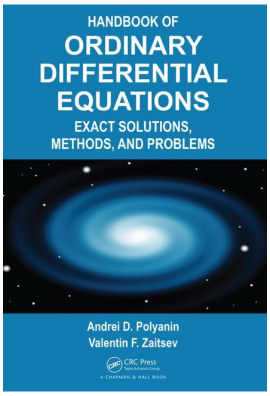

Handbook of Ordinary Differential Equations Exact Solutions, Methods, and Problems

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Chapter 7: Methods for nonlinear systems of ODES

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Chapter 7

Methods for Nonlinear Systems of ODEs

7.1 Solutions and First Integrals. Uniqueness and Existence Theorems

7.1.1 Systems Solved for the Derivative. A Solution and the General Solution

We will be dealing with a system of first-order ordinary differential equations solved for the derivatives

$$y'_k = f_k(x, y_1, \dots, y_n), \qquad k = 1, \dots, n. \tag{7.1.1.1}$$

Throughout the current chapter, the prime denotes a derivative with respect to the independent variable x (unless otherwise stated).

A set of numbers x, $y_1, \dots, y_n$ is convenient to treat as a point in the $(n+1)$-dimensional space.

For brevity, system (7.1.1.1) is conventionally written in vector form:

$$\mathbf{y}' = \mathbf{f}(x, \mathbf{y}),$$

where $\mathbf{y}$ and $\mathbf{f}$ are the vectors defined as $\mathbf{y} = (y_1, \dots, y_n)^T$ and $\mathbf{f} = (f_1, \dots, f_n)^T$.

A *solution* (also an *integral* or an *integral curve*) *of a system of differential equations* (7.1.1.1) is a set of functions $y_1 = y_1(x)$, ..., $y_n = y_n(x)$ such that, when substituted into all equations (7.1.1.1), they turn them into identities. The *general solution of a system of differential equations* is the set of all its solutions. In the general case, the general solution of system (7.1.1.1) depends on n arbitrary constants.

7.1.2 Existence and Uniqueness Theorems

EXISTENCE THEOREM (PEANO). *Let the functions $f_k(x, y_1, \dots, y_n)$ ($k = 1, \dots, n$) be continuous in a domain G of the $(n+1)$-dimensional space of the variables x, y_1, ..., y_n. Then there is at least one integral curve passing through every point $M(x^\circ, y_1^\circ, \dots, y_n^\circ)$ in G. Each of such curves can be extended on both ends up to the boundary of any closed domain completely belonging to G and containing the point M inside.*

Remark 7.1. If there is more than one integral curve passing through the point M, there are infinitely many integral curves passing through M.

UNIQUENESS THEOREM. *There is a unique integral curve passing through the point* $M(x^\circ, y_1^\circ, \dots, y_n^\circ)$ *if the functions* f_k *have partial derivatives with respect to all* y_m, *continuous in* x, y_1, …, y_n *in the domain* G, *or if each function* f_k *in* G *satisfies the Lipschitz condition:*

$$|f_k(x, \bar{y}_1, \dots, \bar{y}_n) - f_k(x, y_1, \dots, y_n)| \le A \sum_{m=1}^{n} |\bar{y}_m - y_m|,$$

where A *is some positive number.*

7.1.3 Reduction of Systems of Equations to a Single Equation or to an Autonomous System of Equations

► **Reduction of systems of equations to a single equation.**

Suppose the right-hand sides of equations (7.1.1.1) are n times differentiable in all variables. Then system (7.1.1.1) can be reduced to a single nth-order equation. Indeed, using the chain rule, let us differentiate the first equation of system (7.1.1.1) with respect to x to get

$$y_1'' = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_1}{\partial y_1} y_1' + \dots + \frac{\partial f_1}{\partial y_n} y_n'. \tag{7.1.3.1}$$

