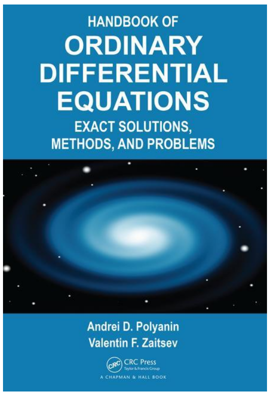

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

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Chapter 5: Methods for Nonlinear ODEs of Arbitrary Order

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Chapter 5

Methods for Nonlinear ODEs of Arbitrary Order

5.1 General Concepts. Cauchy Problem. Uniqueness and Existence Theorems

5.1.1 Equations Solved for the Derivative. General Solution

► Equations solved for the highest derivative. Structure of the general solution.

An *nth-order differential equation solved for the highest derivative* has the form

$$y_x^{(n)} = f(x, y, y'_x, \dots, y_x^{(n-1)}). \tag{5.1.1.1}$$

A *solution of a differential equation* is a function $y(x)$ that, when substituted into the equation, turns it into an identity. The *general solution of a differential equation* is the set of all its solutions.

The general solution of this equation depends on n arbitrary constants $C_1, \dots, C_n$. In some cases, the general solution can be written in explicit form as

$$y = \varphi(x, C_1, \dots, C_n). \tag{5.1.1.2}$$

► Cauchy problem. Existence and uniqueness theorem.

The *Cauchy problem*: find a solution of equation (5.1.1.1) with the *initial conditions*

$$y(x_0) = y_0, \quad y'_x(x_0) = y_0^{(1)}, \quad \dots, \quad y_x^{(n-1)}(x_0) = y_0^{(n-1)}. \tag{5.1.1.3}$$

(At a point x_0, the values of the unknown function $y(x)$ and all its derivatives of orders $\le n-1$ are prescribed.)

EXISTENCE AND UNIQUENESS THEOREM. *Let the function $f(x, y, z_1, \dots, z_{n-1})$ be continuous in all its arguments in a neighborhood of the point $(x_0, y_0, y_0^{(1)}, \dots, y_0^{(n-1)})$ and have bounded derivatives with respect to y, z_1, …, z_{n-1} in this neighborhood. Then a solution of equation (5.1.1.1) satisfying the initial conditions (5.1.1.3) exists and is unique.*

5.1.2 Some Transformations

► **Construction of a differential equation by a given general solution.**

Suppose a general solution (5.1.1.2) of an unknown nth-order ordinary differential equation is given. The equation corresponding to the general solution can be obtained by eliminating the arbitrary constants $C_1, \dots, C_n$ from the identities

$$\begin{aligned} &y = \varphi(x, C_1, \dots, C_n),\\ &y'_x = \varphi'_x(x, C_1, \dots, C_n),\\ &\dots\dots\dots\dots\dots\dots\dots\dots\\ &y_x^{(n)} = \varphi_x^{(n)}(x, C_1, \dots, C_n), \end{aligned}$$

obtained by differentiation from formula (5.1.1.2).

► **Reduction of an nth-order equation to a system of n first-order equations.**

The differential equation (5.1.1.1) is equivalent to the following system of n first-order equations:

$$y'_1 = y_2, \quad y'_2 = y_3, \quad \dots, \quad y'_{n-1} = y_n, \quad y'_n = f(x, y_1, y_2, \dots, y_n), \tag{5.1.2.1}$$

where the notation $y_1 \equiv y$ is adopted.

The initial conditions (5.1.1.3) for equation (5.1.1.1) become the initial conditions

$$y_1(x_0) = y_0, \quad y_2(x_0) = y_0^{(1)}, \quad \dots, \quad y_n(x_0) = y_0^{(n-1)} \tag{5.1.2.2}$$

for system (5.1.2.1).

Remark 5.1. For the numerical integration of equation (5.1.1.1) and system (5.1.2.1), see Sections 7.4.2 and 5.4.1.

