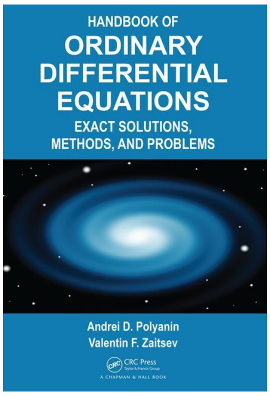

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

Andrei D. Polyanin, Valentin F. Zaitsev

Chapter 6: Methods for linear systems of ODEs

Publication details
https://test.routledgehandbooks.com/doi/10.1201/9781315117638-6
Andrei D. Polyanin, Valentin F. Zaitsev
Published online on: 03 Nov 2017

Chapter 6

Methods for Linear Systems of ODEs

6.1 Systems of Linear Constant-Coefficient Equations

6.1.1 Systems of First-Order Linear Homogeneous Equations. General Solution

1°. In general, a homogeneous linear system of constant-coefficient first-order ordinary differential equations has the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n, \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n, \end{aligned} \tag{6.1.1.1}$$

where a prime stands for the derivative with respect to x. In the sequel, all the coefficients a_{ij} of the system are assumed to be real numbers.

The homogeneous system (6.1.1.1) has the trivial particular solution $y_1 = y_2 = \cdots = y_n = 0$.

Superposition principle for a homogeneous system: any linear combination of particular solutions of system (6.1.1.1) is also a solution of this system.

The general solution of the system of differential equations (6.1.1.1) is the sum of its n linearly independent (nontrivial) particular solutions multiplied by an arbitrary constant.

System (6.1.1.1) can be reduced to a single homogeneous linear constant-coefficient nth-order equation; see Section 7.1.3.

2°. For brevity (and clearness), system (6.1.1.1) is conventionally written in vector-matrix form:

$$\mathbf{y}' = \mathbf{a}\mathbf{y}, \tag{6.1.1.2}$$

where $\mathbf{y} = (y_1, y_2, \dots, y_n)^{\mathrm{T}}$ is the column vector of the unknowns and $\mathbf{a} = (a_{ij})$ is the matrix of the equation coefficients. The superscript T denotes the transpose of a matrix or a vector. So, for example, a row vector is converted into a column vector:

$$(y_1, y_2)^{\mathrm{T}} \equiv \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

The right-hand side of equation (6.1.1.2) is the product of the $n \times n$ square matrix **a** by the $n \times 1$ matrix (column vector) **y**.

Let $\mathbf{y}_k = (y_{k1}, y_{k2}, \dots, y_{kn})^{\mathrm{T}}$ be linearly independent particular solutions* of the homogeneous system (6.1.1.1), where $k = 1, 2, \dots, n$; the first subscript in $y_{km} = y_{km}(x)$ denotes the number of the solution and the second subscript indicates the component of the vector solution. Then the general solution of the homogeneous system (6.1.1.2) is expressed as

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + \cdots + C_n\mathbf{y}_n. \qquad (6.1.1.3)$$

A method for the construction of particular solutions that can be used to obtain the general solution by formula (6.1.1.3) is presented below.

6.1.2 Systems of First-Order Linear Homogeneous Equations. Particular Solutions

Particular solutions to system (6.1.1.1) are determined by the roots of the characteristic equation

$$\Delta(\lambda) = 0, \quad \text{where} \quad \Delta(\lambda) \equiv \begin{vmatrix} a_{11} - \lambda & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} - \lambda & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} - \lambda \end{vmatrix}. \qquad (6.1.2.1)$$

The following cases are possible:

1°. Let $\lambda = \lambda_k$ be a simple real root of the characteristic equation (6.1.2.1). The corresponding particular solution of the homogeneous linear system of equations (6.1.1.1) has the exponential form

$$y_1 = A_1 e^{\lambda x}, \quad y_2 = A_2 e^{\lambda x}, \quad \dots, \quad y_n = A_n e^{\lambda x}, \qquad (6.1.2.2)$$

where the coefficients $A_1, A_2, \dots, A_n$ are determined by solving the associated homogeneous system of algebraic equations obtained by substituting expressions (6.1.2.2) into the differential equation (6.1.1.1) and dividing by $e^{\lambda x}$:

$$\begin{aligned} &(a_{11} - \lambda)A_1 + a_{12}A_2 + \cdots + a_{1n}A_n = 0, \\ &a_{21}A_1 + (a_{22} - \lambda)A_2 + \cdots + a_{2n}A_n = 0, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ &a_{n1}A_1 + a_{n2}A_2 + \cdots + (a_{nn} - \lambda)A_n = 0. \end{aligned} \qquad (6.1.2.3)$$

The solution of this system is unique to within a constant factor.

If all roots of the characteristic equation $\lambda_1, \lambda_2, \dots, \lambda_n$ are real and distinct, then the general solution of system (6.1.1.1) has the form

$$\begin{aligned} y_1 &= C_1 A_{11} e^{\lambda_1 x} + C_2 A_{12} e^{\lambda_2 x} + \cdots + C_n A_{1n} e^{\lambda_n x}, \\ y_2 &= C_1 A_{21} e^{\lambda_1 x} + C_2 A_{22} e^{\lambda_2 x} + \cdots + C_n A_{2n} e^{\lambda_n x}, \\ &\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ y_n &= C_1 A_{n1} e^{\lambda_1 x} + C_2 A_{n2} e^{\lambda_2 x} + \cdots + C_n A_{nn} e^{\lambda_n x}, \end{aligned} \qquad (6.1.2.4)$$

*This means that the condition $\det|y_{mk}(x)| \neq 0$ holds.

where C_1, C_2, ..., C_n are arbitrary constants. The second subscript in A_{mk} indicates a coefficient corresponding to the root λ_k.

2°. For each simple complex root, $\lambda = \alpha \pm i\beta$, of the characteristic equation (6.1.2.1), the corresponding particular solution is obtained in the same way as in the simple real root case; the associated coefficients A_1, A_2, ..., A_n in (6.1.2.2) will be complex. Separating the real and imaginary parts in (6.1.2.2) results in two real particular solutions to system (6.1.1.1); the same two solutions are obtained if one takes the complex conjugate root, $\bar{\lambda} = \alpha - i\beta$.

3°. Let λ be a real root of the characteristic equation (6.1.2.1) of multiplicity m. The corresponding particular solution of system (6.1.1.1) is sought in the form

$$y_1 = P^1_m(x)e^{\lambda x}, \quad y_2 = P^2_m(x)e^{\lambda x}, \quad \dots, \quad y_n = P^n_m(x)e^{\lambda x}, \tag{6.1.2.5}$$

where the $P^k_m(x) = \sum\limits_{i=0}^{m-1} B_{ki}x^i$ are polynomials of degree $m-1$. The coefficients of these polynomials result from the substitution of expressions (6.1.2.5) into equations (6.1.1.1); after dividing by $e^{\lambda x}$ and collecting like terms, one obtains n equations, each representing a polynomial equated to zero. By equating the coefficients of all resulting polynomials to zero, one arrives at a linear algebraic system of equations for the coefficients B_{ki}; the solution to this system will contain m free parameters.