Then change the first derivatives y_k' in (7.1.3.1) to $f_k(x, y_1, \dots, y_n)$ [the right-hand sides of equations (7.1.1.1)] to obtain

$$y_1'' = F_2(x, y_1, \dots, y_n), \tag{7.1.3.2}$$

where $F_2(x, y_1, \dots, y_n) \equiv \dfrac{\partial f_1}{\partial x_1} + \dfrac{\partial f_1}{\partial y_1} f_1 + \dots + \dfrac{\partial f_1}{\partial y_n} f_n$. Now differentiate equation (7.1.3.2) with respect to x and replace the first derivatives y_k' on the right-hand side of the resulting equation by f_k. As a result, we obtain

$$y_1''' = F_3(x, y_1, \dots, y_n),$$

where $F_3(x, y_1, \dots, y_n) \equiv \dfrac{\partial F_2}{\partial x_1} + \dfrac{\partial F_2}{\partial y_1} f_1 + \dots + \dfrac{\partial F_2}{\partial y_n} f_n$. Repeating this procedure as many times as required, one arrives at the following system of equations:

$$\begin{aligned} y_1' &= F_1(x, y_1, \dots, y_n), \\ y_1'' &= F_2(x, y_1, \dots, y_n), \\ &\dots\dots\dots\dots\dots\dots\dots\dots \\ y_1^{(n)} &= F_n(x, y_1, \dots, y_n), \end{aligned}$$

where

$$\begin{aligned} F_1(x, y_1, \dots, y_n) &\equiv f_1(x, y_1, \dots, y_n), \\ F_{k+1}(x, y_1, \dots, y_n) &\equiv \frac{\partial F_k}{\partial x_1} + \frac{\partial F_k}{\partial y_1} f_1 + \dots + \frac{\partial F_k}{\partial y_n} f_n. \end{aligned}$$

Expressing $y_2, y_3, \ldots, y_n$ from the $n-1$ first equations of this system in terms of x, y_1, y_1', $\ldots, y_1^{(n-1)}$ and then substituting the resulting expressions into the last equation of system (7.1.1.1), one finally arrives at an nth-order equation:

$$y_1^{(n)} = \Phi(x, y_1, y_1', \ldots, y_1^{(n-1)}). \tag{7.1.3.3}$$

Remark 7.2. If (7.1.1.1) is a linear system of first-order differential equations, then (7.1.3.3) is a linear nth-order equation.

Remark 7.3. Any equation of the form (7.1.3.3) can be reduced to a system on n first-order equations (see the end of Section 5.1.2).

► **Reduction of the nonautonomous system of equations to an autonomous system of equations.**

In general, the nonautonomous system (7.1.1.1), consisting of n equations, can be reduced to the autonomous system

$$x'_\xi = 1, \quad (y_k)'_\xi = f_k(x, y_1, \ldots, y_n), \quad k = 1, \ldots, n, \tag{7.1.3.4}$$

consisting of $n+1$ equations.

7.1.4 First Integrals. Using Them to Reduce System Dimension

1°. A relation of the form

$$\Psi(x, y_1, \ldots, y_n) = C, \tag{7.1.4.1}$$

where C is an arbitrary constant, is called a *first integral* of system (7.1.1.1) if its left-hand side Φ, generally not identically constant, is turned into a constant by any particular solution, $y_1, \ldots, y_n$, of system (7.1.1.1). In the sequel, we consider only continuously differentiable functions $\Psi(x, y_1, \ldots, y_n)$ in a given domain of variation of its arguments.

THEOREM. *An expression of the form (7.1.4.1) is a first integral of system (7.1.1.1) if and only if the function $\Psi = \Psi(x, y_1, \ldots, y_n)$ satisfies the relation*

$$\frac{\partial \Psi}{\partial x} + \sum_{k=1}^{n} \frac{\partial \Psi}{\partial y_k} f_k(x, y_1, \ldots, y_n) = 0.$$

This relation may be treated as a first-order partial differential equation for Ψ.

Different first integrals of system (7.1.1.1) are called *independent* if the Jacobian of their left-hand sides is nonzero.

System (7.1.1.1) admits n independent first integrals if the conditions of the uniqueness theorem from Section 7.1.2 are met.

2°. Given a first integral (7.1.4.1) of system (7.1.1.1), it may be treated as an implicit specification of one of the unknowns. Solving (7.1.4.1), for example, for y_n yields $y_n = G(x, y_1, \ldots, y_{n-1})$. Substituting this expression into the first $n-1$ equations of system (7.1.1.1), one obtains a system in $n-1$ variables with one arbitrary constant.

Likewise, given m independent first integrals of system (7.1.1.1),

$$\Psi_k(x, y_1, \ldots, y_n) = C_k, \quad k = 1, \ldots, m \quad (m < n),$$

the system may be reduced to a system of $n-m$ first-order equations in $n-m$ unknowns.