⊙ *Literature for Section 5.1*: G. M. Murphy (1960), L. E. El'sgol'ts (1961), N. M. Matveev (1967), I. G. Petrovskii (1970), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), W. E. Boyce and R. C. DiPrima (2004), A. D. Polyanin and A. V. Manzhirov (2007).

5.2 Equations Admitting Reduction of Order

5.2.1 Equations Not Containing y or x Explicitly

► **Equations not containing $y, y'_x, \dots, y_x^{(k)}$ explicitly.**

An equation that does not explicitly contain the unknown function and its derivatives up to order k inclusive can generally be written as

$$F\bigl(x, y_x^{(k+1)}, \dots, y_x^{(n)}\bigr) = 0 \qquad (1 \le k+1 < n). \tag{5.2.1.1}$$

Such equations are invariant under arbitrary translations of the unknown function, $y \to y + \text{const}$ (the form of such equations is also preserved under the transformation $u(x) = y + a_k x^k + \dots + a_1 x + a_0$, where the a_m are arbitrary constants). The substitution $z(x) = y_x^{(k+1)}$ reduces (5.2.1.1) to an equation whose order is by $k+1$ lower than that of the original equation, $F\bigl(x, z, z'_x, \dots, z_x^{(n-k-1)}\bigr) = 0$.

► **Equations not containing x explicitly (autonomous equations).**

An equation that does not explicitly contain x has in the general form

$$F\big(y, y'_x, \dots, y_x^{(n)}\big) = 0. \tag{5.2.1.2}$$

Such equations are invariant under arbitrary translations of the independent variable, $x \to x + \text{const}$. The substitution $y'_x = w(y)$ (where y plays the role of the independent variable) reduces by one the order of an autonomous equation. Higher derivatives can be expressed in terms of w and its derivatives with respect to the new independent variable, $y''_{xx} = ww'_y$, $y'''_{xxx} = w^2 w''_{yy} + w(w'_y)^2, \dots$

► **Related equations.**

Equations of the form

$$F\big(ax + by, y'_x, \dots, y_x^{(n)}\big) = 0$$

are invariant under simultaneous translations of the independent variable and the unknown function, $x \to x + bc$ and $y \to y - ac$, where c is an arbitrary constant.

For $b = 0$, see equation (5.2.1.1). For $b \neq 0$, the substitution $w(x) = y + (a/b)x$ leads to an autonomous equation of the form (5.2.1.2).

5.2.2 Homogeneous Equations

► **Equations homogeneous in the independent variable.**

Equations homogeneous in the independent variable are invariant under scaling of the independent variable, $x \to \alpha x$, where α is an arbitrary constant ($\alpha \neq 0$). In general, such equations can be written in the form

$$F\big(y, xy'_x, x^2 y''_{xx}, \dots, x^n y_x^{(n)}\big) = 0.$$

The substitution $z(y) = xy'_x$ reduces by one the order of this equation.

► **Equations homogeneous in the unknown function.**

Equations homogeneous in the unknown function are invariant under scaling of the unknown function, $y \to \alpha y$, where α is an arbitrary constant ($\alpha \neq 0$). Such equations can be written in the general form

$$F\big(x, y'_x/y, y''_{xx}/y, \dots, y_x^{(n)}/y\big) = 0.$$

The substitution $z(x) = y'_x/y$ reduces by one the order of this equation.

► **Equations homogeneous in both variables.**

Equations homogeneous in both variables are invariant under simultaneous scaling (dilatation) of the independent and dependent variables, $x \to \alpha x$ and $y \to \alpha y$, where α is an arbitrary constant ($\alpha \neq 0$). Such equations can be written in the general form

$$F\big(y/x, y'_x, xy''_{xx}, \dots, x^{n-1} y_x^{(n)}\big) = 0.$$

The transformation $t = \ln|x|$, $w = y/x$ leads to an autonomous equation considered in Section 5.2.1 (see Eq. (5.2.1.2)).

