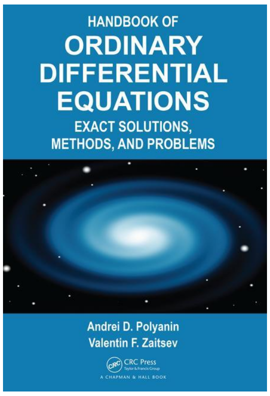

Handbook of Ordinary Differential Equations
Exact Solutions, Methods, and Problems

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Chapter 4: Methods for Linear ODEs of Arbitrary Order

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Chapter 4

Methods for Linear ODEs of Arbitrary Order

4.1 Linear Equations with Constant Coefficients

4.1.1 Homogeneous Linear Equations. General Solution

An nth-order homogeneous linear equation with constant coefficients has the general form

$$y_x^{(n)} + a_{n-1}y_x^{(n-1)} + \cdots + a_1 y_x' + a_0 y = 0. \tag{4.1.1.1}$$

The general solution of this equation is determined by the roots of the characteristic equation

$$P(\lambda) = 0, \qquad \text{where} \quad P(\lambda) = \lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0. \tag{4.1.1.2}$$

The following cases are possible:

1°. All roots λ_1, λ_2, ..., λ_n of the characteristic equation (4.1.1.2) are real and distinct. Then the general solution of the homogeneous linear differential equation (4.1.1.1) has the form

$$y = C_1\exp(\lambda_1 x) + C_2\exp(\lambda_2 x) + \cdots + C_n\exp(\lambda_n x).$$

2°. There are m equal real roots $\lambda_1 = \lambda_2 = \cdots = \lambda_m$ ($m \le n$), and the other roots are real and distinct. In this case, the general solution is given by

$$\begin{aligned} y &= \exp(\lambda_1 x)(C_1 + C_2 x + \cdots + C_m x^{m-1}) \\ &\quad + C_{m+1}\exp(\lambda_{m+1}x) + C_{m+2}\exp(\lambda_{m+2}x) + \cdots + C_n\exp(\lambda_n x). \end{aligned}$$

3°. There are m equal complex conjugate roots $\lambda = \alpha \pm i\beta$ ($2m \le n$), and the other roots are real and distinct. In this case, the general solution is

$$\begin{aligned} y &= \exp(\alpha x)\cos(\beta x)(A_1 + A_2 x + \cdots + A_m x^{m-1}) \\ &\quad + \exp(\alpha x)\sin(\beta x)(B_1 + B_2 x + \cdots + B_m x^{m-1}) \\ &\quad + C_{2m+1}\exp(\lambda_{2m+1}x) + C_{2m+2}\exp(\lambda_{2m+2}x) + \cdots + C_n\exp(\lambda_n x), \end{aligned}$$

where $A_1, \ldots, A_m, B_1, \ldots, B_m, C_{2m+1}, \ldots, C_n$ are arbitrary constants.

4°. In the general case, where there are r different roots $\lambda_1, \lambda_2, \ldots, \lambda_r$ of multiplicities $m_1, m_2, \ldots, m_r$, respectively, the left-hand side of the characteristic equation (4.1.1.2) can be represented as the product

$$P(\lambda) = (\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_r)^{m_r},$$

where $m_1 + m_2 + \cdots + m_r = n$. The general solution of the original equation is given by the formula

$$y = \sum_{k=1}^{r} \exp(\lambda_k x)(C_{k,0} + C_{k,1}x + \cdots + C_{k,m_k-1}x^{m_k-1}),$$

where $C_{k,l}$ are arbitrary constants.

If the characteristic equation (4.1.1.2) has complex conjugate roots, then in the above solution, one should extract the real part on the basis of the relation $\exp(\alpha \pm i\beta) = e^\alpha(\cos\beta \pm i\sin\beta)$.

Example 4.1. Find the general solution of the linear third-order equation

$$y''' + ay'' - y' - ay = 0.$$

Its characteristic equation is $\lambda^3 + a\lambda^2 - \lambda - a = 0$, or, in factorized form,

$$(\lambda + a)(\lambda - 1)(\lambda + 1) = 0.$$

Depending on the value of the parameter a, three cases are possible.

1. Case $a \neq \pm 1$. There are three different roots, $\lambda_1 = -a$, $\lambda_2 = -1$, and $\lambda_3 = 1$. The general solution of the differential equation is expressed as $y = C_1 e^{-ax} + C_2 e^{-x} + C_3 e^x$.

2. Case $a = 1$. There is a double root, $\lambda_1 = \lambda_2 = -1$, and a simple root, $\lambda_3 = 1$. The general solution of the differential equation has the form $y = (C_1 + C_2 x)e^{-x} + C_3 e^x$.

3. Case $a = -1$. There is a double root, $\lambda_1 = \lambda_2 = 1$, and a simple root, $\lambda_3 = -1$. The general solution of the differential equation is expressed as $y = (C_1 + C_2 x)e^x + C_3 e^{-x}$.

Example 4.2. Consider the linear fourth-order equation

$$y''''_{xxxx} - y = 0.$$

Its characteristic equation, $\lambda^4 - 1 = 0$, has four distinct roots, two real and two pure imaginary,

$$\lambda_1 = 1, \quad \lambda_2 = -1, \quad \lambda_3 = i, \quad \lambda_4 = -i.$$

Therefore the general solution of the equation in question has the form (see Item 3°)

$$y = C_1 e^x + C_2 e^{-x} + C_3 \sin x + C_4 \cos x.$$

4.1.2 Nonhomogeneous Linear Equations. General and Particular Solutions

1°. An nth-order nonhomogeneous linear equation with constant coefficients has the general form

$$y_x^{(n)} + a_{n-1}y_x^{(n-1)} + \cdots + a_1 y'_x + a_0 y = f(x). \qquad (4.1.2.1)$$

The general solution of this equation is the sum of the general solution of the corresponding homogeneous equation with $f(x) \equiv 0$ (see Section 4.1.1) and any particular solution of the nonhomogeneous equation (4.1.2.1).

If all the roots $\lambda_1,\ \lambda_2,\ \dots,\ \lambda_n$ of the characteristic equation (4.1.1.2) are different, equation (4.1.2.1) has the general solution:

$$y = \sum_{\nu=1}^{n} C_\nu e^{\lambda_\nu x} + \sum_{\nu=1}^{n} \frac{e^{\lambda_\nu x}}{P'_\lambda(\lambda_\nu)} \int f(x) e^{-\lambda_\nu x}\, dx \qquad (4.1.2.2)$$

(for complex roots, the real part should be taken).

In the general case, if the characteristic equation (4.1.1.2) has multiple roots, the solution to equation (4.1.2.1) can be constructed using formula (4.2.2.2).

2°. Table 4.1 lists the forms of particular solutions corresponding to some special forms of functions on the right-hand side of the linear nonhomogeneous equation.