7.2 Integrable Combinations. Autonomous Systems of Equations

7.2.1 Integrable Combinations

► **Systems of first-order ordinary differential equations.**

In some cases, first integrals of systems of differential equations may be obtained by finding *integrable combinations*. An integrable combination is a differential equation that is easy to integrate and is a consequence of the equations of the system under consideration. Most commonly, an integrable combination is an equation of the form

$$d\,\Psi(x, y_1, \dots, y_n) = 0 \tag{7.2.1.1}$$

or an equation reducible by a change of variables to one of the integrable types of equations in one unknown.

Example 7.1. Consider the nonlinear system

$$ay_1' = (b-c)y_2y_3, \quad by_2' = (c-a)y_1y_3, \quad cy_3' = (a-b)y_1y_2, \tag{7.2.1.2}$$

where a, b, and c are some constants. Such systems arise in the theory of motion of a rigid body.

Let us multiply the first equation by y_1, the second by y_2, and the third by y_3 and add together to obtain

$$ay_1y_1' + by_2y_2' + cy_3y_3' = 0 \implies d(ay_1^2 + by_2^2 + cy_3^2) = 0.$$

Integrating yields a first integral:

$$ay_1^2 + by_2^2 + cy_3^2 = C_1. \tag{7.2.1.3}$$

Now multiply the first equation of the system by ay_1, the second by by_2, and the third by cy_3 and add together to obtain

$$a^2y_1y_1' + b^2y_2y_2' + c^2y_3y_3' = 0 \implies d(a^2y_1^2 + b^2y_2^2 + c^2y_3^2) = 0.$$

Integrating yields another first integral:

$$a^2y_1^2 + b^2y_2^2 + c^2y_3^2 = C_2. \tag{7.2.1.4}$$

If the case $a = b = c$, where system (7.2.1.2) can be integrated directly, does not take place, the above two first integrals (7.2.1.3) and (7.2.1.4) are independent. Hence, using them, one can express y_2 and y_3 in terms of y_1 and then substitute the resulting expressions into the first equation of system (7.2.1.2). As a result, one arrives at a single separable first-order differential equation for y_1.

In this example, the integrable combinations have the form (7.2.1.1).

Example 7.2. A specific example of finding an integrable combination reducible with a change of variables to a simpler, integrable linear equation in one unknown can be found in Section 6.1.6.

► **Systems of second-order ordinary differential equations.**

In relatively few cases, integrals for systems of second-order ordinary differential equations can be found. Let us look at a few examples.

Example 7.3. Consider the *Ermakov system*

$$y''_{xx} + a(x)y = y^{-3} f(z/y), \tag{7.2.1.5}$$

$$z''_{xx} + a(x)z = z^{-3} f(y/z), \tag{7.2.1.6}$$

where $a(x)$, $f(\xi)$, and $g(\eta)$ are arbitrary functions.

Multiplying (7.2.1.5) by z and (7.2.1.6) by $-y$, adding the results together, and using the idenity $zy''_{xx} - yz''_{xx} = (zy'_x - yz'_x)'_x$, we obtain

$$d(zy'_x - yz'_x) = [zy^{-3} f(z/y) - yz^{-3} g(y/x)]\, dx.$$

Multiplying this relation by $(zy'_x - yz'_x)$ and integrating with respect to x, we find that

$$\frac{1}{2}(zy'_x - yz'_x)^2 = \int (zy'_x - yz'_x) zy^{-3} f(z/y)\, dx - \int (zy'_x - yz'_x) yz^{-3} g(y/x)\, dx + C,$$

where C is an arbitrary constant. Using the change of variable $\xi = z/y$ in the first integral and $\eta = y/z$ in the second integral, we arrive at the conservation law

$$\frac{1}{2}(zy'_x - yz'_x)^2 = \int^{z/y} \xi f(\xi)\, d\xi - \int^{y/z} \eta g(\eta)\, d\eta + C,$$

which is independent of $a(x)$.

Remark 7.4. System (7.2.1.5)–(7.2.1.6) admits a class of exact solutions of the form

$$y = y(x), \qquad z = ky(x),$$

where k is a root of the algebraic (or transcendental) equation $f(k) = k^2 g(1/k)$ (to distinct roots there correspond different solutions) and $y = y(x)$ is a solution to the Ermakov (Yermakov) equation $y''_{xx} + a(x)y = f(k)y^{-3}$ (its general solution is expressed in terms of the solution to the truncated linear equation with $f \equiv 0$, see Eq. 14.9.1.2).