5.2.3 Generalized Homogeneous Equations

► **Equations of a special form.**

Generalized homogeneous equations (*equations homogeneous in the generalized sense*) are invariant under simultaneous scaling of the independent variable and the unknown function, $x \to \alpha x$ and $y \to \alpha^k y$, where $\alpha \neq 0$ is an arbitrary constant and k is a given number. Such equations can be written in the general form

$$F\big(x^{-k}y, x^{1-k}y'_x, \dots, x^{n-k}y^{(n)}_x\big) = 0.$$

The transformation $t = \ln x$, $w = x^{-k}y$ leads to an autonomous equation considered in Section 5.2.1 (see Eq. (5.2.1.2)).

► **Equations of the general form.**

The most general form of generalized homogeneous equations is

$$\mathcal{F}\big(x^n y^m, xy'_x/y, \dots, x^n y^{(n)}_x/y\big) = 0.$$

The transformation $z = x^n y^m$, $u = xy'_x/y$ reduces the order of this equation by one.

5.2.4 Equations Invariant under Scaling-Translation Transformations

► **Equations of the first type.**

The equations of the form

$$F\big(e^{\lambda x}y^m,\ y'_x/y,\ y''_{xx}/y,\ \dots,\ y^{(n)}_x/y\big) = 0$$

are invariant under the simultaneous translation and scaling of variables, $x \to x + \alpha$ and $y \to \beta y$, where $\beta = \exp(-\alpha\lambda/m)$ and α is an arbitrary constant. The transformation $z = e^{\lambda x}y^m$, $w = y'_x/y$ leads to an equation of order $n-1$.

► **Equations of the second type.**

The equations of the form

$$F\big(x^m e^{\lambda y},\ xy'_x,\ x^2y''_{xx},\ \dots,\ x^n y^{(n)}_x\big) = 0$$

are invariant under the simultaneous scaling and translation of variables, $x \to \alpha x$ and $y \to y + \beta$, where $\alpha = \exp(-\beta\lambda/m)$ and β is an arbitrary constant. The transformation $z = x^m e^{\lambda y}$, $w = xy'_x$ leads to an equation of order $n-1$.

5.2.5 Other Equations

► **Equations of the form $F\big(x, xy'_x - y, y''_{xx}, \dots, y^{(n)}_x\big) = 0$.**

The substitution $w(x) = xy'_x - y$ reduces the order of this equation by one.

This equation is a special case of the equation

$$F\big(x, xy'_x - my, y^{(m+1)}_x, \dots, y^{(n)}_x\big) = 0, \qquad \text{where} \quad m = 1,\ 2,\ \dots,\ n-1. \qquad (5.2.5.1)$$

The substitution $w(x) = xy'_x - my$ reduces by one the order of equation (5.2.5.1).

► **Nonlinear equations involving linear homogeneous differential forms.**

Consider the nonlinear differential equation

$$F\big(x, \mathrm{L}_1[y], \ldots, \mathrm{L}_k[y]\big) = 0, \tag{5.2.5.2}$$

where the $\mathrm{L}_s[y]$ are linear homogeneous differential forms,

$$\mathrm{L}_s[y] = \sum_{m=0}^{n_s} \varphi_m^{(s)}(x) y_x^{(m)}, \qquad s = 1, \ldots, k.$$

Let $y_0 = y_0(x)$ be a common particular solution of the linear equations

$$\mathrm{L}_s[y_0] = 0 \qquad (s = 1, \ldots, k).$$

Then the substitution

$$w = \psi(x)\big[y_0(x) y'_x - y'_0(x) y\big] \tag{5.2.5.3}$$

with an arbitrary function $\psi(x)$ reduces by one the order of equation (5.2.5.2).

