not work well either is similar situations.

Below we outline a few numerical methods for blow-up problems. We assume that the preliminary numerical (or analytical) analysis has caused a suspicion that the problem may have a blow-up solution.

► **Method based on the hodograph transformation.**

For monotonic blow-up solutions, having made the hodograph transformation, we can solve the Cauchy problem for $x = x(y)$ rather than $y = y(x)$. Since $y_x = 1/x'_y$ and $y''_{xx} = -x''_{yy}/(x'_y)^3$, problem (3.8.1.1)–(3.8.1.2) becomes

$$\begin{gathered} x''_{yy} = -(x'_y)^3 f(x, y, 1/x'_y) \quad (y > y_0), \\ x(y_0) = x_0, \quad x'_y(y_0) = 1/y'_0. \end{gathered} \tag{3.8.7.3}$$

The computation can be carried out using, for example, the explicit fourth-order Runge–Kutta scheme. For sufficiently large y, we find the asymptote $x = x_*$ numerically.

Example 3.33. The hodograph transformation reduces the model problem (3.8.7.1) to

$$x''_{yy} = -2y^3 (x'_y)^3 \quad (y > 1); \quad x(1) = 0, \quad x'_y(1) = 1.$$

The solution of this problem is given by

$$x = 1 - \frac{1}{y};$$

it does not have singularities and monotonically increases for $y > 1$ and tends to the desired limit value $x_* = \lim_{y\to\infty} x(y) = 1$.

*Prior to reading this section, the reader should refer to Section 1.14.4, which discusses blow-up problems for first-order equations.

► Method based on the use of the differential variable $t = y'_x$.

First, assuming the inequalities $f(x, y, y'_x) > 0$ for $y > y_0 > 0$ and $y'_x > y'_0 > 0$ to hold, we rewrite problem (3.8.1.1)–(3.8.1.2) in the parametric form

$$y'_x = t \quad y''_{xx} = f(x, y, t) \quad (t > t_0); \tag{3.8.7.4}$$

$$x(t_0) = x_0, \quad y(t_0) = y_0, \quad t_0 = y'_0. \tag{3.8.7.5}$$

Then, relying on the results of Section 3.8.6, we change to system (3.8.6.6) with $F = t$ and $G = f(x, y, t)$ to arrive at the Cauchy problem for a system of two first-order equations

$$x'_t = \frac{1}{f(x, y, t)}, \quad y'_t = \frac{t}{f(x, y, t)} \quad (t > t_0) \tag{3.8.7.6}$$

subject to the initial conditions (3.8.7.5) . Further, we solve problem (3.8.7.6), (3.8.7.5) numerically using, for example, the Runge–Kutta methods (see Section 7.4.1 for relevant formulas). The resulting solution is also a solution to the original problem (3.8.1.1)–(3.8.1.2) in parametric form. The boundary of the existence domain, $x = x_*$, is determined numerically for sufficiently large t.

Example 3.34. In the model problem (3.8.7.1), the introduction of the auxiliary variable $t = y'_x$ followed by the substitution of $f(x, y, t) = 2y^3$ into (3.8.7.4)–(3.8.7.6) results in the Cauchy problem for a system of two equations

$$x'_t = \frac{1}{2y^3}, \quad y'_t = \frac{t}{2y^3} \quad (t > 1);$$
$$x(1) = 0, \quad y(1) = 1 \quad (t_0 = 1).$$

The exact solution to this problem is

$$x = 1 - \frac{1}{\sqrt{t}}, \quad y = \sqrt{t} \quad (t \geq 1).$$

It does not have singularities; the function $x = x(t)$ monotonically increases for $t > 1$ and tends to the desired limit value $x_* = \lim_{t\to\infty} x(t) = 1$, while $y = y(t)$ monotonically increases without bound.

► Method based on nonlocal transformations. Monotonic blow-up solutions.