7.2.2 Autonomous Systems and Their Reduction to Systems of Lower Dimension

1°. A system of equations is called *autonomous* if the right-hand sides of the equations do not depend explicitly on x. In general, such systems have the form

$$y'_k = f_k(y_1, \dots, y_n), \qquad k = 1, \dots, n. \tag{7.2.2.1}$$

If $\mathbf{y}(x)$ is a solution of the autonomous system (7.2.2.1), then the function $\mathbf{y}(x + C)$, where C is an arbitrary constant, is also a solution of this system.

A point $\mathbf{y}^\circ = (y_1^\circ, \dots, y_n^\circ)$ is called an *equilibrium point* (or a *stationary point*) of the autonomous system (7.2.2.1) if

$$f_k(y_1^\circ, \dots, y_n^\circ) = 0, \qquad k = 1, \dots, n.$$

To an equilibrium point there corresponds a special, simplest particular solution when all unknowns are constant:

$$y_1 = y_1^\circ, \quad \dots, \quad y_n = y_n^\circ, \qquad k = 1, \dots, n.$$

2°. Any n-dimensional autonomous system of the form (7.2.2.1) can be reduced to an $(n-1)$-dimensional system of equations independent of x. To this end, one should select one of the equations and divide the other $n - 1$ equations of the system by it.

Example 7.4. The autonomous system of two first-order equations

$$y'_x = f_1(y,z), \qquad z'_x = f_2(y,z) \tag{7.2.2.2}$$

is reduced by dividing the first equation by the second to a single equation for $y = y(z)$:

$$y'_z = \frac{f_1(y,z)}{f_2(y,z)}. \tag{7.2.2.3}$$

If the general solution of equation (7.2.2.3) is obtained in the form

$$y = \varphi(z, C_1), \tag{7.2.2.4}$$

then $z = z(x)$ is found in implicit form from the second equation in (7.2.2.2) by quadrature:

$$\int \frac{dz}{f_2(\varphi(z,C_1),z)} = x + C_2. \tag{7.2.2.5}$$

Formulas (7.2.2.4)–(7.2.2.5) determine the general solution of system (7.2.2.2) with two arbitrary constants, C_1 and C_2.

Remark 7.5. The dependent variables y and z in the autonomous system (7.2.2.2) are often called *phase variables*; the plane y, z they form is called a *phase plane*, which serves to display integral curves of equation (7.2.2.3).

⊙ *Literature for Section 7.2*: E. Kamke (1977), J. R. Ray and J. L. Reid (1979), A. D. Polyanin and A. V. Manzhirov (2007).

7.3 Elements of Stability Theory

7.3.1 Lyapunov Stability. Asymptotic Stability. Unstable Solutions

1°. In many applications, time t plays the role of the independent variable, and the associated system of differential equations is conventionally written in the following notation:

$$x'_k = f_k(t, x_1, \dots, x_n), \qquad k = 1, \dots, n. \tag{7.3.1.1}$$

Here the $x_k = x_k(t)$ are unknown functions that may be treated as coordinates of a moving point in an n-dimensional space.

Let us supply system (7.3.1.1) with initial conditions

$$x_k = x_k^\circ \quad \text{at} \quad t = t^\circ \qquad (k = 1, \dots, n). \tag{7.3.1.2}$$

Denote by

$$x_k = \varphi_k(t; t^\circ, x_1^\circ, \dots, x_n^\circ), \qquad k = 1, \dots, n, \tag{7.3.1.3}$$

the solution of system (7.3.1.1) with the initial conditions (7.3.1.2).

A solution (7.3.1.3) of system (7.3.1.1) is called *Lyapunov stable* if for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$|x_k^\circ - \widetilde{x}_k^\circ| < \delta, \qquad k = 1, \dots, n, \tag{7.3.1.4}$$

then the following inequalities hold for $t^\circ \le t < \infty$:

$$\bigl|\varphi_k(t; t^\circ, x_1^\circ, \dots, x_n^\circ) - \varphi_k(t; t^\circ, \widetilde{x}_1^\circ, \dots, \widetilde{x}_n^\circ)\bigr| < \varepsilon, \qquad k = 1, \dots, n.$$

Any solution which is not stable is called *unstable*. Solution (7.3.1.3) is called unperturbed and the solution $\varphi_k(t; t^\circ, \widetilde{x}_1^\circ, \dots, \widetilde{x}_n^\circ)$ is called perturbed. Geometrically, Lyapunov stability means that the trajectory of the perturbed solution stays at all times $t \ge t^\circ$ within a small neighborhood of the associated unperturbed solution.