TABLE 4.1

Forms of particular solutions of the constant-coefficient nonhomogeneous linear equation $y_x^{(n)} + a_{n-1} y_x^{(n-1)} + \dots + a_1 y'_x + a_0 y = f(x)$ that correspond to some special forms of the function $f(x)$

Form of the function $f(x)$	Roots of the characteristic equation $\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$	Form of a particular solution $y = \widetilde{y}(x)$
$P_m(x)$	Zero is not a root of the characteristic equation (i.e., $a_0 \neq 0$)	$\widetilde{P}_m(x)$
	Zero is a root of the characteristic equation (multiplicity r)	$x^r \widetilde{P}_m(x)$
$P_m(x)e^{\alpha x}$ (α is a real constant)	α is not a root of the characteristic equation	$\widetilde{P}_m(x)e^{\alpha x}$
	α is a root of the characteristic equation (multiplicity r)	$x^r \widetilde{P}_m(x)e^{\alpha x}$
$P_m(x)\cos\beta x + Q_n(x)\sin\beta x$	$i\beta$ is not a root of the characteristic equation	$\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x$
	$i\beta$ is a root of the characteristic equation (multiplicity r)	$x^r[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]$
$[P_m(x)\cos\beta x + Q_n(x)\sin\beta x]e^{\alpha x}$	$\alpha + i\beta$ is not a root of the characteristic equation	$[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]e^{\alpha x}$
	$\alpha + i\beta$ is a root of the characteristic equation (multiplicity r)	$x^r[\widetilde{P}_\nu(x)\cos\beta x + \widetilde{Q}_\nu(x)\sin\beta x]e^{\alpha x}$

Notation: P_m and Q_n are polynomials of degrees m and n with given coefficients; $\widetilde{P}_m$, $\widetilde{P}_\nu$, and $\widetilde{Q}_\nu$ are polynomials of degrees m and ν whose coefficients are determined by substituting the particular solution into the basic equation; $\nu = \max(m,\ n)$; and α and β are real numbers, $i^2 = -1$.

3°. Consider the Cauchy problem for equation (4.1.2.1) subject to the homogeneous initial conditions

$$y(0) = y'_x(0) = \dots = y_x^{(n-1)}(0) = 0. \qquad (4.1.2.3)$$

Let $y(x)$ be the solution of problem (4.1.2.1), (4.1.2.3) for arbitrary $f(x)$ and let $u(x)$ be the solution of the auxiliary, simpler problem (4.1.2.1), (4.1.2.3) with $f(x) \equiv 1$, so that $u(x) = y(x)|_{f(x)\equiv 1}$. Then the formula

$$y(x) = \int_0^x f(t)u'_x(x-t)\,dt$$

holds. It is called the *Duhamel integral*.

⊙ *Literature for Section 4.1*: G. M. Murphy (1960), L. E. El'sgol'ts (1961), N. M. Matveev (1967), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

4.2 Linear Equations with Variable Coefficients

4.2.1 Homogeneous Linear Equations. General Solution. Order Reduction. Liouville Formula

► **Structure of the general solution.**

The general solution of the nth-order homogeneous linear differential equation

$$f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y'_x + f_0(x)y = 0 \qquad (4.2.1.1)$$

has the form

$$y = C_1y_1(x) + C_2y_2(x) + \cdots + C_ny_n(x). \qquad (4.2.1.2)$$

Here $y_1(x)$, $y_2(x)$, …, $y_n(x)$ is a fundamental system of solutions (the y_k are linearly independent particular solutions, $y_k \not\equiv 0$); $C_1, C_2, \ldots, C_n$ are arbitrary constants.

► **Utilization of particular solutions for reducing the order of the equation.**

1°. Let $y_1 = y_1(x)$ be a nontrivial particular solution of equation (4.2.1.1). The substitution

$$y = y_1(x)\int z(x)\,dx$$

results in a linear equation of order $n-1$ for the function $z(x)$.

2°. Let $y_1 = y_1(x)$ and $y_2 = y_2(x)$ be two nontrivial linearly independent solutions of equation (4.2.1.1). The substitution

$$y = y_1\int y_2w\,dx - y_2\int y_1w\,dx$$

results in a linear equation of order $n-2$ for $w(x)$.

3°. Suppose that m linearly independent solutions $y_1(x)$, $y_2(x)$, …, $y_m(x)$ of equation (4.2.1.1) are known. Then one can reduce the order of the equation to $n-m$ by successive application of the following procedure. The substitution $y = y_m(x)\int z(x)\,dx$ leads to an equation of order $n-1$ for the function $z(x)$ with known linearly independent solutions:

$$z_1 = \left(\frac{y_1}{y_m}\right)'_x, \quad z_2 = \left(\frac{y_2}{y_m}\right)'_x, \quad \ldots, \quad z_{m-1} = \left(\frac{y_{m-1}}{y_m}\right)'_x.$$

The substitution $z = z_{m-1}(x)\int w(x)\,dx$ yields an equation of order $n-2$. Repeating this procedure m times, we arrive at a homogeneous linear equation of order $n-m$.

► **Wronskian determinant and Liouville formula.**

The *Wronskian determinant* (or simply, *Wronskian*) is the function defined as

$$W(x) = \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \cdots & \cdots & \cdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}, \tag{4.2.1.3}$$

where $y_1(x)$, ..., $y_n(x)$ is a fundamental system of solutions of the homogeneous equation (4.2.1.1); $y_k^{(m)}(x) = \dfrac{d^m y_k}{dx^m}$, $m = 1, \ldots, n-1$; $k = 1, \ldots, n$.

The following *Liouville formula* holds:

$$W(x) = W(x_0)\exp\left[-\int_{x_0}^{x} \frac{f_{n-1}(t)}{f_n(t)}\,dt\right].$$

4.2.2 Nonhomogeneous Linear Equations. General Solution. Superposition Principle

► **Construction of the general solution.**

1°. The general nonhomogeneous nth-order linear differential equation has the form

$$f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y_x' + f_0(x)y = g(x). \tag{4.2.2.1}$$

The general solution of the nonhomogeneous equation (4.2.2.1) can be represented as the sum of its particular solution and the general solution of the corresponding homogeneous equation (4.2.1.1).

2°. Let $y_1(x)$, ..., $y_n(x)$ be a fundamental system of solutions of the homogeneous equation (4.2.1.1), and let $W(x)$ be the Wronskian determinant (4.2.1.3). Then the general solution of the nonhomogeneous linear equation (4.2.2.1) can be represented as

$$y = \sum_{\nu=1}^{n} C_\nu y_\nu(x) + \sum_{\nu=1}^{n} y_\nu(x)\int \frac{W_\nu(x)\,dx}{f_n(x)W(x)}, \tag{4.2.2.2}$$

where $W_\nu(x)$ is the determinant of the matrix (4.2.1.3) in which the νth column is replaced by the column vector with the elements $0, 0, \ldots, 0, g$.

► **Superposition principle.**

The solution of a nonhomogeneous linear equation

$$\mathbf{L}[y] = \sum_{k=1}^{m} g_k(x), \qquad \mathbf{L}[y] \equiv f_n(x)y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y_x' + f_0(x)y$$

is determined by adding together the solutions,

$$y = \sum_{k=1}^{m} y_k,$$

of m (simpler) equations,

$$\mathbf{L}[y_k] = g_k(x), \qquad k = 1,\, 2,\, \ldots,\, m,$$

corresponding to respective nonhomogeneous terms in the original equation.