4°. For a multiple complex, $\lambda = \alpha + i\beta$, of multiplicity m, the corresponding particular solution is sought, just as in the case of a multiple real root, in the form (6.1.2.5); here the coefficients B_{ki} of the polynomials $P^k_m(x)$ will be complex. Finally, in order to obtain real solutions of the original system (6.1.1.1), one separates the real and imaginary parts in formulas (6.1.2.5), thus obtaining two particular solutions with m free parameters each. The two solutions correspond to the complex conjugate roots $\lambda = \alpha \pm i\beta$.

5°. In the general case, where the characteristic equation (6.1.2.1) has simple and multiple, real and complex roots (see Items 1°–4°), the general solution to system (6.1.1.1) is obtained as the sum of all particular solutions multiplied by arbitrary constants.

Example 6.1. Consider the homogeneous system of two linear differential equations

$$y_1' = y_1 + 4y_2,$$
$$y_2' = y_1 + y_2.$$

The associated characteristic equation,

$$\begin{vmatrix} 1-\lambda & 4 \\ 1 & 1-\lambda \end{vmatrix} = \lambda^2 - 2\lambda - 3 = 0,$$

has two distinct real roots:

$$\lambda_1 = 3, \qquad \lambda_2 = -1.$$

The system of algebraic equations (6.1.2.3) for solution coefficients becomes

$$\begin{aligned} (1-\lambda)A_1 + 4A_2 &= 0, \\ A_1 + (1-\lambda)A_2 &= 0. \end{aligned} \tag{6.1.2.6}$$

Substituting the first root, $\lambda = 3$, into system (6.1.2.6) yields $A_1 = 2A_2$. We can set $A_1 = 2$ and $A_2 = 1$, since the solution is determined to within a constant factor. Thus the first particular solution of the homogeneous system of linear ordinary differential equations (6.1.2.6) has the form

$$y_1 = 2e^{3x}, \qquad y_2 = e^{3x}. \tag{6.1.2.7}$$

The second particular solution, corresponding to $\lambda = -1$, is found in the same way:

$$y_1 = -2e^{-x}, \quad y_2 = e^{-x}. \tag{6.1.2.8}$$

The sum of the two particular solutions (6.1.2.7), (6.1.2.8) multiplied by arbitrary constants, C_1 and C_2, gives the general solution to the original homogeneous system of linear ordinary differential equations:

$$y_1 = 2C_1e^{3x} - 2C_2e^{-x}, \quad y_2 = C_1e^{3x} + C_2e^{-x}.$$

Example 6.2. Consider the system of ordinary differential equations

$$\begin{aligned} y_1' &= -y_2, \\ y_2' &= 2y_1 + 2y_2. \end{aligned} \tag{6.1.2.9}$$

The characteristic equation,

$$\begin{vmatrix} -\lambda & -1 \\ 2 & 2-\lambda \end{vmatrix} = \lambda^2 - 2\lambda + 2 = 0$$

has complex conjugate roots:

$$\lambda_1 = 1 + i, \quad \lambda_2 = 1 - i.$$

The algebraic system (6.1.2.3) for the complex coefficients A_1 and A_2 becomes

$$\begin{aligned} -\lambda A_1 - A_2 &= 0, \\ 2A_1 + (2-\lambda)A_2 &= 0. \end{aligned}$$

With $\lambda = 1 + i$, one nonzero solution is given by $A_1 = 1$ and $A_2 = -1 - i$. The corresponding complex solution to system (6.1.2.9) has the form

$$y_1 = e^{(1+i)x}, \quad y_2 = (-1-i)e^{(1+i)x}.$$

Separating the real and imaginary parts, taking into account the formulas

$$\begin{aligned} e^{(1+i)x} &= e^x(\cos x + i\sin x) = e^x\cos x + ie^x\sin x, \\ (-1-i)e^{(1+i)x} &= -(1+i)e^x(\cos x + i\sin x) = e^x(\sin x - \cos x) - ie^x(\sin x + \cos x), \end{aligned}$$

and making linear combinations from them, one arrives at the general solution to the original system (6.1.2.9):

$$\begin{aligned} y_1 &= C_1e^x\cos x + C_2e^x\sin x, \\ y_2 &= C_1e^x(\sin x - \cos x) - C_2e^x(\sin x + \cos x). \end{aligned}$$

Remark 6.1. Systems of two homogeneous linear constant-coefficient second-order differential equations are treated in detail in Section 6.1.8.

6.1.3 Nonhomogeneous Systems of Linear First-Order Equations

1°. In general, a nonhomogeneous linear system of constant-coefficient first-order ordinary differential equations has the form

$$\begin{aligned} y_1' &= a_{11}y_1 + a_{12}y_2 + \cdots + a_{1n}y_n + f_1(x), \\ y_2' &= a_{21}y_1 + a_{22}y_2 + \cdots + a_{2n}y_n + f_2(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n' &= a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{nn}y_n + f_n(x). \end{aligned} \tag{6.1.3.1}$$

For brevity, the conventional vector notation will also be used:

$$\mathbf{y}' = \mathbf{a}\mathbf{y} + \mathbf{f}(x),$$

where $\mathbf{f}(x) = (f_1(x), f_2(x), \dots, f_n(x))^{\mathrm{T}}$.

The general solution of this system is the sum of the general solution to the corresponding homogeneous system with $f_k(x) \equiv 0$ [see system (6.1.1.1)] and any particular solution of the nonhomogeneous system (6.1.3.1).

System (6.1.3.1) can also be reduced to a single nonhomogeneous linear constant-coefficient nth-order equation; see Section 7.1.3.

2°. Let $\mathbf{y}_m = (D_{m1}(x), D_{m1}(x), \dots, D_{mn}(x))^{\mathrm{T}}$ represent particular solutions to the homogeneous linear system of constant-coefficient first-order ordinary differential equations (6.1.1.1) that satisfy the special initial conditions

$$y_m(0) = 1, \quad y_k(0) = 0 \quad \text{if} \quad k \neq m; \qquad m,\, k = 1,\, \dots,\, n.$$

Then the general solution to the nonhomogeneous system (6.1.3.1) is expressed as

$$y_k(x) = \sum_{m=1}^{n} \int_0^x f_m(t) D_{mk}(x-t)\, dt + \sum_{m=1}^{n} C_m D_{mk}(x), \qquad k = 1, \dots, n. \tag{6.1.3.2}$$

Alternatively, the general solution to the nonhomogeneous linear system of equations (6.1.3.1) can be obtained using the formulas from Section 6.2.2.

The solution of the Cauchy problem for the nonhomogeneous system (6.1.3.1) with arbitrary initial conditions,

$$y_1(0) = y_1^\circ, \quad y_2(0) = y_2^\circ, \quad \dots, \quad y_n(0) = y_n^\circ, \tag{6.1.3.3}$$

is determined by formulas (6.1.3.2) with $C_m = y_m^\circ$, $m = 1, \dots, n$.

6.1.4 Homogeneous Linear Systems of Higher-Order Differential Equations

An arbitrary system of homogeneous linear systems of constant-coefficient ordinary differential equations consists of n equations, each representing a linear combination of unknowns, y_k, and their derivatives, y_k', y_k'', $\dots$, $y_k^{(m_k)}$, $k = 1,\, 2,\, \dots,\, n$.