2°. A solution (7.3.1.3) of system (7.3.1.1) is called *asymptotically stable* if it is Lyapunov stable and, in addition, with inequalities (7.3.1.4) met, satisfies the conditions

$$\lim_{t\to\infty}\left|\varphi_k(t;t^\circ,x_1^\circ,\dots,x_n^\circ)-\varphi_k(t;t^\circ,\widetilde{x}_1^\circ,\dots,\widetilde{x}_n^\circ)\right|=0,\qquad k=1,\,\dots,\,n. \tag{7.3.1.5}$$

3°. In stability analysis, it is normally assumed, without loss of generality, that $t^\circ = x_1^\circ = \cdots = x_n^\circ = 0$ (this can be achieved by shifting each of the variables by a constant value). Further, with the changes of variables

$$z_k = x_k - \varphi_k(t;t^\circ,x_1^\circ,\dots,x_n^\circ)\qquad (k=1,\,\dots,\,n),$$

the stability analysis of any solution is reduced to that of the zero solution $z_1=\cdots=z_n=0$.

7.3.2 Theorems of Stability and Instability by First Approximation

► **Statement of the problem.**

In studying stability of the trivial solution $x_1=\cdots=x_n=0$ of system (7.3.1.1) the following method is often employed. The right-hand sides of the equations are approximated by the principal (linear) terms of the expansion into Taylor series about the equilibrium point:

$$f_k(t,x_1,\dots,x_n)\approx a_{k1}(t)x_1+\cdots+a_{kn}(t)x_n,$$
$$a_{km}(t)=\frac{\partial f_k}{\partial x_m}\bigg|_{x_1=\cdots=x_n=0},\qquad k=1,\,\dots,\,n.$$

Then a stability analysis of the resulting simplified, linear system is performed. The question arises: Is it possible to draw correct conclusions about the stability of the original nonlinear system (7.3.1.1) from the analysis of the linearized system? Two theorems stated below give a partial answer to this question.

► **Stability by first approximation.**

THEOREM (STABILITY BY FIRST APPROXIMATION). *Suppose in the system*

$$x_k' = a_{k1}x_1+\cdots+a_{kn}x_n+\psi_k(t,x_1,\dots,x_n),\qquad k=1,\dots,n, \tag{7.3.2.1}$$

the functions ψ_k are defined and continuous in a domain $t\ge 0$, $|x_k|\le b$ $(k=1,\dots,n)$ and, in addition, the inequality

$$\sum_{k=1}^n|\psi_k|\le A\sum_{k=1}^n|x_k| \tag{7.3.2.2}$$

holds for some constant A. In particular, this implies that $\psi_k(t,0,\dots,0)=0$, and therefore

$$x_1=\cdots=x_n=0 \tag{7.3.2.3}$$

is a solution of system (7.3.2.1). Suppose further that

$$\frac{\sum\limits_{k=1}^n|\psi_k|}{\sum\limits_{k=1}^n|x_k|}\to 0\qquad\text{as}\qquad \sum_{k=1}^n|x_k|\to 0\quad\text{and}\quad t\to\infty, \tag{7.3.2.4}$$

and the real parts of all roots of the characteristic equation

$$\det|a_{ij} - \lambda\delta_{ij}| = 0, \qquad \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j \end{cases} \tag{7.3.2.5}$$

are negative. Then solution (7.3.2.3) is stable.

Remark 7.6. Necessary and sufficient conditions for the real parts of all roots of the characteristic equation (7.3.2.5) to be negative are established by Hurwitz's theorem, which allows avoiding its solution.

Remark 7.7. In the above system, the a_{ij}, x_k, and ψ_k may be complex valued.

▶ Instability by first approximation.

THEOREM (INSTABILITY BY FIRST APPROXIMATION). *(instability by first approximation). Suppose conditions (7.3.2.2) and (7.3.2.4) are met and the conditions for the functions ψ_k from the previous theorem are also met. If at least one root of the characteristic equation (7.3.2.5) has a positive real part, then the equilibrium point (7.3.2.3) of system (7.3.2.1) is unstable.*

Example 7.5. Consider the following two-dimensional system of the form (7.3.2.1) with real coefficients:

$$\begin{aligned} x'_t &= a_{11}x + a_{12}y + \psi_1(t, x, y), \\ y'_t &= a_{21}x + a_{22}y + \psi_2(t, x, y). \end{aligned} \tag{7.3.2.6}$$

We assume that the functions ψ_1 and ψ_2 satisfy conditions (7.3.2.2) and (7.3.2.4).