► **Euler equation.**

1°. The nonhomogeneous Euler equation has the form

$$x^n y_x^{(n)} + a_{n-1}x^{n-1}y_x^{(n-1)} + \cdots + a_1 x y'_x + a_0 y = f(x).$$

The substitution $x = be^t$ ($b \neq 0$) leads to a constant-coefficient linear equation of the form (4.1.2.1).

2°. Particular solutions of the homogeneous Euler equation [with $f(x) \equiv 0$] are sought in the form $y = x^k$. If all k are real and distinct, its general solution is expressed as

$$y(x) = C_1|x|^{k_1} + C_2|x|^{k_2} + \cdots + C_n|x|^{k_n}.$$

Remark 4.1. To a pair of complex conjugate values $k = \alpha \pm i\beta$ there corresponds a pair of particular solutions: $y = |x|^\alpha \sin(\beta|x|)$ and $y = |x|^\alpha \cos(\beta|x|)$.

4.2.3 Nonhomogeneous Linear Equations. Cauchy Problem. Reduction to Integral Equations

► **Cauchy problem. Cauchy formula.**

Let $y(x,\sigma)$ be the solution to the Cauchy problem for the homogeneous equation (4.2.1.1) with nonhomogeneous initial conditions at $x = \sigma$:

$$y(\sigma) = y'_x(\sigma) = \cdots = y_x^{(n-2)}(\sigma) = 0, \quad y_x^{(n-1)}(\sigma) = 1,$$

where σ is an arbitrary parameter. Then a particular solution of the nonhomogeneous linear equation (4.2.2.1) with homogeneous boundary conditions

$$y(x_0) = y'_x(x_0) = \cdots = y_x^{(n-1)}(x_0) = 0$$

is given by the *Cauchy formula*

$$\bar{y}(x) = \int_{x_0}^{x} y(x,\sigma)\frac{g(\sigma)}{f_n(\sigma)}\, d\sigma.$$

► Reduction of the Cauchy problem for ODEs to integral equations.

1°. Integral equations play an important role in the theory of ordinary differential equations. The reduction of Cauchy and boundary value problems to integral equations allows for the application of iteration and finite-difference methods of solving integral equations. These methods are, as a rule, substantially simpler than those used for solving differential equations. Moreover, many delicate proofs and qualitative results of the theory of differential equations have been obtained by the investigation of the corresponding integral equations.

2°. Consider the Cauchy problem for nth order ODE (4.2.2.1) with the homogeneous initial conditions at the point $x = a$:

$$y(a) = y'_x(a) = \cdots = y_x^{(n-1)}(a) = 0. \tag{4.2.3.1}$$

Introducing a new unknown function by

$$y(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1}u(t)\,dt \tag{4.2.3.2}$$

and differentiating (4.2.3.2) n times, we get

$$\begin{aligned} y_x^{(k)}(x) &= \frac{1}{(n-k-1)!}\int_a^x (x-t)^{n-k-1}u(t)\,dt, \qquad k = 1, \dots, n-1; \\ y_x^{(n)}(x) &= u(x). \end{aligned} \tag{4.2.3.3}$$

Obviously, the function (4.2.3.2) satisfies the initial conditions (4.2.3.1). By substituting (4.2.3.3) into the left-hand side of equation (4.2.2.1), we obtain

$$f_n(x)u(x) + \int_a^x K(x,t)u(t)\,dt = g(x), \tag{4.2.3.4}$$

where

$$K(x,t) = f_{n-1}(x) + f_{n-2}(x)\frac{x-t}{1!} + \cdots + f_0(x)\frac{(x-t)^{n-1}}{(n-1)!}. \tag{4.2.3.5}$$

Thus, the Cauchy problem (4.2.2.1)–(4.2.3.1) has been reduced to the integral equation (4.2.3.4)–(4.2.3.5), which is a *Volterra equation of the second kind*. Finding the function $u(x)$ from (4.2.3.4) and using formula (4.2.3.2) we obtain the desired solution $y(x)$.

The solution of the integral equation (4.2.3.4) can be obtained using, for example, the method of successive approximations with the recurrence relation

$$u_{m+1}(x) + \frac{1}{f_n(x)}\int_a^x K(x,t)u_m(t)\,dt = \frac{g(x)}{f_n(x)}, \tag{4.2.3.6}$$

where $m = 0,\ 1,\ 2,\ \ldots$ The function $u_0(x) = 0$ can be taken as the zeroth approximation; then $u_1(x) = g(x)/f_n(x)$.

For more efficient numerical methods for integral equations of the form (4.2.3.4), see the book by Polyanin & Manzhirov (2008).

Remark 4.2. The Cauchy problem for equation (4.2.2.1) with nonhomogeneous boundary conditions

$$y(a) = b_0, \quad y'_x(a) = b_1, \quad \ldots, \quad y_x^{(n-1)}(a) = b_{n-1}$$

can be reduced to a Cauchy problem with homogeneous boundary conditions for another function $w(x)$ with the help of the substitution

$$y(x) = w(x) + \sum_{k=1}^{n-1} b_k \frac{(x-a)^k}{k!}.$$

⊙ *Literature for Section 4.2*: G. M. Murphy (1960), L. E. El'sgol'ts (1961), N. M. Matveev (1967), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1980), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007, 2008).

4.3 Laplace Transform and the Laplace Integral. Applications to Linear ODEs

4.3.1 Laplace Transform and the Inverse Laplace Transform

► **Laplace transform.**

The *Laplace transform* of an arbitrary (complex-valued) function $f(x)$ of a real variable x ($x \geq 0$) is defined by

$$\widetilde{f}(p) = \int_0^\infty e^{-px} f(x)\,dx, \tag{4.3.1.1}$$

where $p = s + i\sigma$ is a complex variable.

The Laplace transform exists for any continuous or piecewise-continuous function satisfying the condition $|f(x)| < Me^{\sigma_0 x}$ with some $M > 0$ and $\sigma_0 \geq 0$. In the following, σ_0 often means the greatest lower bound of the possible values of σ_0 in this estimate; this value is called the *growth exponent* of the function $f(x)$.

For any $f(x)$, the transform $\widetilde{f}(p)$ is defined in the half-plane $\operatorname{Re} p > \sigma_0$ and is analytic there.

For brevity, we shall write formula (4.3.1.1) as follows:

$$\widetilde{f}(p) = \mathfrak{L}\{f(x)\}.$$

► **Inverse Laplace transform.**

Given the transform $\widetilde{f}(p)$, the function $f(x)$ can be found by means of the inverse Laplace transform

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \widetilde{f}(p) e^{px}\,dp, \qquad i^2 = -1, \tag{4.3.1.2}$$

where the integration path is parallel to the imaginary axis and lies to the right of all singularities of $\widetilde{f}(p)$, which corresponds to $c > \sigma_0$.

The integral in inversion formula (4.3.1.2) is understood in the sense of the Cauchy principal value:

$$\int_{c-i\infty}^{c+i\infty} \widetilde{f}(p)e^{px}\,dp = \lim_{\omega\to\infty}\int_{c-i\omega}^{c+i\omega} \widetilde{f}(p)e^{px}\,dp.$$

In the domain $x < 0$, formula (4.3.1.2) gives $f(x) \equiv 0$.