The general solution of such systems is a linear combination of particular solutions multiplied by arbitrary constants. In total, such a system has $m_1 + m_2 + \dots + m_n$ linearly independent particular solutions (the system is assumed to be consistent and nondegenerate, so that the constituent equations are linearly independent).

Particular solutions of the system are sought in the form (6.1.2.2). On substituting these expressions into the differential equations and dividing by $e^{\lambda x}$, one obtains a homogeneous linear algebraic system for coefficients A_1, A_2, $\dots$, A_n. For this system to have nontrivial solutions, the determinant of the system must vanish. This results in an algebraic equation for the exponent λ; in physics, this equation is called a *dispersion equation*. To different roots of the dispersion equation there correspond different particular solutions of the original system of equations. For simple real and complex-conjugate roots, the procedure of finding particular solutions is the same as in the case of a linear system of first-order equations (6.1.1.1).

Example 6.3. Consider the linear system of constant-coefficient second-order equations

$$\begin{aligned} y_1'' + y_2' + ay_2 &= 0, \\ y_2'' + y_1' + ay_1 &= 0. \end{aligned} \tag{6.1.4.1}$$

Particular solutions are sought in the form

$$y_1 = A_1 e^{\lambda x}, \quad y_2 = A_2 e^{\lambda x}. \tag{6.1.4.2}$$

Substituting (6.1.4.2) into (6.1.4.1) yields a homogeneous linear algebraic system for the coefficients A_1 and A_2:

$$\begin{aligned} \lambda^2 A_1 + (\lambda + a)A_2 &= 0, \\ (\lambda + a)A_1 + \lambda^2 A_2 &= 0. \end{aligned} \tag{6.1.4.3}$$

For this system to have nontrivial solutions, its determinant must vanish. This results in the dispersion equation

$$\begin{vmatrix} \lambda^2 & \lambda + a \\ \lambda + a & \lambda^2 \end{vmatrix} = \lambda^4 - (\lambda + a)^2 = 0.$$

Its roots are

$$\lambda_{1,2} = \tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} + a}, \quad \lambda_{3,4} = -\tfrac{1}{2} \pm \sqrt{\tfrac{1}{4} - a}. \tag{6.1.4.4}$$

Let us confine ourselves to the simplest case of $-\frac{1}{4} < a < \frac{1}{4}$, where all roots of the dispersion equation are real and distinct. It follows from the system of algebraic equations (6.1.4.3) that $A_1 = \lambda + a$ and $A_2 = -\lambda^2$, where $\lambda = \lambda_n$. Substituting these values into (6.1.4.2) yields the particular solutions $y_{1n} = (\lambda_n + a)e^{\lambda_n x}$, $y_{2n} = -\lambda_n^2 e^{\lambda_n x}$ $(n = 1, 2, 3, 4)$. A linear combination of the particular solutions gives the general solution of system (6.1.4.1):

$$\begin{aligned} y_1 &= C_1(\lambda_1 + a)e^{\lambda_1 x} + C_2(\lambda_2 + a)e^{\lambda_2 x} + C_3(\lambda_3 + a)e^{\lambda_3 x} + C_4(\lambda_4 + a)e^{\lambda_4 x}, \\ y_2 &= -C_1\lambda_1^2 e^{\lambda_1 x} - C_2\lambda_2^2 e^{\lambda_2 x} - C_3\lambda_3^2 e^{\lambda_3 x} - C_4\lambda_4^2 e^{\lambda_4 x}, \end{aligned}$$

where C_1, C_2, C_3, and C_4 are arbitrary constants, and the roots λ_n are determined by formulas (6.1.4.4).

Remark 6.2. Section 6.1.7 (see Item 2°) presents a method for the solution of systems of arbitrary homogeneous linear constant-coefficients ordinary differential equations using the Laplace transform.

6.1.5 Normal Coordinates and Natural Oscillations

Small undamped oscillations of mechanical or electrical systems are often described by a system of n linear constant-coefficient ordinary differential equations of the second order

$$\sum_{k=1}^{n} \bigl(b_{jk} y_k'' + a_{jk} y_k\bigr) = 0 \qquad (j = 1,\, 2,\, \dots,\, n). \tag{6.1.5.1}$$

Both matrices, (a_{jk}) and (b_{jk}), are symmetric and positive definite; in addition, they have the property that the characteristic equation, obtained by substituting a solution of the form (6.1.2.2) into (6.1.5.1), has $2n$ different nonzero pure imaginary roots: $\pm i\omega_1$, $\pm i\omega_2$, …, $\pm i\omega_n$.

System (6.1.5.1) can be simplified with so-called *normal coordinates* $\bar{y}_1$, $\bar{y}_2$, …, $\bar{y}_n$ using a linear transformation of the form

$$y_k = \sum_{m=1}^{n} c_{km} \bar{y}_m \qquad (k = 1,\, 2,\, \dots,\, n), \tag{6.1.5.2}$$

with the coefficients c_{km} chosen so as to reduce both matrices, (a_{jk}) and (b_{jk}), to a diagonal form simultaneously. As a result, system (6.1.5.1) becomes

$$\bar{y}_m'' + \omega_m^2 \bar{y}_m = 0 \qquad (m = 1,\, 2,\, \dots,\, n), \tag{6.1.5.3}$$

where all of the equations are isolated and independent of one another. The general solution of system (6.1.5.3) can be written as

$$\bar{y}_m = A_m \cos(\omega_m x) + B_m \sin(\omega_m x) \qquad (m = 1,\, 2,\, \dots,\, n), \tag{6.1.5.4}$$

where A_m and B_m are arbitrary constants.

◆ *Very often, normal coordinates have a clear physical meaning. For details of the method for determining the coefficients c_{km} in the transformation (6.1.5.2), see, for example, the handbooks by Korn & Korn (2000) and Polyanin & Cheroutsan (2011).*

Example 6.4. Let us look at the system

$$\begin{aligned} y_1'' + \omega^2 y_1 + \sigma^2(y_1 - y_2) &= 0, \\ y_2'' + \omega^2 y_2 - \sigma^2(y_1 - y_2) &= 0. \end{aligned} \tag{6.1.5.5}$$

It is not difficult to show that

$$\bar{y}_1 = y_1 + y_2, \qquad \bar{y}_2 = y_1 - y_2$$

are normal coordinates for this system. In terms of the normal coordinates, the system becomes

$$\bar{y}_1'' + \omega^2 y_1 = 0, \qquad \bar{y}_2'' + (\omega^2 + 2\sigma^2) y_2 = 0. \tag{6.1.5.6}$$

Under the initial conditions

$$y_1(0) = 1, \quad y_2(0) = 0, \quad y_1'(0) = y_2'(0) = 0,$$

which are equivalent to

$$\bar{y}_1(0) = \bar{y}_2(0) = 1, \quad \bar{y}_1'(0) = \bar{y}_2'(0) = 0$$

in the normal coordinates, the solution of (6.1.5.6) is

$$\bar{y}_1 = \cos(\omega x), \qquad \bar{y}_2 = \cos\bigl(\sqrt{\omega^2 + 2\sigma^2}\, x\bigr).$$