The characteristic equation of the linearized system (obtained by setting $\psi_1 = \psi_2 = 0$) is given by

$$\lambda^2 - b\lambda + c = 0, \quad \text{where} \quad b = a_{11} + a_{22}, \quad c = a_{11}a_{22} - a_{12}a_{21}. \tag{7.3.2.7}$$

1. Using the theorem of stability by first approximation and examining the roots of the quadratic equation (7.3.2.7), we obtain two sufficient stability conditions for system (7.3.2.6):

$$\begin{aligned} &b < 0, \quad 0 < \tfrac{1}{4}b^2 < c \quad \text{(complex roots with negative real part);} \\ &b < 0, \quad 0 < c < \tfrac{1}{4}b^2 \quad \text{(negative real roots).} \end{aligned}$$

The two conditions can be combined into one:

$$b < 0, \quad c > 0, \qquad \text{or} \qquad a_{11} + a_{22} < 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0.$$

These inequalities define the second quadrant in the plane b, c; see Fig. 7.1.

2. Using the theorem of instability by first approximation and examining the roots of the quadratic equation (7.3.2.7), we get three sufficient instability conditions for system (7.3.2.6):

$$\begin{aligned} &b > 0, \quad 0 < \tfrac{1}{4}b^2 < c \quad \text{(complex roots with positive real part);} \\ &b > 0, \quad 0 < c < \tfrac{1}{4}b^2 \quad \text{(positive real roots);} \\ &c < 0, \quad b \text{ is any} \quad \text{(real roots with different signs).} \end{aligned}$$

The first two conditions can be combined into one:

$$b > 0, \quad c > 0, \qquad \text{or} \qquad a_{11} + a_{22} > 0, \quad a_{11}a_{22} - a_{12}a_{21} > 0.$$

The domain of instability of system (7.3.2.6) covers the first, third, and fourth quadrants in the plane b, c (shaded in Fig. 7.1).

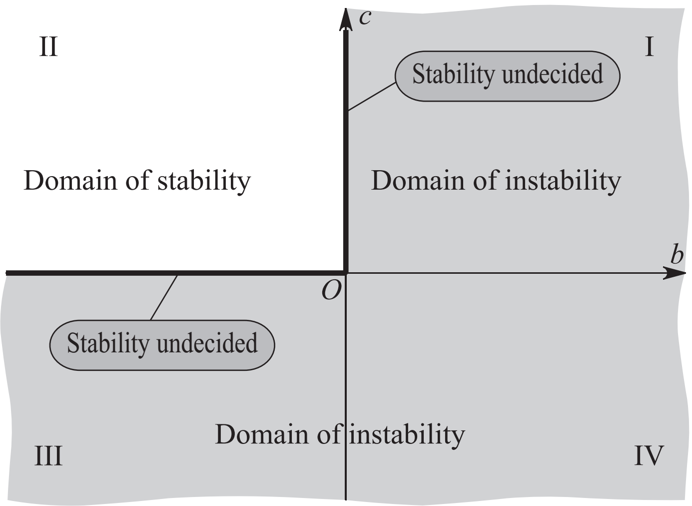

Figure 7.1: Domains of stability and instability of the trivial solution of system (7.3.2.6).

3. The conditions obtained above in Items 1 and 2 do not cover the whole domain of variation of the parameters a_{ij}. Stability or instability is not established for the boundary of the second quarter (shown by thick solid line in Fig. 7.1). This corresponds to the cases

$$b = 0, \quad c \geq 0 \quad \text{(two pure imaginary or two zero roots);}$$
$$c = 0, \quad b \leq 0 \quad \text{(one zero root and one negative real or zero root).}$$

Specific examples of such systems are considered below in Section 7.3.3.

Remark 7.8. When the conditions of Item 1 or 2 hold, the phase trajectories of the nonlinear system (7.3.2.6) have the same qualitative arrangement in a neighborhood of the equilibrium point $x = y = 0$ as that of the phase trajectories of the linearized system (with $\psi_2 = \psi_1 = 0$). A detailed classification of equilibrium points of linear systems with associated arrangements of the phase trajectories can be found in Section 6.1.8.

7.3.3 Lyapunov Function. Theorems of Stability and Instability

► **Lyapunov function.**

In the cases where the theorems of stability and instability by first approximation fail to resolve the issue of stability for a specific system of nonlinear differential equations, more subtle methods must be used. Such methods are considered below.