Formula (4.3.1.2) holds for continuous functions. If $f(x)$ has a (finite) jump discontinuity at a point $x = x_0 > 0$, then the left-hand side of (4.3.1.2) is equal to $\frac{1}{2}[f(x_0 - 0) + f(x_0 + 0)]$ at this point (for $x_0 = 0$, the first term in the square brackets must be omitted).

For brevity, we write the Laplace inversion formula (4.3.1.2) as follows:

$$f(x) = \mathfrak{L}^{-1}\{\widetilde{f}(p)\}.$$

There are tables of direct and inverse Laplace transforms (see Sections S3.1 and S3.2, which are handy in solving linear differential and integral equations.

4.3.2 Main Properties of the Laplace Transform. Inversion Formulas for Some Functions

► Main properties of the Laplace transform.

1°. The main properties of the correspondence between functions and their Laplace transforms are gathered in Table 4.2.

2°. The Laplace transforms of some functions are listed in Table 4.3; for more detailed tables see Section S3.1 and the list of references at the end of this section.

TABLE 4.2
Main properties of the Laplace transform

No.	Function	Laplace transform	Operation
1	$af_1(x) + bf_2(x)$	$a\widetilde{f}_1(p) + b\widetilde{f}_2(p)$	Linearity
2	$f(x/a),\ a > 0$	$a\widetilde{f}(ap)$	Scaling
3	$f(x-a)$, $f(\xi) \equiv 0$ for $\xi < 0$	$e^{-ap}\widetilde{f}(p)$	Shift of the argument
4	$x^n f(x);\ n = 1, 2, \ldots$	$(-1)^n \widetilde{f}_p^{(n)}(p)$	Differentiation of the transform
5	$\dfrac{1}{x} f(x)$	$\displaystyle\int_p^\infty \widetilde{f}(q)\,dq$	Integration of the transform
6	$e^{ax} f(x)$	$\widetilde{f}(p-a)$	Shift in the complex plane
7	$f'_x(x)$	$p\widetilde{f}(p) - f(+0)$	Differentiation
8	$f_x^{(n)}(x)$	$p^n \widetilde{f}(p) - \sum\limits_{k=1}^n p^{n-k} f_x^{(k-1)}(+0)$	Differentiation
9	$x^m f_x^{(n)}(x),\ m = 1, 2, \ldots$	$(-1)^m \dfrac{d^m}{dp^m}\Big[p^n \widetilde{f}(p) - \sum\limits_{k=1}^n p^{n-k} f_x^{(k-1)}(+0)\Big]$	Differentiation
10	$\dfrac{d^n}{dx^n}\big[x^m f(x)\big],\ \ m \geq n$	$(-1)^m p^n \dfrac{d^m}{dp^m}\widetilde{f}(p)$	Differentiation
11	$\displaystyle\int_0^x f(t)\,dt$	$\dfrac{\widetilde{f}(p)}{p}$	Integration
12	$\displaystyle\int_0^x f_1(t) f_2(x-t)\,dt$	$\widetilde{f}_1(p)\widetilde{f}_2(p)$	Convolution

TABLE 4.3
The Laplace transforms of some functions

No.	Function, $f(x)$	Laplace transform, $\widetilde{f}(p)$	Remarks
1	1	$1/p$	
2	x^n	$\dfrac{n!}{p^{n+1}}$	$n = 1,\ 2,\ \dots$
3	x^a	$\Gamma(a+1)p^{-a-1}$	$a > -1$
4	e^{-ax}	$(p+a)^{-1}$	
5	$x^a e^{-bx}$	$\Gamma(a+1)(p+b)^{-a-1}$	$a > -1$
6	$\sinh(ax)$	$\dfrac{a}{p^2-a^2}$	
7	$\cosh(ax)$	$\dfrac{p}{p^2-a^2}$	
8	$\ln x$	$-\dfrac{1}{p}(\ln p + \mathcal{C})$	$\mathcal{C} = 0.5772\dots$ is the Euler constant
9	$\sin(ax)$	$\dfrac{a}{p^2+a^2}$	
10	$\cos(ax)$	$\dfrac{p}{p^2+a^2}$	
11	$\operatorname{erfc}\Bigl(\dfrac{a}{2\sqrt{x}}\Bigr)$	$\dfrac{1}{p}\exp\bigl(-a\sqrt{p}\,\bigr)$	$a \ge 0$
12	$J_0(ax)$	$\dfrac{1}{\sqrt{p^2+a^2}}$	$J_0(x)$ is the Bessel function

► **Inverse transforms of rational functions.**

Consider the important case in which the transform is a rational function of the form

$$\widetilde{f}(p) = \frac{R(p)}{Q(p)}, \tag{4.3.2.1}$$

where $Q(p)$ and $R(p)$ are polynomials in the variable p and the degree of $Q(p)$ exceeds that of $R(p)$.

Assume that the zeros of the denominator are simple, i.e.,

$$Q(p) \equiv \text{const}\,(p-\lambda_1)(p-\lambda_2)\dots(p-\lambda_n).$$

Then the inverse transform can be determined by the formula

$$f(x) = \sum_{k=1}^{n} \frac{R(\lambda_k)}{Q'(\lambda_k)} \exp(\lambda_k x), \tag{4.3.2.2}$$

where the primes denote the derivatives.

If $Q(p)$ has multiple zeros, i.e.,

$$Q(p) \equiv \text{const}\,(p-\lambda_1)^{s_1}(p-\lambda_2)^{s_2}\dots(p-\lambda_m)^{s_m},$$

then

$$f(x) = \sum_{k=1}^{m} \frac{1}{(s_k-1)!} \lim_{p\to s_k} \frac{d^{s_k-1}}{dp^{s_k-1}}\bigl[(p-\lambda_k)^{s_k}\widetilde{f}(p)e^{px}\bigr]. \tag{4.3.2.3}$$

Example 4.3. The transform

$$\widetilde{f}(p) = \frac{b}{p^2 - a^2} \qquad (a, b \text{ real numbers})$$

can be represented as the fraction (4.3.2.1) with $R(p) = b$ and $Q(p) = (p - a)(p + a)$. The denominator $Q(p)$ has two simple roots, $\lambda_1 = a$ and $\lambda_2 = -a$. Using formula (4.3.2.2) with $n = 2$ and $Q'(p) = 2p$, we obtain the inverse transform in the form

$$f(x) = \frac{b}{2a}e^{ax} - \frac{b}{2a}e^{-ax} = \frac{b}{a}\sinh(ax).$$

Example 4.4. The transform

$$\widetilde{f}(p) = \frac{b}{p^2 + a^2} \qquad (a, b \text{ real numbers})$$

can be written as the fraction (4.3.2.1) with $R(p) = b$ and $Q(p) = (p - ia)(p + ia)$, $i^2 = -1$. The denominator $Q(p)$ has two simple pure imaginary roots, $\lambda_1 = ia$ and $\lambda_2 = -ia$. Using formula (4.3.2.2) with $n = 2$, we find the inverse transform:

$$f(x) = \frac{b}{2ia}e^{iax} - \frac{b}{2ia}e^{-iax} = -\frac{bi}{2a}\big[\cos(ax) + i\sin(ax)\big] + \frac{bi}{2a}\big[\cos(ax) - i\sin(ax)\big] = \frac{b}{a}\sin(ax).$$