Consequently, the solution of the original system (6.1.5.5) is expressed as

$$\begin{gathered} y_1 = \tfrac{1}{2}(\bar{y}_1 + \bar{y}_2) = \cos(px)\cos(qx), \qquad y_2 = \tfrac{1}{2}(\bar{y}_1 - \bar{y}_2) = \sin(px)\sin(qx), \\ p = \tfrac{1}{2}\bigl(\sqrt{\omega^2 + 2\sigma^2} + \omega\bigr), \qquad q = \tfrac{1}{2}\bigl(\sqrt{\omega^2 + 2\sigma^2} - \omega\bigr). \end{gathered}$$

6.1.6 Nonhomogeneous Higher-Order Linear Systems. D’Alembert’s Method

Consider the system of two linear constant-coefficient mth-order differential equations

$$\begin{aligned} y_1^{(m)} &= a_{11} y_1 + a_{12} y_2 + f_1(x), \\ y_2^{(m)} &= a_{21} y_1 + a_{22} y_2 + f_2(x). \end{aligned} \tag{6.1.6.1}$$

Let us multiply the second equation of system (6.1.6.1) by k and add it termwise to the first equation to obtain, after rearrangement,

$$(y_1 + k y_2)^{(m)} = (a_{11} + k a_{21})\left(y_1 + \frac{a_{12} + k a_{22}}{a_{11} + k a_{21}} y_2\right) + f_1(x) + k f_2(x). \tag{6.1.6.2}$$

Let us take the constant k so that $\dfrac{a_{12}+ka_{22}}{a_{11}+ka_{21}}=k$, which results in a quadratic equation for k:

$$a_{21}k^2+(a_{11}-a_{22})k-a_{12}=0. \tag{6.1.6.3}$$

In this case, (6.1.6.2) is a nonhomogeneous linear constant-coefficient equation for $z=y_1+ky_2$:

$$z^{(m)}=(a_{11}+ka_{21})z+f_1(x)+kf_2(x).$$

Integrating this equation yields

$$y_1+ky_2=C_1\varphi_1(x,k)+\cdots+C_m\varphi_m(x,k)+\psi(x,k).$$

It follows that if the roots of the quadratic equation (6.1.6.3) are distinct, we have two relations,

$$\begin{aligned} y_1+k_1y_2&=C_1\varphi_1(x,k_1)+\cdots+C_m\varphi_m(x,k_1)+\psi(x,k_1),\\ y_1+k_2y_2&=C_{m+1}\varphi_1(x,k_2)+\cdots+C_{2m}\varphi_m(x,k_2)+\psi(x,k_2), \end{aligned}$$

which represent a linear algebraic system of equations for the functions y_1 and y_2.

Remark 6.3. The above method for the solution of system (6.1.6.1) is known as *D Alembert's method*. The quantity $z=y_1+ky_2$ in the above reasoning gives an example of an integrable combination (see Section 7.2.1).

Remark 6.4. The more complicated system where y_1 and y_2 on the right-hand side are replaced by the derivatives of the same order, $y_1^{(n)}$ and $y_2^{(n)}$, can be treated likewise.

Remark 6.5. System (6.1.6.1) can be solved using the Laplace transform (see Section 6.1.7).

6.1.7 Usage of the Laplace Transform for Solving Linear Systems of Equations

1°. To solve the Cauchy problem for the nonhomogeneous linear system of differential equations (6.1.3.1) with the initial conditions (6.1.3.3), one can use the Laplace transform, based on the following formulas (for details, see Section 4.3):

$$\widetilde{y}_k(p)=\mathfrak{L}\{y_k(x)\},\quad \widetilde{f}_k(p)=\mathfrak{L}\{f_k(x)\},\quad \text{where}\quad \mathfrak{L}\{f(x)\}\equiv\int_0^\infty e^{-px}f(x)\,dx.$$

To this end, one should multiply each equation in (6.1.3.1) by e^{-px} and then integrate with respect to x from zero to infinity. In view of the differentiation rule $\mathfrak{L}\{y_k'(x)\}=p\widetilde{y}_k(p)-y_k(0)$ and the initial conditions (6.1.3.3), one arrives at a nonhomogeneous linear system of algebraic equations for the transforms $\widetilde{y}_k(p)$:

$$\begin{aligned} (a_{11}-p)\widetilde{y}_1+a_{12}\widetilde{y}_2+\cdots+a_{1n}\widetilde{y}_n&=-\widetilde{f}_1(p)-y_1^\circ,\\ a_{21}\widetilde{y}_1+(a_{22}-p)\widetilde{y}_2+\cdots+a_{2n}\widetilde{y}_n&=-\widetilde{f}_2(p)-y_2^\circ,\\ &\cdots\cdots\cdots\\ a_{n1}\widetilde{y}_1+a_{n2}\widetilde{y}_2+\cdots+(a_{nn}-p)\widetilde{y}_n&=-\widetilde{f}_n(p)-y_n^\circ. \end{aligned} \tag{6.1.7.1}$$

The solution of this system is obtained by Kramer's rule and is given by

$$\widetilde{y}_k=\frac{\Delta_k(p)}{\Delta(p)};\qquad k=1,\ldots,n, \tag{6.1.7.2}$$

where $\Delta(p)$ is the determinant of the basic matrix of system (6.1.7.1), coinciding with the determinant in (6.1.2.1) with $\lambda = p$, and $\Delta_k(p)$ is the determinant of the matrix obtained from the basic matrix by replacing its kth column with the column of free terms of system (6.1.7.1).

On applying the Laplace inversion formula (see Section 4.3.1) to (6.1.7.2), one obtains a solution to the Cauchy problem (6.1.3.1), (6.1.3.3) in the form

$$y_k(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{\Delta_k(p)}{\Delta(p)} e^{px}\,dp; \qquad k = 1, \ldots, n.$$

The formulas from Section 4.3.2 and tables from Section S3.2 can be used to find the inverse Laplace transform of the function $\Delta_k(p)/\Delta(p)$.

2°. The Laplace transform is also suitable for the solution of systems of second- and higher-order ordinary differential equations with constant coefficients.

Example 6.5. Consider the Cauchy problem for the nonhomogeneous linear system of constant-coefficient second-order differential equations

$$\sum_{k=1}^{n}\bigl(a_{mk}y_k'' + b_{mk}y_k' + c_{mk}y_k\bigr) = f_m(x), \qquad m = 1,\, 2,\, \ldots,\, n,$$

subject to the initial conditions

$$y_k(0) = \alpha_k, \quad y_k'(0) = \beta_k, \qquad k = 1,\, 2,\, \ldots,\, n.$$

The Laplace transform reduces this problem to a linear system of algebraic equations for the transform $\widetilde{y}_k(p)$:

$$\sum_{k=1}^{n}\bigl(a_{mk}p^2 + b_{mk}p + c_{mk}\bigr)\widetilde{y}_k(p) = \widetilde{f}_m(p) + \sum_{k=1}^{n}\bigl[(a_{mk}p + b_{mk})\alpha_k + \beta_k\bigr], \qquad m = 1,\, 2,\, \ldots,\, n.$$

The solution to this system can be obtained using Kramer’s rule. By applying then the inverse Laplace transform to the resulting expressions of $\widetilde{y}_k(p)$, one obtains the solution to the Cauchy problem.