A *Lyapunov function* for system of equations (7.3.1.1) is a differentiable function $V = V(x_1, \ldots, x_n)$ such that

$$1) \quad V > 0 \quad \text{if} \quad \sum_{k=1}^{n} x_k^2 \neq 0, \qquad V = 0 \quad \text{if} \quad x_1 = \cdots = x_n = 0;$$

$$2) \quad \frac{dV}{dt} = \sum_{k=1}^{n} f_k(t, x_1, \ldots, x_n) \frac{\partial V}{\partial x_k} \leq 0 \quad \text{for} \quad t \geq 0.$$

Remark 7.9. The derivative with respect to t in the definition of a Lyapunov function is taken along an integral curve of system (7.3.1.1).

► **Theorems of stability and instability.**

THEOREM (STABILITY, LYAPUNOV). *Let system (7.3.1.1) have the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. This solution is stable if there exists a Lyapunov function for the system.*

THEOREM (ASYMPTOTIC STABILITY, LYAPUNOV). *Let system (7.3.1.1) have the trivial solution $x_1 = \cdots = x_n = 0$. This solution is asymptotically stable if there exists a Lyapunov function satisfying the additional condition*

$$\frac{dV}{dt} \le -\beta < 0 \quad \text{with} \quad \sum_{k=1}^{n} x_k^2 \ge \varepsilon_1 > 0, \quad t \ge \varepsilon_2 \ge 0,$$

where ε_1 and ε_2 are any positive numbers.

Example 7.6. Let us perform a stability analysis of the two-dimensional system

$$x'_t = -ay - x\varphi(x,y), \qquad y'_t = bx - y\psi(x,y),$$

where $a > 0$, $b > 0$, $\varphi(x,y) \ge 0$, and $\psi(x,y) \ge 0$ (φ and ψ are continuous functions).

A Lyapunov function will be sought in the form $V = Ax^2 + By^2$, where A and B are constants to be determined. The first condition characterizing a Lyapunov function will be satisfied automatically if $A > 0$ and $B > 0$ (it will be shown later that these inequalities do hold). To verify the second condition, let us compute the derivative:

$$\begin{aligned}\frac{dV}{dt} &= f_1(x,y)\frac{\partial V}{\partial x} + f_2(x,y)\frac{\partial V}{\partial y} = -2Ax[ay + x\varphi(x,y)] + 2By[bx - y\psi(x,y)]\\ &= 2(Bb - Aa)xy - 2Ax^2\varphi(x,y) - 2By^2\psi(x,y).\end{aligned}$$

Setting here $A = b > 0$ and $B = a > 0$ (thus satisfying the first condition), we obtain the inequality

$$\frac{dV}{dt} = -2bx^2\varphi(x,y) - 2ay^2\psi(x,y) \le 0.$$

This means that the second condition characterizing a Lyapunov function is also met. Hence, the trivial solution of the system in question is stable.

Example 7.7. Let us perform a stability analysis for the trivial solution of the nonlinear system

$$x'_t = -xy^2, \qquad y'_t = yx^4.$$

Let us show that the $V(x,y) = x^4 + y^2$ is a Lyapunov function for the system. Indeed, both conditions are satisfied:

$$\begin{aligned}&1)\quad x^4 + y^2 > 0 \quad \text{if} \quad x^2 + y^2 \ne 0, \qquad V(0,0) = 0 \quad \text{if} \quad x = y = 0;\\ &2)\quad \frac{dV}{dt} = -4x^4y^2 + 2x^4y^2 = -2x^4y^2 \le 0.\end{aligned}$$

Hence the trivial solution of the system is stable.

Remark 7.10. No stability analysis of the systems considered in Examples 7.6 and 7.7 is possible based on the theorem of stability by first approximation.

THEOREM (INSTABILITY, CHETAEV). *Suppose there exists a differentiable function $W = W(x_1, \ldots, x_n)$ that possesses the following properties:*

1. In an arbitrarily small domain R containing the origin of coordinates, there exists a subdomain $R_+ \subset R$ in which $W > 0$, with $W = 0$ on part of the boundary of R_+ in R.

2. The condition

$$\frac{dW}{dt} = \sum_{k=1}^{n} f_k(t, x_1, \dots, x_n)\frac{\partial W}{\partial x_k} > 0$$

holds in R_+ and, moreover, in the domain of the variables where $W \ge \alpha > 0$, the inequality $\dfrac{dW}{dt} \ge \beta > 0$ holds.