First, equation (3.8.1.1) can be represented as a system of two equations

$$y'_x = t, \quad t'_x = f(x, y, t),$$

and then we introduce a nonlocal variable of general form by the formula

$$\xi = \int_{x_0}^{x} g(x, y, t)\, dx, \quad y = y(x), \quad t = t(x), \tag{3.8.7.7}$$

where $g = g(x, y, t)$ is a *regularizing function* which can be varied. As a result, the Cauchy problem (3.8.1.1)–(3.8.1.2) can be transformed to the following equivalent problem for an autonomous system of three equations:

$$x'_\xi = \frac{1}{g(x, y, t)}, \quad y'_\xi = \frac{t}{g(x, y, t)}, \quad t'_\xi = \frac{f(x, y, t)}{g(x, y, t)} \quad (\xi > 0);$$
$$x(0) = x_0, \quad y(0) = y_0, \quad t(0) = y_1. \tag{3.8.7.8}$$

With a suitably chosen function $g = g(x, y, t)$ (subject to not-very-restrictive conditions), the Cauchy problem (3.8.7.8) can be numerically integrated using standard numerical methods, without the fear of getting blow-up solutions.

Here are a few possible ways of how the regularizing function g in system (3.8.7.8) can be chosen.

1°. The special case $g = t$ is equivalent to the hodograph transformation with an additional translation in the dependent variable, which gives $\xi = y - y_0$.

2°. We can take $g = \bigl(c + |t|^s + |f|^s\bigr)^{1/s}$ with $c \ge 0$ and $s > 0$. The case $c = 1$ and $s = 2$ corresponds to the method of arc length transformation.

3°. By taking $g = f$ in (3.8.7.8), after the integration of the third equation, we arrive at system (3.8.7.6). It follows that the method based on the nonlocal transformation (3.8.7.7) is a generalization of the method based on the differential variable.

4°. Also, we can take $g = f/y$, $g = f/t$, or $g = t/y$ (in the last two cases, system (3.8.7.8) is simplified, since one of its equations is directly integrated).

Remark 3.33. It follows from Items 1°, 2°, and 3° that the method based on the hodograph transformation, the method of arc length transformation, and the method based on the differential variable are special cases of the method based on a nonlocal transformation of general form.

Remark 3.34. One does not have to compute integrals of the form (3.8.7.7) to apply nonlocal transformations.

Example 3.35. For the test problem (3.8.7.1), in which $f = 2y^3$, we set $g = t/y$ (see Item 4° with $g = t/y$). Substituting these functions into (3.8.7.8), we arrive at the Cauchy problem

$$\begin{gathered} x'_\xi = \frac{y}{t}, \quad y'_\xi = y, \quad t'_\xi = \frac{2y^4}{t} \quad (\xi > 0); \\ x(0) = 0, \quad y(0) = a, \quad t(0) = a^2. \end{gathered} \tag{3.8.7.9}$$

The exact solution of this problem is

$$x = \frac{1}{a}\bigl(1 - e^{-\xi}\bigr), \quad y = ae^{\xi}, \quad t = a^2 e^{2\xi}.$$

One can see that the unknown $x = x(\xi)$ exponentially tends to the asymptotic value $x = x_* = 1/a$ as $\xi \to \infty$.

Figure 3.5 displays a numerical solution of the Cauchy problem (3.8.7.9) in parametric form and compares the numerical solution with the exact solution (3.8.7.2).

Remark 3.35. The method based on the use of the special case of system (3.8.7.8) with $g = t/y$ (see Item 4° with $g = t/y$ above) is more efficient as compared to the methods based on the hodograph transformation, arc length transformation, and differential variable.

► Problems with non-monotonic blow-up solutions.

For problems with non-monotonic blow-up solutions, it is reasonable to choose regularizing functions of the form

$$g = G(|t|, |f|), \tag{3.8.7.10}$$

where $f = f(x, y, t)$ is the right-hand side of equation (3.8.1.1) and $t = y'_x$. We impose the following conditions on the function $G = G(u, v)$:

$$G > 0; \quad G_u \ge 0, \quad G_v \ge 0, \quad G \to \infty \text{ as } u + v \to \infty, \tag{3.8.7.11}$$