Example 5.1. Consider the third-order equation

$$y'''_{xxx} = y + f(y'_x - y).$$

It can be represented in the form (5.2.5.2) with

$$k = 2, \quad F(x, u, w) = u - f(w), \quad u = \mathrm{L}_1[y] = y'''_{xxx} - y, \quad w = \mathrm{L}_2[y] = y'_x - y.$$

The linear equations $\mathrm{L}_k[y] = 0$ are

$$y'''_{xxx} - y = 0, \quad y'_x - y = 0.$$

These equations have a common particular solution $y_0 = e^x$. Therefore, the substitution $w = y'_x - y$ (see formula (5.2.5.3) with $\psi(x) = e^{-x}$) leads to a second-order autonomous equation: $w''_{xx} + w'_x + w = f(w)$.

Example 5.2. Consider the other third-order equation

$$x y'''_{xxx} = f(x y'_x - 2y).$$

It can be represented in the form (5.2.5.2) with

$$k = 2, \quad F(x, u, w) = xu - f(w), \quad u = \mathrm{L}_1[y] = y'''_{xxx}, \quad w = \mathrm{L}_2[y] = x y'_x - 2y.$$

The linear equations $\mathrm{L}_k[y] = 0$ are

$$y'''_{xxx} = 0, \quad x y'_x - 2y = 0.$$

These equations have a common particular solution $y_0 = x^2$. Therefore, the substitution $w = x y'_x - 2y$ (see formula (5.2.5.3) with $\psi(x) = 1/x$) leads to a second-order autonomous equation: $w''_{xx} = f(w)$. For the solution of this equation, see Example 3.1 in Section 3.2.1.

Example 5.3. The $2n$th-order equation

$$y_x^{(2n)} = f(x, y''_{xx} - y) + y \tag{5.2.5.4}$$

can be represented in the form (5.2.5.2) with

$$k = 2, \quad F(x, u, w) = u - f(x, w), \quad \mathrm{L}_1[y] = y_x^{(2n)} - y, \quad \mathrm{L}_2[y] = y''_{xx} - y.$$

Consider the linear equations

$$\mathrm{L}_1[y] \equiv y_x^{(2n)} - y = 0, \quad \mathrm{L}_2[y] \equiv y''_{xx} - y = 0. \tag{5.2.5.5}$$

There are two cases.

1°. Equations (5.2.5.5) have a common particular solution, $y_0 = e^x$. Therefore, the substitution $w = y'_x - y$ (see formula (5.2.5.3) with $\varphi(x) = e^{-x}$) takes Eq. (5.2.5.4) to an $(n-1)$st-order equation.

2°. Equations (5.2.5.5) also have another common particular solution, $y_0 = e^{-x}$. Therefore, the substitution $w = y'_x + y$ (see formula (5.2.5.3) with $\varphi(x) = e^x$) leads Eq. (5.2.5.4) to an $(n-1)$st-order equation.

Both of the above cases can be combined together. Specifically, the substitution $u = y''_{xx} - y$ reduces Eq. (5.2.5.4) to an $(n-2)$nd-order equation.

In particular, a fourth-order equation of the form

$$y_x^{(4)} = f(y''_{xx} - y) + y$$

can be reduced with the substitution $u = y''_{xx} - y$ to the second-order autonomous equation $u''_{xx} = f(u) - u$, whose general solution can be represented in implicit form (see Example 3.1).

⊙ *Literature for Section 5.2*: G. M. Murphy (1960), N. M. Matveev (1967), E. Kamke (1977), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and V. F. Zaitsev (2003), A. D. Polyanin and A. V. Manzhirov (2007).