6.1.8 Classification of Equilibrium Points of Two-Dimensional Linear Systems

► **Two linear constant-coefficient coupled equations. Characteristic equation.**

Let us study the behavior of solutions near the equilibrium (also called stationary, steady-state, or fixed) point $x = y = 0$ for the system of two homogeneous linear constant-coefficient equations

$$\begin{aligned} x_t' &= a_{11}x + a_{12}y,\\ y_t' &= a_{21}x + a_{22}y. \end{aligned} \tag{6.1.8.1}$$

By convention, for clearness and convenience of interpretation of the results, t will be used to designate the independent variable and will be treated as time. A solution $x = x(t)$, $y = y(t)$ of system (6.1.8.1) plotted in the plane x, y (the phase plane) is called a (phase) trajectory of the system.

A solution to system (6.1.8.1) will be sought in the form

$$x = k_1 e^{\lambda t}, \qquad y = k_2 e^{\lambda t}. \tag{6.1.8.2}$$

On substituting (6.1.8.2) into (6.1.8.1), one obtains the characteristic equation for the exponent λ:

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0, \quad \text{or} \quad \lambda^2 - (a_{11}+a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0. \tag{6.1.8.3}$$

The coefficients k_1 and k_2 are found as

$$k_1 = Ca_{12}, \qquad k_2 = C(\lambda - a_{11}), \tag{6.1.8.4}$$

where C is an arbitrary constant. To two different roots of the quadratic equation (6.1.8.3) there correspond two pairs of coefficients (6.1.8.4).

▶ Discriminant of the characteristic equation. Classification of equilibrium points.

Denote the discriminant of the quadratic equation (6.1.8.3) by

$$D = (a_{11} - a_{22})^2 + 4a_{12}a_{21}. \tag{6.1.8.5}$$

Three situations are possible.

1°. If $D > 0$, the roots of the characteristic equation (6.1.8.3) are real and distinct ($\lambda_1 \neq \lambda_2$):

$$\lambda_{1,2} = \tfrac{1}{2}(a_{11}+a_{22}) \pm \tfrac{1}{2}\sqrt{D}.$$

The general solution of system (6.1.8.1) is expressed as

$$\begin{aligned} x &= C_1 a_{12} e^{\lambda_1 t} + C_2 a_{12} e^{\lambda_2 t}, \\ y &= C_1(\lambda_1 - a_{11}) e^{\lambda_1 t} + C_2(\lambda_2 - a_{11}) e^{\lambda_2 t}, \end{aligned} \tag{6.1.8.6}$$

where C_1 and C_2 are arbitrary constants. For $C_1 = 0$, $C_2 \neq 0$ and $C_2 = 0$, $C_1 \neq 0$, the trajectories in the phase plane x, y are straight lines. Four cases are possible here.

1.1. Two negative real roots, $\lambda_1 < 0$ and $\lambda_2 < 0$. This corresponds to $a_{11} + a_{22} < 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. The equilibrium point is asymptotically stable and all trajectories starting within a small neighborhood of the origin tend to the origin as $t \to \infty$. To $C_1 = 0$, $C_2 \neq 0$ and $C_2 = 0$, $C_1 \neq 0$ there correspond straight lines passing through the origin. Fig. 6.1*a* depicts the arrangement of the phase trajectories near an equilibrium point called a *stable node* (or a *sink*). The direction of motion along the trajectories with increasing t is shown by arrows.

1.2. $\lambda_1 > 0$ and $\lambda_2 > 0$. This corresponds to $a_{11} + a_{22} > 0$ and $a_{11}a_{22} - a_{12}a_{21} > 0$. The phase trajectories in the vicinity of the equilibrium point have the same pattern as in the preceding case; however, the trajectories go in the opposite direction, away from the equilibrium point; see Fig. 6.1*b*. An equilibrium point of this type is called an *unstable node* (or a *source*).

1.3. $\lambda_1 > 0$ and $\lambda_2 < 0$ (or $\lambda_1 < 0$ and $\lambda_2 > 0$). This corresponds to $a_{11}a_{22} - a_{12}a_{21} < 0$. In this case, the equilibrium point is also unstable, since the trajectory (6.1.8.6) with $C_2 = 0$ goes beyond a small neighborhood of the origin as t increases. If $C_1C_2 \neq 0$, then the

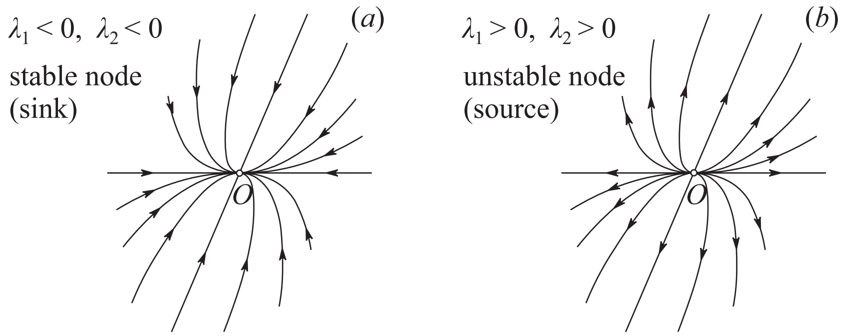

Figure 6.1: Phase trajectories of a system of differential equations near an equilibrium point of the node type.

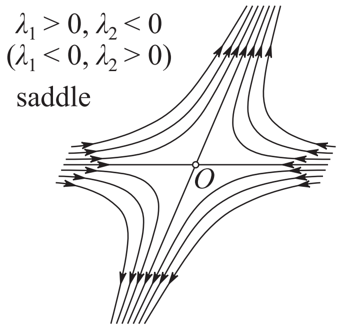

Figure 6.2: Phase trajectories of a system of differential equations near an equilibrium point of the saddle type.

trajectories leave the neighborhood of the origin as $t \to -\infty$ and $t \to \infty$. An equilibrium point of this type is called a *saddle* (or a *hyperbolic point*); see Fig. 6.2.

1.4. $\lambda_1 = 0$ and $\lambda_2 = a_{11} + a_{22} \neq 0$. This corresponds to $a_{11}a_{22} - a_{12}a_{21} = 0$. The general solution of system (6.1.8.1) is expressed as

$$x = C_1 a_{12} + C_2 a_{12} e^{(a_{11}+a_{22})t},$$
$$y = -C_1 a_{11} + C_2 a_{22} e^{(a_{11}+a_{22})t}, \qquad (6.1.8.7)$$

where C_1 and C_2 are arbitrary constants. By eliminating time t from (6.1.8.7), one obtains a family of parallel lines defined by the equation $a_{22}x - a_{12}y = a_{12}(a_{11} + a_{22})C_1$. To $C_2 = 0$ in (6.1.8.7) there corresponds a one-parameter family of equilibrium points that lie on the straight line $a_{11}x + a_{12}y = 0$.

(i) If $\lambda_2 < 0$, then the trajectories approach the equilibrium point lying as $t \to \infty$; see Fig. 6.3. The equilibrium point $x = y = 0$ is stable (or neutrally stable)—there is no asymptotic stability.

(ii) If $\lambda_2 > 0$, the trajectories have the same pattern as in case (i), but they go, as $t \to \infty$, in the opposite direction, away from the equilibrium point. The point $x = y = 0$ is unstable.