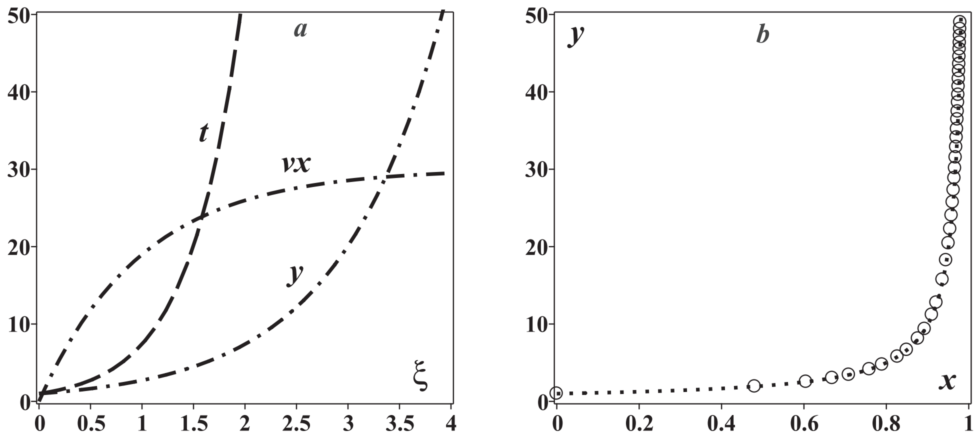

Figure 3.5: (a) numerical solution $t = t(\xi)$, $x = x(\xi)$, $y = y(\xi)$ of the Cauchy problem (3.8.7.9) with $a = 1$ ($\nu = 30$); (b) exact solution (3.8.7.2) with $a = 1$, solid dots; numerical solution of the Cauchy problem (3.8.7.9), open circles.

where $u \geq 0$, $v \geq 0$. By selecting a suitable function G, we can ensure that the Cauchy problem (3.8.7.8) has no blow-up singularity on the half-line $0 \leq \xi < \infty$; this problem can be solved by applying standard fixed-step numerical methods.

Example 3.36. Consider a three-parameter Cauchy problem for the nonlinear second-order autonomous ODE:

$$y''_{xx} - 3yy'_x - 2\lambda y'_x + y^3 + 2\lambda y^2 + (\beta^2 + \lambda^2)y = 0; \quad (3.8.7.12)$$

$$y(0) = b\beta, \qquad y'_x(0) = 2b\beta\lambda + b^2\beta^2. \quad (3.8.7.13)$$

The exact solution of the problem is

$$y = \frac{b[\lambda\sin(\beta x) + \beta\cos(\beta x)]}{e^{-\lambda x} - b\sin(\beta x)}. \quad (3.8.7.14)$$

This solution can change the sign and, for certain values of the parameters, is a non-monotonic blow-up solution.

For problem (3.8.7.12)–(3.8.7.13), we choose a regularizing function in the form $g = (1+|t|)^{1/3}$. Substituting it into (3.8.7.8), we arrive at the Cauchy problem

$$x'_\xi = \frac{1}{(1+|t|+|f|)^{1/3}}, \quad y'_\xi = \frac{t}{(1+|t|+|f|)^{1/3}}, \quad t'_\xi = \frac{f}{(1+|t|+|f|)^{1/3}}; \quad (3.8.7.15)$$
$$x(0) = 0, \quad y(0) = b\beta, \quad t(0) = 2b\beta\lambda + b^2\beta^2,$$

where $f = 3yt + 2\lambda t - y^3 - 2\lambda y^2 - (\beta^2 + \lambda^2)y$.

The numerical solutions of problem (3.8.7.15) obtained using two sets of parameters, $b = 0.9$, $\beta = 8$, $\lambda = 0.3$ and $b = 0.5$, $\beta = 5$, $\lambda = 0.1$, and the fourth-order Runge–Kutta method with the fixed step size $h = 0.01$ are shown by open circles in Fig. 3.6a and Fig. 3.6b. For this step size, the maximum difference between the exact solution (3.8.7.14) and the numerical solution of the Cauchy problem for system (3.8.7.15) at $y = 50$ was found to be 0.0002500% for the first set of parameters and 0.0011033% for the second set. The solution for the first set of parameters exists in a finite region $0 \leq x < x_* = 0.9112959$, while that for the second set of parameters displays a pronounced non-monotonic sawtooth behavior with six local maxima and exists in a finite region $0 \leq x < x_* = 7.7730738$ (see Fig. 3.6b).