5.3 Method for Construction of Solvable Equations of General Form

5.3.1 Description of the Method

Consider a function

$$y = f(x, C_1, C_2, \dots, C_{n+1}) \tag{5.3.1.1}$$

depending on $n+1$ free parameters C_k. Differentiating relation (5.3.1.1) n times, we obtain the following sequence of equations:

$$y_x^{(k)} = f_x^{(k)}(x, C_1, C_2, \dots, C_{n+1}), \qquad k = 1,\ 2,\ \dots,\ n. \tag{5.3.1.2}$$

Treating relations (5.3.1.1), (5.3.1.2) as an algebraic (transcendental) system of equations for the parameters C_1, C_2, $\dots$, C_{n+1} and solving this system, we obtain

$$C_k = \varphi_k\bigl(x, y, y'_x, \dots, y_x^{(n)}\bigr), \qquad k = 1,\ 2,\ \dots,\ n+1. \tag{5.3.1.3}$$

Consider a general nth-order equation of the form

$$F(\varphi_1, \varphi_2, \dots, \varphi_{n+1}) = 0, \tag{5.3.1.4}$$

where F is an arbitrary function of $(n+1)$ variables and $\varphi_k = \varphi_k\bigl(x, y, y'_x, \dots, y_x^{(n)}\bigr)$ are the functions from (5.3.1.3). Equation (5.3.1.4) is satisfied by the function (5.3.1.1), where the $(n+1)$ arbitrary parameters C_1, C_2, $\dots$, C_{n+1} are related by a single constraint:

$$F(C_1, C_2, \dots, C_{n+1}) = 0.$$

Remark 5.2. Equation (5.3.1.4) may also have singular solutions depending on a smaller number of arbitrary constants. In order to examine these solutions, one should differentiate equation (5.3.1.4); see Example 5.4.

Remark 5.3. Instead of (5.3.1.4), one can consider a more general equation

$$F(\psi_1, \psi_2, \dots, \psi_{n+1}) = 0, \qquad \text{where} \quad \psi_k = \psi_k(\varphi_1, \varphi_2, \dots, \varphi_{n+1}).$$

Remark 5.4. The original expression (5.3.1.1) can be specified in an implicit form.

Remark 5.5. The original expression (5.3.1.1) can be written as an mth-order differential equation ($m < n$) with $n-m+1$ free parameters C_k. The solution of the nth-order differential equation obtained in this way can be expressed in terms of the solution of an mth-order differential equation (see Example 5.7).

5.3.2 Illustrative Examples

Example 5.4. Consider the function

$$y = -C_1 e^{-x} + C_2. \tag{5.3.2.1}$$

By differentiation we obtain

$$y'_x = C_1 e^{-x}. \tag{5.3.2.2}$$

Let us solve equations (5.3.2.1)–(5.3.2.2) for the parameters C_1 and C_2. We have

$$C_1 = e^x y'_x, \quad C_2 = y'_x + y.$$

Using the above method, we construct an equation in accordance with (5.3.1.4):

$$F\big(e^x y'_x,\, y'_x + y\big) = 0. \tag{5.3.2.3}$$

This equation admits a solution of the form (5.3.2.1) with constants C_1 and C_2 related by the constraint $F(C_1, C_2) = 0$.

Singular solution. Differentiating equation (7) with respect to x, we get

$$(y''_{xx} + y'_x)(e^x F_u + F_v) = 0, \tag{5.3.2.4}$$

where the subscripts u and v indicate the respective partial derivatives of the function $F = F(u, v)$. Equating the first factor in (5.3.2.4) to zero, we obtain solution (5.3.2.1). Equating the second factor to zero, we obtain an expression which, combined with equation (5.3.2.3), yields a singular solution in parametric form:

$$F(u, v) = 0, \quad e^x F_u + F_v = 0, \qquad \text{where} \quad u = e^x t, \quad v = t + y.$$

One should eliminate $t = y'_x$ from these expressions.