2°. If $D < 0$, the characteristic equation (6.1.8.3) has complex-conjugate roots:

$$\lambda_{1,2} = \alpha \pm i\beta, \qquad \alpha = \tfrac{1}{2}(a_{11} + a_{22}), \quad \beta = \tfrac{1}{2}\sqrt{|D|}, \quad i^2 = -1.$$

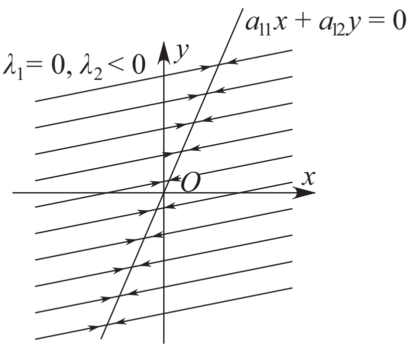

Figure 6.3: Phase trajectories of a system of differential equations near a set of equilibrium points located on a straight line.

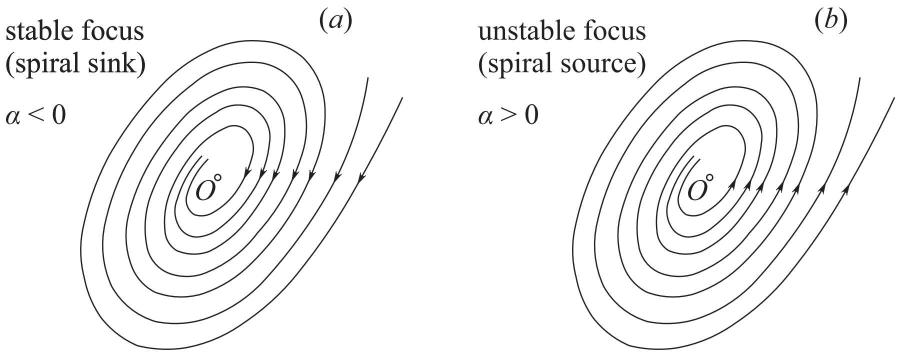

Figure 6.4: Phase trajectories of a system of differential equations near an equilibrium point of the focus type.

The general solution of system (6.1.8.1) has the form

$$x = e^{\alpha t}[C_1 \cos(\beta t) + C_2 \sin(\beta t)],$$
$$y = e^{\alpha t}[C_1^* \cos(\beta t) + C_2^* \sin(\beta t)], \tag{6.1.8.8}$$

where C_1 and C_2 are arbitrary constants, and C_1^* and C_2^* are defined by linear combinations of C_1 and C_2. The following cases are possible.

2.1. For $\alpha < 0$, the trajectories in the phase plane are spirals asymptotically approaching the origin of coordinates (the equilibrium point) as $t \to \infty$; see Fig. 6.4*a*. Therefore the equilibrium point is asymptotically stable and is called a *stable focus* (also a *stable spiral point* or a *spiral sink*). A focus is characterized by the fact that the tangent to a trajectory changes its direction all the way to the equilibrium point.

2.2. For $\alpha > 0$, the phase trajectories are also spirals, but unlike the previous case they spiral away from the origin as $t \to \infty$; see Fig. 6.4*b*. Therefore such an equilibrium point is called an *unstable focus* (also an *unstable spiral point* or a *spiral source*).

2.3. At $\alpha = 0$, the phase trajectories are closed curves, containing the equilibrium point inside (see Fig. 6.5). Such an equilibrium point is called a *center*. A center is a stable equilibrium point. Note that there is no asymptotic stability in this case.

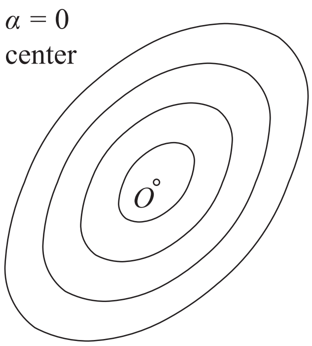

Figure 6.5: Phase trajectories of a system of differential equations near an equilibrium point of the center type.

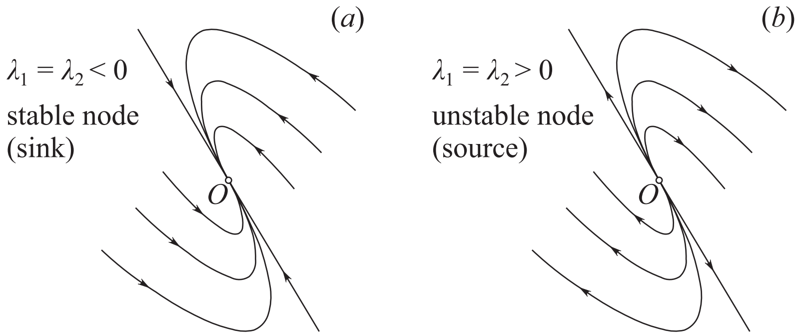

Figure 6.6: Phase trajectories of a system of differential equations near an equilibrium point of the node type in the case of a double root, $\lambda_1 = \lambda_2$.

3°. If $D = 0$, the characteristic equation (6.1.8.3) has a double real root, $\lambda_1 = \lambda_2 = \frac{1}{2}(a_{11} + a_{22})$. The following cases are possible.

3.1. If $\lambda_1 = \lambda_2 = \lambda < 0$, the general solution of system (6.1.8.1) has the form

$$x = a_{12}(C_1 + C_2 t)e^{\lambda t},$$
$$y = [(\lambda - a_{11})C_1 + C_2 + C_2(\lambda - a_{11})t]e^{\lambda t}, \qquad (6.1.8.9)$$

where C_1 and C_2 are arbitrary constants.

Since there is a rapidly decaying factor, $e^{\lambda t}$, all trajectories tend to the equilibrium point as $t \to \infty$; see Fig. 6.6*a*. To $C_2 = 0$ there corresponds a straight line in the phase plane x, y. The equilibrium point is asymptotically stable and is called a *stable node* (a *sink*). Such a node is in intermediate position between a node from Item 1.1 and a focus from Item 2.1.

3.2. If $\lambda_1 = \lambda_2 = \lambda > 0$, the general solution of system (6.1.8.1) is determined by formulas (6.1.8.9). The phase trajectories are similar to those from Item 3.1, but they go in the opposite direction, as $t \to \infty$, rapidly away from the equilibrium point. Such an equilibrium point is called an *unstable node* (a *source*); see Fig. 6.6*b*.

3.3. If $\lambda_1 = \lambda_2 = 0$, which corresponds to

$$a_{11} + a_{22} = 0 \quad \text{and} \quad a_{11}a_{22} - a_{12}a_{21} = 0$$

simultaneously, the general solution of system (6.1.8.1) is obtained by substituting $\lambda = 0$ into (6.1.8.9) and has the form

$$x = a_{12}C_1 + a_{12}C_2 t,$$
$$y = C_2 - a_{11}C_1 - a_{11}C_2 t.$$

For $a_{12} \neq 0$ all trajectories are parallel straight lines. As $t \to \pm\infty$, the trajectories go away from the origin. The equilibrium point is unstable.

For clearness, the classification results for equilibrium points of systems of two linear constant-coefficient differential equations (6.1.8.1) are summarized in Table 6.1.