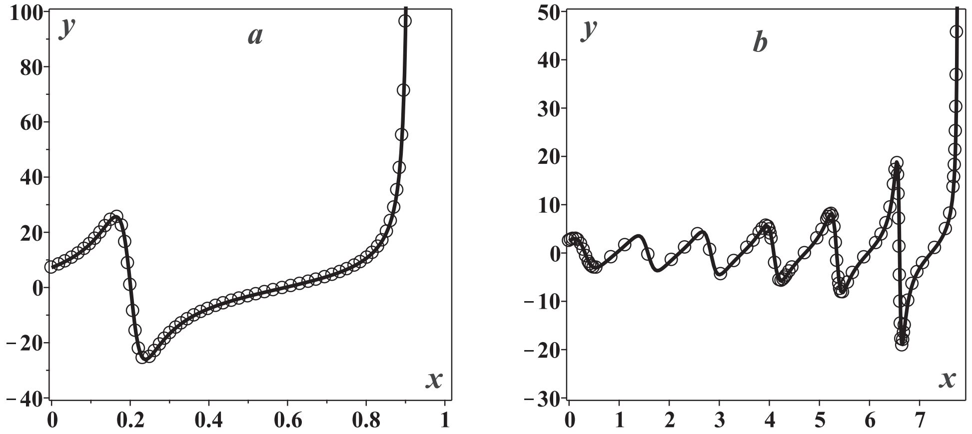

Figure 3.6: The exact solution (3.8.7.14) of the original problem (3.8.7.12)–(3.8.7.13) (solid line) and the numerical solution of the transformed problem (3.8.7.15) (open circles) for two sets of parameters: (a) $b = 0.9$, $\beta = 8$, $\lambda = 0.3$ and (b) $b = 0.5$, $\beta = 5$, $\lambda = 0.1$.

Table 3.4 compares the efficiency of various functions g used in the numerical integration of the transformed problem (3.8.7.8) in order to solve the original problem (3.8.7.12)–(3.8.7.13) with $b = 0.5$, $\beta = 5$, and $\lambda = 0.1$. The maximum allowed error was set to be 0.01% at $y = 100$. The main integration parameters (largest interval $0 \le \xi \le \xi_{\max}$, step size h, and number of grid points N) used to achieve the required accuracy are specified in the table.

TABLE 3.4
A comparison of the efficiency of various regularizing functions g in the transformed problem (3.8.7.8), used for the numerical solution of the original problem (3.8.7.12)–(3.8.7.13), with the prescribed maximum error 0.01% at $y = 100$, for $b = 0.5$, $\beta = 5$, and $\lambda = 0.1$ ($x_* = 7.7730738$)

No.	Regularizing function	$\xi_{\max}$	Step size h	N
1	$g = (1 + t^2)^{1/2}$	274.050	0.0029000000	94,500
2	$g = (1 + t^2 + f^2)^{1/2}$	11,742.300	0.1800000000	65,235
3	$g = (1 + \lvert t\rvert)^{1/2}$	35.764	0.0029593683	12,085
4	$g = \frac{1}{2}(1 + \lvert t\rvert)^{1/3} + \frac{1}{2}(1 + \lvert f\rvert)^{1/3}$	28.442	0.0090899000	3,129
5	$g = (1 + \lvert t\rvert + \lvert f\rvert)^{1/3}$	39.702	0.0185090000	2,145

⊙ *Literature for Section 3.8*: M. Abramowitz and I. A. Stegun (1964), S. K. Godunov and V. S. Ryaben'kii (1973), J. D. Lambert (1973), H. B. Keller (1976), N. S. Bakhvalov (1977), N. N. Kalitkin (1978), S. Moriguti, C. Okuno, R. Suekane, M. Iri, and K. Takeuchi (1979), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), J. C. Butcher (1987), E. Hairer, C. Lubich, and M. Roche (1989), M. Stuart and M. S. Floater (1990), W. E. Schiesser (1994), V. F. Zaitsev and A. D. Polyanin (1993), L. F. Shampine (1994), K. E. Brenan, S. L. Campbell, and L. R. Petzold (1996), J. R. Dormand (1996), E. Hairer and G. Wanner (1996), D. Zwillinger (1997), U. M. Ascher and L. R. Petzold (1998), G. A. Korn and T. M. Korn (2000), G. Acosta, G. Durán, and J. D. Rossi (2002), P. J. Rabier and W. C. Rheinboldt (2002), A. D. Polyanin and V. F. Zaitsev (2003), H. J. Lee and W. E. Schiesser (2004), A. D. Polyanin and A. V. Manzhirov (2007), S. C. Chapra and R. P. Canale (2010), M. Mizuguchi, and S. Oishi (2017), A. D. Polyanin and A. I. Zhurov (2017b), A. D. Polyanin and I. K. Shingareva (2017a,b,c,d,e).