Example 5.5. Consider the function

$$y = C_1 x^2 + C_2 x + C_3. \tag{5.3.2.5}$$

Differentiating this function twice, we get

$$\begin{aligned} y'_x &= 2C_1 x + C_2, \\ y''_{xx} &= 2C_1. \end{aligned} \tag{5.3.2.6}$$

Solving (5.3.2.5)–(5.3.2.6) for the parameters C_k, we find that

$$C_1 = \tfrac{1}{2} y''_{xx}, \quad C_2 = y'_x - x y''_{xx}, \quad C_3 = y - x y'_x + \tfrac{1}{2} x^2 y''_{xx}.$$

These relations lead to a second-order equation of general form:

$$F\big(\tfrac{1}{2} y''_{xx},\, y'_x - x y''_{xx},\, y - x y'_x + \tfrac{1}{2} x^2 y''_{xx}\big) = 0,$$

which has a solution of the type (5.3.2.5) with the three constants C_1, C_2, and C_3 related by the constraint $F(C_1, C_2, C_3) = 0$.

Example 5.6. In Example 5.5, one can choose the functions ψ_k of the form (see Remark 5.3)

$$\psi_1 = 2\varphi_1, \quad \psi_2 = -\varphi_2, \quad \psi_3 = 4\varphi_1\varphi_3 - \varphi_2^2,$$

where $\varphi_1 = \frac{1}{2}y''_{xx}$, $\varphi_2 = y'_x - xy''_{xx}$, $\varphi_3 = y - xy'_x + \frac{1}{2}x^2 y''_{xx}$. As a result, we obtain the differential equation:

$$\mathcal{F}\big(y''_{xx},\, xy''_{xx} - y'_x,\, 2yy''_{xx} - (y'_x)^2\big) = 0.$$

Its solution is given by (5.3.2.5) with three constants C_1, C_2, and C_3 related by a single constraint $\mathcal{F}(2C_1, -C_2, 4C_1C_3 - C_3^2) = 0$.

Example 5.7. Consider the autonomous equation

$$y''_{xx} = C_1 y^{-a} + C_2. \tag{5.3.2.7}$$

Its solution can be represented in implicit form (see Example 3.1 and Eq. 14.9.1.1). Differentiating (5.3.2.7), we obtain

$$y'''_{xxx} = -aC_1 y^{-a-1} y'_x. \tag{5.3.2.8}$$

Let us solve equations (5.3.2.7)–(5.3.2.8) for the parameters C_1 and C_2:

$$C_1 = -y^{a+1}\frac{y'''_{xxx}}{ay'_x}, \quad C_2 = y''_{xx} + y\frac{y'''_{xxx}}{ay'_x}.$$

Taking $\psi_1 = -a\varphi_1$ and $\psi_2 = a\varphi_2$ (see Remark 5.3), we obtain the equation:

$$F\left(y^{a+1}\frac{y'''_{xxx}}{y'_x},\, y\frac{y'''_{xxx}}{y'_x} + ay''_{xx}\right) = 0.$$

This equation is satisfied by the solutions of a second-order autonomous equation of the form (5.3.2.7), where the constants C_1 and C_2 are related by the constraint $F(-aC_1, aC_2) = 0$.

⊙ *Literature for Section 5.3*: A. D. Polyanin and V. F. Zaitsev (2003).

5.4 Numerical Integration of n-order Equations

5.4.1 Numerical Solution of the Cauchy Problem for n-order ODEs

The Cauchy problem for the nth-order equation (5.1.1.1) subject to the initial conditions (5.1.1.3) is solved numerically in two stages. First, equation (5.1.1.1) is reduced to the equivalent system of n first-order equations (5.1.2.1) with the initial conditions (5.1.2.2). In the second stage, the resulting system (5.1.2.1) is integrated numerically with standard methods outlined in Section 7.4.2.

5.4.2 Numerical Solution of Equations Defined Implicitly or Parametrically

► **Numerical integration of equations defined parametrically.**

Below we describe a numerical method for solving the Cauchy problem for the n-order equation represented in parametric form by two relations

$$y_x^{(m)} = F(x, y, y'_x, \ldots, y_x^{(m-1)}, t), \quad y_x^{(n)} = G(x, y, y'_x, \ldots, y_x^{(n-1)}, t), \quad m < n, \tag{5.4.2.1}$$

subject to the initial conditions (5.1.1.3), with t being a functional parameter.