TABLE 6.1
Classification of equilibrium points for systems of constant-coefficient equations (6.1.8.1); the symbols ∘ and ∗ indicate stable and unstable equilibrium points, respectively, where not clearly stated

Discriminant, formula (6.1.8.5)	Roots of quadratic equation (6.1.8.3), λ_1 and λ_2	Conditions for coefficients a_{ij} of homogeneous liner ordinary differential equations (6.1.8.1)	Type of equilibrium points or shape of phase trajectories
$D>0$	$\lambda_1<0,\ \lambda_2<0,\ \lambda_1\neq\lambda_2$	$a_{11}+a_{22}<0,\ a_{11}a_{22}-a_{12}a_{21}>0$	stable node
	$\lambda_1>0,\ \lambda_2>0,\ \lambda_1\neq\lambda_2$	$a_{11}+a_{22}>0,\ a_{11}a_{22}-a_{12}a_{21}>0$	unstable node
	roots have unlike signs	$a_{11}a_{22}-a_{12}a_{21}<0$	saddle∗
	$\lambda_1=0,\ \lambda_2<0$	$a_{11}+a_{22}<0,\ a_{11}a_{22}-a_{12}a_{21}=0$	parallel lines∘
	$\lambda_1=0,\ \lambda_2>0$	$a_{11}+a_{22}>0,\ a_{11}a_{22}-a_{12}a_{21}=0$	parallel lines∗
$D<0$	$\lambda_{1,2}=\alpha\pm i\beta,\ \alpha>0$	$a_{11}+a_{22}<0,\ (a_{11}-a_{22})^2+4a_{12}a_{21}<0$	stable focus
	$\lambda_{1,2}=\alpha\pm i\beta,\ \alpha<0$	$a_{11}+a_{22}>0,\ (a_{11}-a_{22})^2+4a_{12}a_{21}<0$	unstable focus
	$\lambda_{1,2}=\pm i\beta$, imaginary roots	$a_{11}+a_{22}=0,\ a_{11}a_{22}-a_{12}a_{21}>0$	center∘
$D=0$	$\lambda_1=\lambda_2<0$	$a_{11}+a_{22}<0,\ (a_{11}-a_{22})^2+4a_{12}a_{21}=0$	stable node
	$\lambda_1=\lambda_2>0$	$a_{11}+a_{22}>0,\ (a_{11}-a_{22})^2+4a_{12}a_{21}=0$	unstable node
	$\lambda_1=\lambda_2=0$	$a_{11}+a_{22}=0,\ a_{11}a_{22}-a_{12}a_{21}=0$	saddle∗
			parallel lines∗

Remark 6.6. For general definitions of a stable and an unstable equilibrium point, see Section 7.3.1.

⊙ *Literature for Section 6.1*: G. M. Murphy (1960), V. A. Ditkin and A. P. Prudnikov (1965), G. A. Korn and T. M. Korn (2000), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and A. V. Manzhirov (2007).

6.2 Systems of Linear Variable-Coefficient Equations

6.2.1 Homogeneous Systems of Linear First-Order Equations

► **Superposition principle for a homogeneous system.**

In general, a homogeneous linear system of variable-coefficient first-order ordinary differential equations has the form

$$\begin{aligned} y_1' &= f_{11}(x)y_1 + f_{12}(x)y_2 + \cdots + f_{1n}(x)y_n,\\ y_2' &= f_{21}(x)y_1 + f_{22}(x)y_2 + \cdots + f_{2n}(x)y_n,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ y_n' &= f_{n1}(x)y_1 + f_{n2}(x)y_2 + \cdots + f_{nn}(x)y_n, \end{aligned} \qquad (6.2.1.1)$$

where the prime denotes a derivative with respect to x. It is assumed further on that the functions $f_{ij}(x)$ are continuous of an interval $a \le x \le b$ (intervals are allowed with $a = -\infty$ or/and $b = +\infty$).

Any homogeneous linear system of the form (6.2.1.1) has the trivial particular solution $y_1 = y_2 = \cdots = y_n = 0$.

Superposition principle for a homogeneous system: any linear combination of particular solutions to system (6.2.1.1) is also a solution to this system.

► Wronskian determinant. General solution of the homogeneous system.

Let

$$\mathbf{y}_k = (y_{k1}, y_{k2}, \dots, y_{kn})^{\mathrm{T}}, \qquad y_{km} = y_{km}(x); \qquad k,\ m = 1,\ 2,\ \dots,\ n \tag{6.2.1.2}$$

be nontrivial particular solutions of the homogeneous system of equations (6.2.1.1). Solutions (6.2.1.2) are linearly independent if the *Wronskian determinant* is nonzero:

$$W(x) \equiv \begin{vmatrix} y_{11}(x) & y_{12}(x) & \cdots & y_{1n}(x) \\ y_{21}(x) & y_{22}(x) & \cdots & y_{2n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(x) & y_{n2}(x) & \cdots & y_{nn}(x) \end{vmatrix} \neq 0. \tag{6.2.1.3}$$

If condition (6.2.1.3) is satisfied, the general solution of the homogeneous system (6.2.1.1) is expressed as

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + \cdots + C_n\mathbf{y}_n, \tag{6.2.1.4}$$

where $C_1, C_2, \dots, C_n$ are arbitrary constants. The vector functions $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ in (6.2.1.4) are called *fundamental solutions* of system (6.2.1.1).

► Liouville formula.

Suppose condition (6.2.1.3) is met. Then the *Liouville formula*

$$W(x) = W(x_0) \exp\left[\int_{x_0}^{x} \left(\sum_{s=1}^{n} f_{ss}(t)\right) dt\right]$$

holds.

COROLLARY. *Particular solutions (6.2.1.2) are linearly independent on the interval $[a, b]$ if and only if there exists a point $x_0 \in [a, b]$ such that the Wronskian determinant is nonzero at x_0: $W(x_0) \neq 0$.*

► Reduction of the number of unknowns.

Suppose a nontrivial particular solution of system (6.2.1.1),

$$\mathbf{y}_1 = (u_1, u_2, \dots, u_n)^{\mathrm{T}}, \qquad u_m = u_m(x), \qquad m = 1,\ 2,\ \dots,\ n,$$

is known. Then the number of unknowns can be reduced by one. To this end, one considers the auxiliary homogeneous linear system of $n-1$ equations

$$z_k' = \sum_{q=2}^{n} \left[f_{kq}(x) - \frac{u_k(x)}{u_1(x)} f_{1q}(x) \right] z_q, \qquad k = 2, \dots, n. \tag{6.2.1.5}$$

Let

$$\mathbf{z}_p = (z_{p2}, z_{p3}, \dots, z_{pn})^{\mathrm{T}}, \quad z_{mk} = z_{mk}(x); \quad p = 2, \dots, n,$$

be a fundamental system of solutions to system (6.2.1.5) and let

$$\mathbf{Z}_p = (0, z_{p2}, z_{p3}, \dots, z_{pn})^{\mathrm{T}}, \quad F_p(x) = \int \left(\frac{1}{u_1(x)} \sum_{s=2}^{n} f_{1s}(x) z_{ps}(x) \right) dx, \quad p = 2, \dots, n,$$

the vector $\mathbf{Z}_p$ having an additional component compared with $\mathbf{z}_p$. Then the vector functions $\mathbf{y}_p = F_p(x)\mathbf{y}_1 + \mathbf{z}_p$ $(p = 2, \dots, n)$ together with $\mathbf{y}_1$ form a fundamental system of solutions to the initial homogeneous system of equations (6.2.1.1).