Example 4.5. The transform

$$\widetilde{f}(p) = ap^{-n},$$

where n is a positive integer, can be written as the fraction (.2.2.1) with $R(p) = a$ and $Q(p) = p^n$. The denominator $Q(p)$ has one root of multiplicity n, $\lambda_1 = 0$. By formula (.2.2.3) with $m = 1$ and $s_1 = n$, we find the inverse transform:

$$f(x) = \frac{a}{(n-1)!}x^{n-1}.$$

◆ *Detailed tables of inverse Laplace transforms can be found in Section S3.2.*

4.3.3 Limit Theorems. Representation of Inverse Transforms as Convergent Series and Asymptotic Expansions

► **Limit theorems.**

THEOREM 1. *Let $0 \le x < \infty$ and $\widetilde{f}(p) = \mathfrak{L}\{f(x)\}$ be the Laplace transform of $f(x)$. If a limit of $f(x)$ as $x \to 0$ exists, then*

$$\lim_{x\to 0} f(x) = \lim_{p\to\infty}\big[p\widetilde{f}(p)\big].$$

THEOREM 2. *If a limit of $f(x)$ as $x \to \infty$ exists, then*

$$\lim_{x\to\infty} f(x) = \lim_{p\to 0}\big[p\widetilde{f}(p)\big].$$

► **Representation of inverse transforms as convergent series.**

THEOREM 1. *Suppose the transform $\widetilde{f}(p)$ can be expanded into series in negative powers of p,*

$$\widetilde{f}(p) = \sum_{n=1}^{\infty}\frac{a_n}{p^n},$$

convergent for $|p| > R$, where R is an arbitrary positive number; note that the transform tends to zero as $|p| \to \infty$. Then the inverse transform can be obtained by the formula

$$f(x) = \sum_{n=1}^{\infty} \frac{a_n}{(n-1)!} x^{n-1},$$

where the series on the right-hand side is convergent for all x.

THEOREM 2. *Suppose the transform $\widetilde{f}(p)$, $|p| > R$, is represented by an absolutely convergent series,*

$$\widetilde{f}(p) = \sum_{n=0}^{\infty} \frac{a_n}{p^{\lambda_n}}, \tag{4.3.3.1}$$

where $\{\lambda_n\}$ is any positive increasing sequence, $0 < \lambda_0 < \lambda_1 < \cdots \to \infty$. Then it is possible to proceed termwise from series (4.3.3.1) to the following inverse transform series:

$$f(x) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(\lambda_n)} x^{\lambda_n - 1}, \tag{4.3.3.2}$$

where $\Gamma(\lambda)$ is the Gamma function. Series (4.3.3.2) is convergent for all real and complex values of x other than zero (if $\lambda_0 \geq 1$, the series is convergent for all x).

► Representation of inverse transforms as asymptotic expansions as $x \to \infty$.

1°. Let $p = p_0$ be a singular point of the Laplace transform $\widetilde{f}(p)$ with the greatest real part (it is assumed there is only one such point). If $\widetilde{f}(p)$ can be expanded near $p = p_0$ into an absolutely convergent series,

$$\widetilde{f}(p) = \sum_{n=0}^{\infty} c_n (p - p_0)^{\lambda_n} \qquad (\lambda_0 < \lambda_1 < \cdots \to \infty) \tag{4.3.3.3}$$

with arbitrary λ_n, then the inverse transform $f(x)$ can be expressed in the form of the asymptotic expansion

$$f(x) \sim e^{p_0 x} \sum_{n=0}^{\infty} \frac{c_n}{\Gamma(-\lambda_n)} x^{-\lambda_n - 1} \quad \text{as} \quad x \to \infty. \tag{4.3.3.4}$$

The terms corresponding to nonnegative integer λ_n must be omitted from the summation, since $\Gamma(0) = \Gamma(-1) = \Gamma(-2) = \cdots = \infty$.

2°. If the transform $\widetilde{f}(p)$ has several singular points, $p_1, \ldots, p_m$, with the same greatest real part, $\operatorname{Re} p_1 = \cdots = \operatorname{Re} p_m$, then expansions of the form (4.3.3.3) should be obtained for each of these points and the resulting expressions must be added together.

► Post–Widder formula.

In applications, one can find $f(x)$ if the Laplace transform $\widetilde{f}(t)$ on the real semiaxis is known for $t = p \geq 0$. To this end, one uses the Post–Widder formula

$$f(x) = \lim_{n \to \infty} \left[\frac{(-1)^n}{n!} \left(\frac{n}{x} \right)^{n+1} \widetilde{f}_t^{(n)} \left(\frac{n}{x} \right) \right]. \tag{4.3.3.5}$$

Approximate inversion formulas are obtained by taking sufficiently large positive integer n in (4.3.3.5) instead of passing to the limit.

4.3.4 Solution of the Cauchy Problem for Constant-Coefficient Linear ODEs. Applications to Integro-Differential Equations

► **Cauchy problem for constant-coefficient linear ODEs.**

Consider the Cauchy problem for equation (4.1.2.1) with arbitrary initial conditions

$$y(0) = y_0, \quad y'_x(0) = y_1, \quad \dots, \quad y_x^{(n-1)}(0) = y_{n-1}, \tag{4.3.4.1}$$

where $y_0, y_1, \dots, y_{n-1}$ are given constants.

Problem (4.1.2.1), (4.3.4.1) can be solved using the Laplace transform based on the formulas (for details, see Section 4.3.1)

$$\widetilde{y}(p) = \mathfrak{L}\{y(x)\}, \quad \widetilde{f}(p) = \mathfrak{L}\{f(x)\}, \quad \text{where} \quad \mathfrak{L}\{f(x)\} \equiv \int_0^\infty e^{-px} f(x)\,dx.$$

To this end, let us multiply equation (4.1.2.1) by e^{-px} and then integrate with respect to x from zero to infinity. Taking into account the differentiation rule

$$\mathfrak{L}\{y_x^{(n)}(x)\} = p^n \widetilde{y}(p) - \sum_{k=1}^{n} p^{n-k} y_x^{(k-1)}(+0)$$

and the initial conditions (4.3.4.1), we arrive at a linear algebraic equation for the transform $\widetilde{y}(p)$:

$$P(p)\widetilde{y}(p) - Q(p) = \widetilde{f}(p), \tag{4.3.4.2}$$

where

$$P(p) = p^n + a_{n-1}p^{n-1} + \dots + a_1 p + a_0, \qquad Q(p) = b_{n-1}p^{n-1} + \dots + b_1 p + b_0,$$
$$b_k = y_{n-k-1} + a_{n-1}y_{n-k-2} + \dots + a_{k+2}y_1 + a_{k+1}y_0, \qquad k = 0,\ 1,\ \dots,\ n-1.$$

The polynomial $P(p)$ coincides with the characteristic polynomial (4.1.1.2) at $\lambda = p$.