We start directly from equations (5.4.2.1). Consider two auxiliary Cauchy problems

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(m-1)}, t),$$
$$y(x_0) = y_0,\ y'_x(x_0) = y_0^{(1)},\ \dots,\ y_x^{(m-1)}(x_0) = y_0^{(m-1)} \quad \text{(1st problem)}; \qquad (5.4.2.2)$$
$$y_x^{(n)} = G(x, y, y'_x, \dots, y_x^{(n-1)}, t),$$
$$y(x_0) = y_0,\ y'_x(x_0) = y_0^{(1)},\ \dots,\ y_x^{(n-1)}(x_0) = y_0^{(n-1)} \quad \text{(2nd problem)}. \qquad (5.4.2.3)$$

Let $y_F = y_F(x,t)$ and $y_G = y_G(x,t)$ denote their respective solutions. Introduce the difference of these solutions

$$\Delta(x,t) = y_G(x,t) - y_F(x,t). \qquad (5.4.2.4)$$

By fixing a value of the parameter, $t = t_k$, we compute the solutions $y_F(x,t_k)$ and $y_G(x,t_k)$ using, for example, the Runge–Kutta method. Further, by varying x, we find a x_k at which the right-hand side of equation (5.4.2.3) becomes zero: $\Delta(x_k,t_k) = 0$. To this x_k there corresponds the value $y_k = y_F(x_k,t_k) = y_G(x_k,t_k)$. Thus, to each t_k there corresponds a point (x_k,y_k) in the (x,y) plane. By choosing a different value t_{k+1}, we find a new point (x_{k+1},y_{k+1}). The combination of discrete points (x_k,y_k) with $k = 0, 1, 2, \dots$ determines an approximation to the solution $y = y(x)$ of the original problem (5.4.2.1), (5.1.1.3).

The initial value of the parameter, $t = t_0$, is determined from the algebraic (or transcendental) equation

$$y_0^{(m)} = F(x_0, y_0, y_0^{(1)}, \dots, y_0^{(m-1)}, t_0), \qquad (5.4.2.5)$$

where x_0, y_0, $y_0^{(1)}$, $\dots$, $y_0^{(m)}$ are the values appearing in the initial conditions (5.4.2.2)–(5.4.2.3), obtained from (5.1.1.3)).

Remark 5.6. In general, the algebraic (or transcendental) equation (5.4.2.5) can have one, two, or more different roots, in which case the original problem (5.4.2.1), (5.1.1.3) will have the same number of different solutions.

► Numerical integration of equations defined implicitly.

Consider the Cauchy problem for the implicitly defined equation

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(n-1)}, y_x^{(n)}), \qquad m < n \qquad (5.4.2.6)$$

subject to the initial conditions (5.1.1.3).

The substitution $y_x^{(n)} = t$ reduces equation (5.4.2.6) to the parametric equation

$$y_x^{(m)} = F(x, y, y'_x, \dots, y_x^{(n-1)}, t), \qquad y_x^{(n)} = t \qquad (5.4.2.7)$$

with the initial conditions (5.1.1.3).

Problem (5.4.2.7), (5.1.1.3) is a special case of problem (5.4.2.1), (5.1.1.3) in which $G(\dots) = t$; hence, the above method is suitable for its solution.

⊙ *Literature for Section 5.4*: N. N. Kalitkin (1978), J. C. Butcher (1987), E. Hairer, C. Lubich, and M. Roche (1989), W. E. Schiesser (1994), L. F. Shampine (1994), K. E. Brenan, S. L. Campbell, and L. R. Petzold (1996), J. R. Dormand (1996), E. Hairer and G. Wanner (1996), D. Zwillinger (1997), U. M. Ascher and L. R. Petzold (1998), G. A. Korn and T. M. Korn (2000), P. J. Rabier and W. C. Rheinboldt (2002), S. C. Chapra and R. P. Canale (2010), A. D. Polyanin (2016b).

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