6.2.2 Nonhomogeneous Systems of Linear First-Order Equations

► Nonhomogeneous linear systems. Existence and uniqueness theorem.

In general, a nonhomogeneous linear system of variable-coefficient first-order differential equations has the form

$$\begin{aligned} y_1' &= f_{11}(x)y_1 + f_{12}(x)y_2 + \cdots + f_{1n}(x)y_n + g_1(x), \\ y_2' &= f_{21}(x)y_1 + f_{22}(x)y_2 + \cdots + f_{2n}(x)y_n + g_2(x), \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ y_n' &= f_{n1}(x)y_1 + f_{n2}(x)y_2 + \cdots + f_{nn}(x)y_n + g_n(x). \end{aligned} \tag{6.2.2.1}$$

Alternatively, the system can be written in the short vector-matrix notation as

$$\mathbf{y}' = \mathbf{f}(x)\mathbf{y} + \mathbf{g}(x),$$

with $\mathbf{f}(x) = \big(f_{ij}(x)\big)$ being the matrix of equation coefficients and $\mathbf{g}(x) = \big(g_1(x), g_2(x), \dots, g_n(x)\big)^{\mathrm{T}}$ being the vector function defining the nonhomogeneous part of the equations.

EXISTENCE AND UNIQUENESS THEOREM. *Let the functions $f_{ij}(x)$ and $g_i(x)$ be continuous on an interval $a < x < b$. Then, for any set of values x°, y_1°, …, y_n°, where $a < x^\circ < b$, there exists a unique solution $y_1 = y_1(x), \dots, y_n = y_n(x)$ satisfying the initial conditions*

$$y_1(x^\circ) = y_1^\circ, \quad \dots, \quad y_n(x^\circ) = y_n^\circ,$$

and this solution is defined on the whole interval $a < x < b$.

► General solution.

Let

$$\bar{\mathbf{y}} = (\bar{y}_1, \bar{y}_2, \dots, \bar{y}_n)^{\mathrm{T}}, \quad \bar{y}_k = \bar{y}_k(x); \quad k = 1, 2, \dots, n,$$

be a particular solution to the nonhomogeneous system of equations (6.2.2.1). The general solution of this system is the sum of the general solution of the corresponding homogeneous system (6.2.1.1), which corresponds to $g_k(x) \equiv 0$ in (6.2.2.1), and any particular solution of the nonhomogeneous system (6.2.2.1), or

$$\mathbf{y} = C_1\mathbf{y}_1 + C_2\mathbf{y}_2 + \cdots + C_n\mathbf{y}_n + \bar{\mathbf{y}}, \qquad (6.2.2.2)$$

where $\mathbf{y}_1, \ldots, \mathbf{y}_n$ are linearly independent solutions of the homogeneous system (6.2.1.1).

► A particular solution.

Given a fundamental system of solutions $y_{km}(x)$ (6.2.1.2) of the homogeneous system (6.2.1.1), a particular solution of the nonhomogeneous system (6.2.2.1) is found as

$$\bar{y}_k = \sum_{m=1}^{n} y_{mk}(x) \int \frac{W_m(x)}{W(x)}\,dx, \qquad k = 1,\ 2,\ \ldots,\ n,$$

where $W_m(x)$ is the determinant obtained by replacing the mth row in the Wronskian determinant (6.2.1.3) by the row of free terms, $g_1(x),\ g_2(x),\ \ldots,\ g_n(x)$, of equation (6.2.2.1). The general solution of the nonhomogeneous system (6.2.2.1) is given by (6.2.2.2).

► Superposition principle for a nonhomogeneous system.

A particular solution, $\mathbf{y} = \bar{\mathbf{y}}$, of the nonhomogeneous system of linear differential equations,

$$\mathbf{y}' = \mathbf{f}(x)\mathbf{y} + \sum_{k=1}^{m} \mathbf{g}_k(x),$$

is given by the sum

$$\mathbf{y} = \sum_{k=1}^{m} \mathbf{y}_k,$$

where the $\mathbf{y}_k$ are particular solutions of m (simpler) systems of equations

$$\mathbf{y}_k' = \mathbf{f}(x)\mathbf{y}_k + \mathbf{g}_k(x), \qquad k = 1,\ 2,\ \ldots,\ m,$$

corresponding to individual nonhomogeneous terms of the original system.

6.2.3 Euler System of Ordinary Differential Equations

► Euler system of ODEs. Reduction to a constant-coefficient linear system.

A homogeneous Euler system is a homogeneous linear system of ordinary differential equations composed by linear combinations of the following terms:

$$y_k, \quad xy_k', \quad x^2y_k'', \quad \ldots, \quad x^{m_k}y_k^{(m_k)}; \qquad k = 1,\ 2,\ \ldots,\ n.$$

Such a system is invariant under scaling in the independent variable (i.e., it preserves its form under the change of variable $x \to \alpha x$, where α is any nonzero number). A nonhomogeneous Euler system contains additional terms, given functions.

The substitution $x = be^t$ ($b \neq 0$) brings an Euler system, both homogeneous and nonhomogeneous, to a constant-coefficient linear system of equations.

Example 6.6. In general, a nonhomogeneous Euler system of second-order equations has the form

$$\sum_{k=1}^{n}\left(a_{mk}x^2\frac{d^2y_k}{dx^2}+b_{mk}x\frac{dy_k}{dx}+c_{mk}y_k\right)=f_m(x),\qquad m=1,\,2,\,\dots,\,n. \tag{6.2.3.1}$$

The substitutions $x=\pm e^t$ bring this system to a constant-coefficient linear system,

$$\sum_{k=1}^{n}\left[a_{mk}\frac{d^2y_k}{dt^2}+(b_{mk}-a_{mk})\frac{dy_k}{dt}+c_{mk}y_k\right]=f_m(\pm e^t),\qquad m=1,\,2,\,\dots,\,n,$$

which can be solved using, for example, the Laplace transform (see Example 6.5 from Section 6.1.7).

► Particular solutions.

Particular solutions to a homogeneous Euler system (for system (6.2.3.1), corresponding to $f_m(x)\equiv 0$) are sought in the form of power functions:

$$y_1=A_1x^\sigma,\quad y_2=A_2x^\sigma,\quad \dots,\quad y_n=A_nx^\sigma, \tag{6.2.3.2}$$

where the coefficients $A_1,\,A_2,\,\dots,\,A_n$ are determined by solving the associated homogeneous system of algebraic equations obtained by substituting expressions (6.2.3.2) into the differential equations of the system in question and dividing by x^σ. Since the system is homogeneous, for it to have nontrivial solutions, its determinant must vanish. This results in a dispersion equation for the exponent σ.

⊙ *Literature for Section 6.2*: G. M. Murphy (1960), E. Kamke (1977), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), D. Zwillinger (1997), G. A. Korn and T. M. Korn (2000), A. D. Polyanin and A. V. Manzhirov (2007).