The solution of equation (4.3.4.2) is given by the formula

$$\widetilde{y}(p) = \frac{\widetilde{f}(p) + Q(p)}{\widetilde{P}(p)}. \tag{4.3.4.3}$$

On applying the Laplace inversion formula (4.3.1.2) to (4.3.4.3), we obtain a solution to problem (4.1.2.1), (4.3.4.1) in the form

$$y(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\widetilde{f}(p) + Q(p)}{\widetilde{P}(p)} e^{px}\,dp. \tag{4.3.4.4}$$

Since the transform $\widetilde{y}(p)$ (4.3.4.3) is a rational function, the inverse Laplace transform (4.3.4.4) can be obtained using the formulas from Section 4.3.2 or the tables of Section S3.2.

Remark 4.3. In practice, the solution method for the Cauchy problem based on the Laplace transform leads to the solution faster than the direct application of general formulas like (4.1.2.2), where one has to determine the coefficients $C_1, \dots, C_n$.

Example 4.6. Consider the following Cauchy problem for a homogeneous fourth-order equation:

$$y''''_{xxxx} + a^4 y = 0; \qquad y(0) = y'_x(0) = y'''_{xxx}(0) = 0, \quad y''_{xx}(0) = b.$$

Using the Laplace transform reduces this problem to a linear algebraic equation for the $\widetilde{y}(p)$: $(p^4 + a^4)\widetilde{y}(p) - bp = 0$. It follows that

$$\widetilde{y}(p) = \frac{bp}{p^4 + a^4}.$$

In order to invert this expression, let us use the table of inverse Laplace transforms (see Section S3.2.2, row 52) and take into account that a constant multiplier can be taken outside the transform operator to obtain the solution to the original Cauchy problem in the form

$$y(x) = \frac{b}{a^2} \sin\Bigl(\frac{ax}{\sqrt{2}}\Bigr) \sinh\Bigl(\frac{ax}{\sqrt{2}}\Bigr).$$

► Cauchy problem for integro-differential equations.

The Laplace transform can also be effective in solving some linear integro-differential equations. This is illustrated below with a specific example:

Example 4.7. Consider the Cauchy problem for the linear integro-differential equation

$$\frac{dy}{dx} + \int_0^x K(x-t)y(t)\,dt = f(x) \qquad (0 \le x < \infty) \tag{4.3.4.5}$$

with the initial condition

$$y = a \quad \text{at} \quad x = 0. \tag{4.3.4.6}$$

Multiply equation (4.3.4.5) by e^{-px} and then integrate with respect to x from zero to infinity. Using properties 7 and 12 of the Laplace transform (Table 4.2) and taking into account the initial condition (4.3.4.6), we obtain a linear algebraic equation for the transform $\widetilde{y}(p)$:

$$p\widetilde{y}(p) - a + \widetilde{K}(p)\widetilde{y}(p) = \widetilde{f}(p).$$

It follows that

$$\widetilde{y}(p) = \frac{\widetilde{f}(p) + a}{p + \widetilde{K}(p)}.$$

By the inversion formula (4.3.1.2), the solution to the original problem (4.3.4.5)–(4.3.4.6) is found in the form

$$y(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\widetilde{f}(p) + a}{p + \widetilde{K}(p)} e^{px}\,dp, \qquad i^2 = -1. \tag{4.3.4.7}$$

Consider the special case of $a = 0$ and $K(x) = \cos(bx)$. From row 10 of Table 4.3 it follows that $\widetilde{K}(p) = \dfrac{p}{p^2 + b^2}$. Rearrange the integrand of (4.3.4.7):

$$\frac{\widetilde{f}(p)}{p + \widetilde{K}(p)} = \frac{p^2 + b^2}{p(p^2 + b^2 + 1)}\widetilde{f}(p) = \Bigl(\frac{1}{p} - \frac{1}{p(p^2 + b^2 + 1)}\Bigr)\widetilde{f}(p).$$

In order to invert this expression, let us use the convolution theorem (see formula 16 of Section S3.2.1) as well as formulas 1 and 28 for the inversion of rational functions, Section S3.2.2. As a result, we arrive at the solution in the form

$$y(x) = \int_0^x \frac{b^2 + \cos\bigl(t\sqrt{b^2 + 1}\bigr)}{b^2 + 1} f(x-t)\,dt.$$

4.3.5 Solution of Linear Equations with Polynomial Coefficients Using the Laplace Transform

► **Solution of equations using the Laplace transform. General description.**

1°. Some classes of equations (4.2.1.1) or (4.2.2.1) with polynomial coefficients

$$f_k(x)=\sum_{m=0}^{s_k} a_{km}x^m$$

may be solved using the Laplace transform (see Sections 4.3.1, 4.3.2, and S3.1). To this end, one uses the following formula for the Laplace transform of the product of a power function and a derivative of the unknown function:

$$\mathfrak{L}\left\{x^m y_x^{(n)}(x)\right\}=(-1)^m\frac{d^m}{dp^m}\left[p^n\widetilde{y}(p)-\sum_{k=1}^{n}p^{n-k}y_x^{(k-1)}(+0)\right]. \qquad (4.3.5.1)$$

The right-hand side contains initial data $y_x^{(m)}(+0)$, $m=0,\,1,\,\ldots,\,n-1$ (specified in the Cauchy problem). As a result, one arrives at a linear ordinary differential equation, with respect to p, for the transform $\widetilde{y}(p)$; the order of this equation is equal to $\max\limits_{1\le k\le n}\{s_k\}$, the highest degree of the polynomials that determine the equation coefficients. In some cases, the equation for $\widetilde{y}(p)$ turns out to be simpler than the initial equation for $y(x)$ and can be solved in closed form. The desired function $y(x)$ is found by inverting the transform $\widetilde{y}(p)$ using the formulas from Section 4.3.2 or the tables from Section S3.2.

► **Application to the Laplace equation.**

Consider the *Laplace equation*

$$(a_n+b_nx)y_x^{(n)}+(a_{n-1}+b_{n-1}x)y_x^{(n-1)}+\cdots+(a_1+b_1x)y'_x+(a_0+b_0x)y=0, \qquad (4.3.5.2)$$

whose coefficients are linear functions of the independent variable x. The application of the Laplace transform, in view of formulas (4.3.5.1), brings it to a linear first-order ordinary differential equation for the transform $\widetilde{y}(p)$.

Example 4.8. Consider a special case of equation (4.3.5.2):

$$xy''_{xx}+y'_x+axy=0. \qquad (4.3.5.3)$$

Denote $y(0)=y_0$ and $y'_x(0)=y_1$. Let us apply the Laplace transform to this equation using formulas (4.3.5.1). On rearrangement, we obtain a linear first-order equation for $\widetilde{y}(p)$:

$$-(p^2\widetilde{y}-y_0p-y_1)'_p+(p\widetilde{y}-y_0)-a\widetilde{y}'_p=0 \qquad\Longrightarrow\qquad (p^2+a)\widetilde{y}'_p+p\widetilde{y}=0.$$

Its general solution is expressed as

$$\widetilde{y}=\frac{C}{\sqrt{p^2+a}}, \qquad (4.3.5.4)$$

where C is an arbitrary constant. Applying the inverse Laplace transform to (4.3.5.4) and taking into account formulas 19 and 20 from Section S3.2.3, we find a solution to the original equation (4.3.5.3):

$$y(x)=\begin{cases} CJ_0(x\sqrt{a}\,) & \text{if } a>0,\\ CI_0(x\sqrt{-a}\,) & \text{if } a<0,\end{cases} \qquad (4.3.5.5)$$

where $J_0(x)$ is the Bessel function of the first kind and $I_0(x)$ is the modified Bessel function of the first kind.

In this case, only one solution (4.3.5.5) has been obtained. This is due to the fact that the other solution goes to infinity as $x \to 0$, and hence formula (4.3.5.1) cannot be applied to it; this formula is only valid for finite initial values of the function and its derivatives.

4.3.6 Solution of Linear Equations with Polynomial Coefficients Using the Laplace Integral

► **Solution of equations using the Laplace integral. General description.**

Solutions to linear differential equations with polynomial coefficients can sometimes be represented as a *Laplace integral* in the form

$$y(x) = \int_{\mathcal{K}} e^{px} u(p)\, dp. \tag{4.3.6.1}$$

For now, no assumptions are made about the domain of integration $\mathcal{K}$; it could be a segment of the real axis or a curve in the complex plane.

Let us exemplify the usage of the Laplace integral (4.3.6.1) by considering equation (4.3.5.2). It follows from (4.3.6.1) that

$$\begin{aligned} y_x^{(k)}(x) &= \int_{\mathcal{K}} e^{px} p^k u(p)\, dp, \\ x y_x^{(k)}(x) &= \int_{\mathcal{K}} x e^{px} p^k u(p)\, dp = \Big[e^{px} p^k u(p)\Big]_{\mathcal{K}} - \int_{\mathcal{K}} e^{px} \frac{d}{dp}\Big[p^k u(p)\Big]\, dp. \end{aligned}$$

Substituting these expressions into (4.3.5.2) yields

$$\int_{\mathcal{K}} e^{px} \left\{ \sum_{k=0}^{n} a_k p^k u(p) - \sum_{k=0}^{n} b_k \frac{d}{dp}\Big[p^k u(p)\Big] \right\} dp + \sum_{k=0}^{n} b_k \Big[e^{px} p^k u(p)\Big]_{\mathcal{K}} = 0. \tag{4.3.6.2}$$

This equation is satisfied if the expression in braces vanishes, thus resulting in a linear first-order ordinary differential equation for $u(p)$:

$$u(p) \sum_{k=0}^{n} a_k p^k - \frac{d}{dp}\left[u(p) \sum_{k=0}^{n} b_k p^k\right] = 0. \tag{4.3.6.3}$$

The remaining term in (4.3.6.2) must also vanish:

$$\left[\sum_{k=0}^{n} b_k e^{px} p^k u(p)\right]_{\mathcal{K}} = 0. \tag{4.3.6.4}$$

This condition can be met by appropriately selecting the path of integration $\mathcal{K}$. Consider the example below to illustrate the aforesaid.

► **Application to the second-order Laplace equation of the special form.**

Consider the linear variable-coefficient second-order equation

$$x y''_{xx} + (x + a + b) y'_x + a y = 0 \qquad (a > 0,\ b > 0), \tag{4.3.6.5}$$

that is a special case of equation (4.3.5.2) with $n=2$, $a_2=0$, $a_1=a+b$, $a_0=a$, $b_2=b_1=1$, and $b_0=0$. On substituting these values into (4.3.6.3), we arrive at an equation for $u(p)$:

$$p(p+1)u'_p - [(a+b-2)p+a-1]u = 0.$$

Its solution is given by

$$u(p) = p^{a-1}(p+1)^{b-1}. \tag{4.3.6.6}$$

It follows from condition (4.3.6.4), in view of formula (4.3.6.6), that

$$\Big[e^{px}(p+p^2)u(p)\Big]_\alpha^\beta = \Big[e^{px}p^a(p+1)^b\Big]_\alpha^\beta = 0, \tag{4.3.6.7}$$

where a segment of the real axis, $\mathcal{K}=[\alpha,\beta]$, has been chosen to be the path of integration. Condition (4.3.6.7) is satisfied if we set $\alpha=-1$ and $\beta=0$. Consequently, one of the solutions to equation (4.3.6.5) has the form

$$y(x) = \int_{-1}^{0} e^{px}p^{a-1}(p+1)^{b-1}\,dp. \tag{4.3.6.8}$$

Remark 4.4. If a is noninteger, it is necessary to separate the real and imaginary parts in (4.3.6.8) to obtain real solutions.

Remark 4.5. By setting $\alpha=-\infty$ and $\beta=0$ in (4.3.6.7), one can find a second solution to equation (4.3.6.5) (at least for $x>0$).

⊙ *Literature for Section 4.3*: G. Doetsch (1950, 1956, 1974), H. Bateman and A. Erdélyi (1954), G. M. Murphy (1960), V. A. Ditkin and A. P. Prudnikov (1965), J. W. Miles (1971), F. Oberhettinger and L. Badii (1973), E. Kamke (1977), W. R. LePage (1980), R. Bellman and R. Roth (1984), A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev (1992*a,b*), M. Ya. Antimirov (1993), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

4.4 Asymptotic Solutions of Linear Equations

This section presents asymptotic solutions, as $\varepsilon\to 0$ ($\varepsilon>0$), of some higher-order linear ordinary differential equations containing arbitrary functions (sufficiently smooth), with the independent variable being real.

4.4.1 Fourth-Order Linear Differential Equations

▶ **Binomial equation.**

1°. Consider the equation

$$\varepsilon^4 y''''_{xxxx} - f(x)y = 0$$

on a closed interval $a\le x\le b$. With the condition $f>0$, the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon\to 0$, are given by the formulas

$$y_1 = [f(x)]^{-3/8}\exp\biggl\{-\frac{1}{\varepsilon}\int[f(x)]^{1/4}\,dx\biggr\},\quad y_2 = [f(x)]^{-3/8}\exp\biggl\{\frac{1}{\varepsilon}\int[f(x)]^{1/4}\,dx\biggr\},$$
$$y_3 = [f(x)]^{-3/8}\cos\biggl\{\frac{1}{\varepsilon}\int[f(x)]^{1/4}\,dx\biggr\},\quad y_4 = [f(x)]^{-3/8}\sin\biggl\{\frac{1}{\varepsilon}\int[f(x)]^{1/4}\,dx\biggr\}.$$

► **Trinomial equation.**

Now consider the “biquadratic” equation

$$\varepsilon^4 y''''_{xxxx} - 2\varepsilon^2 g(x) y''_{xx} - f(x) y = 0. \qquad (4.4.1.1)$$

Introduce the notation

$$D(x) = [g(x)]^2 + f(x).$$

In the range where the conditions $f(x) \neq 0$ and $D(x) \neq 0$ are satisfied, the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (4.4.1.1) are described by the formulas

$$y_k = [\lambda_k(x)]^{-1/2} [D(x)]^{-1/4} \exp\biggl\{\frac{1}{\varepsilon}\int \lambda_k(x)\,dx - \frac{1}{2}\int \frac{[\lambda_k(x)]'_x}{\sqrt{D(x)}}\,dx\biggr\}; \quad k = 1,\ 2,\ 3,\ 4,$$

where

$$\lambda_1(x) = \sqrt{g(x) + \sqrt{D(x)}}, \qquad \lambda_2(x) = -\sqrt{g(x) + \sqrt{D(x)}},$$
$$\lambda_3(x) = \sqrt{g(x) - \sqrt{D(x)}}, \qquad \lambda_4(x) = -\sqrt{g(x) - \sqrt{D(x)}}.$$

4.4.2 Higher-Order Linear Differential Equations

► **Binomial equation.**

Consider an equation of the form

$$\varepsilon^n y_x^{(n)} - f(x) y = 0$$

on a closed interval $a \le x \le b$. Assume that $f \neq 0$. Then the leading terms of the asymptotic expansions of the fundamental system of solutions, as $\varepsilon \to 0$, are given by

$$y_m = \bigl[f(x)\bigr]^{-\frac{1}{2} + \frac{1}{2n}} \exp\biggl\{\frac{\omega_m}{\varepsilon}\int \bigl[f(x)\bigr]^{\frac{1}{n}}\,dx\biggr\}\bigl[1 + O(\varepsilon)\bigr],$$

where $\omega_1, \omega_2, \ldots, \omega_n$ are roots of the equation $\omega^n = 1$:

$$\omega_m = \cos\Bigl(\frac{2\pi m}{n}\Bigr) + i\sin\Bigl(\frac{2\pi m}{n}\Bigr), \qquad m = 1,\ 2,\ \ldots,\ n.$$

► **More complex equation.**

Now consider an equation of the form

$$\varepsilon^n y_x^{(n)} + \varepsilon^{n-1} f_{n-1}(x) y_x^{(n-1)} + \cdots + \varepsilon f_1(x) y'_x + f_0(x) y = 0 \qquad (4.4.2.1)$$

on a closed interval $a \le x \le b$. Let $\lambda_m = \lambda_m(x)$ $(m = 1, 2, \ldots, n)$ be the roots of the characteristic equation

$$P(x, \lambda) \equiv \lambda^n + f_{n-1}(x)\lambda^{n-1} + \cdots + f_1(x)\lambda + f_0(x) = 0.$$

Let all the roots of the characteristic equation be different on the interval $a \le x \le b$, i.e., the conditions $\lambda_m(x) \neq \lambda_k(x)$, $m \neq k$, are satisfied, which is equivalent to the fulfillment of the conditions $P_\lambda(x,\lambda_m) \neq 0$. Then the leading terms of the asymptotic expansions of the fundamental system of solutions of equation (4.4.2.1), as $\varepsilon \to 0$, are given by

$$y_m = \exp\left\{\frac{1}{\varepsilon}\int \lambda_m(x)\,dx - \frac{1}{2}\int [\lambda_m(x)]'_x \frac{P_{\lambda\lambda}\big(x,\lambda_m(x)\big)}{P_\lambda\big(x,\lambda_m(x)\big)}\,dx\right\},$$

where

$$P_\lambda(x,\lambda) \equiv \frac{\partial P}{\partial\lambda} = n\lambda^{n-1} + (n-1)f_{n-1}(x)\lambda^{n-2} + \cdots + 2\lambda f_2(x) + f_1(x),$$

$$P_{\lambda\lambda}(x,\lambda) \equiv \frac{\partial^2 P}{\partial\lambda^2} = n(n-1)\lambda^{n-2} + (n-1)(n-2)f_{n-1}(x)\lambda^{n-3} + \cdots + 6\lambda f_3(x) + 2f_2(x).$$

⊙ *Literature for Section 4.4*: W. Wasov (1965), M. V. Fedoryuk (1993), A. D. Polyanin and V. F. Zaitsev (2003).

4.5 Collocation Method

4.5.1 Statement of the Problem. Approximate Solution

1°. Consider the linear boundary value problem defined by the equation

$$\mathrm{L}y \equiv y_x^{(n)} + f_{n-1}(x)y_x^{(n-1)} + \cdots + f_1(x)y'_x + f_0(x)y = g(x), \qquad -1 < x < 1, \quad (4.5.1.1)$$

and the boundary conditions

$$\sum_{j=0}^{n-1}\big[\alpha_{ij}y_x^{(j)}(-1) + \beta_{ij}y_x^{(j)}(1)\big] = 0, \qquad i = 1, \dots, n. \qquad (4.5.1.2)$$

2°. We seek an approximate solution to problem (4.5.1.1)–(4.5.1.2) in the form

$$y_m(x) = A_1\varphi_1(x) + A_2\varphi_2(x) + \cdots + A_m\varphi_m(x),$$

where $\varphi_k(x)$ is a polynomial of degree $n+k-1$ that satisfies the boundary conditions (4.5.1.2). The coefficients A_k are determined by the linear system of algebraic equations

$$\big[\mathrm{L}y_m - g(x)\big]_{x=x_i} = 0, \qquad i = 1, \dots, m, \qquad (4.5.1.3)$$

with *Chebyshev nodes* $x_i = \cos\Big(\dfrac{2i-1}{2m}\pi\Big)$, $i = 1,\ \dots,\ m$.

4.5.2 Convergence Theorem

THEOREM. *Let the functions $f_j(x)$ ($j = 0, \dots, n-1$) and $g(x)$ be continuous on the interval $[-1, 1]$ and let the boundary value problem (4.5.1.1)–(4.5.1.2) have a unique solution, $y(x)$. Then there exists an m_0 such that system (4.5.1.3) is uniquely solvable for $m \geq m_0$; and the limit relations*

$$\max_{-1\leq x\leq 1}\big|y_m^{(k)}(x) - y^{(k)}(x)\big| \leq cE_m\big(y^{(n)}\big) \to 0, \qquad k = 0,\ 1,\ \dots,\ n-1;$$

$$\left\{\int_{-1}^{1}\frac{\big|y_m^{(n)}(x) - y^{(n)}(x)\big|^2}{\sqrt{1-x^2}}\,dx\right\}^{1/2} \leq cE_m\big(y^{(n)}\big) \to 0$$

hold for $m \to \infty$. *Here* $c = \text{const}$ *and*

$$E_m(v) = \min_{b_0,\dots,b_{m-1}} \max_{-1\le x\le 1} \biggl| v(x) - \sum_{j=0}^{m-1} b_j x^j \biggr|.$$

Remark 4.6. A similar result holds true if the nodes are roots of some orthogonal polynomials with some weight function. If the nodes are equidistant, the method diverges.

⊙ *Literature for Section 4.5*: R. D. Russell and L. F. Shampine (1972), C. de Boor and B. Swartz (1993), Mathematical Encyclopedia (1979, p. 951), A. D. Polyanin and A. V. Manzhirov (2007).