Then the trivial solution $x_1 = \cdots = x_n = 0$ of system (7.3.1.1) is unstable.

Example 7.8. Perform a stability analysis of the nonlinear system

$$x'_t = y^3\varphi(x, y, t) + x^5, \qquad y'_t = x^3\varphi(x, y, t) + y^5,$$

where $\varphi(x, y, t)$ is an arbitrary continuous function.

Let us show that the $W = x^4 - y^4$ satisfies the conditions of the Chetaev theorem. We have:

1. $W > 0$ for $|x| > |y|$, $W = 0$ for $|x| = |y|$.
2. $\dfrac{dW}{dt} = 4x^3[y^3\varphi(x, y, t) + x^5] - 4y^3[x^3\varphi(x, y, t) + y^5] = 4(x^8 - y^8) > 0$ for $|x| > |y|$.

Moreover, if $W \ge \alpha > 0$, we have $\dfrac{dW}{dt} = 4\alpha(x^4 + y^4) \ge 4\alpha^2 = \beta > 0$. It follows that the equilibrium point $x = y = 0$ of the system in question is unstable.

7.4 Numerical Integration

7.4.1 Systems of Two Equations

▶ Preliminary remarks.

The majority of the numerical methods for single first-order equations discussed in Section 1.13 generate analogous numerical methods for solving systems of first-order equations (7.1.1.1).

We illustrate this with the Cauchy problem described by the system of first-order differential equations

$$y'_x = f(x, y, z), \qquad z'_x = g(x, y, z) \tag{7.4.1.1}$$

with the initial conditions

$$y(x_0) = y_0, \qquad z(x_0) = z_0. \tag{7.4.1.2}$$

It is required to find $y = y(x)$ and $z = z(x)$.

▶ Method of Euler polygonal lines.

The unknowns are calculated successively by the formulas

$$y_{k+1} = y_k + hf(x_k, y_k, z_k), \qquad z_{k+1} = z_k + hg(x_k, y_k, z_k),$$

where

$$x_k = x_0 + kh, \qquad y_k = y(x_k), \qquad z_k = z(x_k), \qquad k = 0,\ 1,\ 2,\ \dots\ .$$

The Euler method is the simplest explicit method of the first-order approximation (with respect to the step size h).

► Modified Euler method.

The *modified Euler method* is more accurate than the method of Euler polygonal lines. One first calculates the intermediate quantities

$$x_{k+\frac{1}{2}} = x_k + \tfrac{1}{2}h, \qquad y_{k+\frac{1}{2}} = y_k + \tfrac{1}{2}hf(x_k, y_k, z_k), \qquad z_{k+\frac{1}{2}} = z_k + \tfrac{1}{2}hg(x_k, y_k, z_k)$$

and then finds y_{k+1} and z_{k+1} by the formulas

$$y_{k+1} = y_k + hf\bigl(x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}, z_{k+\frac{1}{2}}\bigr), \qquad z_{k+1} = z_k + hg\bigl(x_{k+\frac{1}{2}}, y_{k+\frac{1}{2}}, z_{k+\frac{1}{2}}\bigr).$$

The modified Euler method is of the second order of accuracy.

► Runge–Kutta method of the fourth-order approximation.

The unknown values y_k and z_k are successively found by the formulas

$$y_{k+1} = y_k + \tfrac{1}{6}h(\varphi_1 + 2\varphi_2 + 2\varphi_3 + \varphi_4), \qquad z_{k+1} = z_k + \tfrac{1}{6}h(\psi_1 + 2\psi_2 + 2\psi_3 + \psi_4),$$

where

$$\begin{aligned}
&\varphi_1 = f(x_k,\, y_k,\, z_k), \qquad \psi_1 = g(x_k,\, y_k,\, z_k),\\
&\varphi_2 = f(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_1,\, z_k + \tfrac{1}{2}h\psi_1),\\
&\psi_2 = g(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_1,\, z_k + \tfrac{1}{2}h\psi_1),\\
&\varphi_3 = f(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_2,\, z_k + \tfrac{1}{2}h\psi_2),\\
&\psi_3 = g(x_k + \tfrac{1}{2}h,\, y_k + \tfrac{1}{2}h\varphi_2,\, z_k + \tfrac{1}{2}h\psi_2),\\
&\varphi_4 = f(x_k + h,\, y_k + h\varphi_3,\, z_k + h\psi_3),\\
&\psi_4 = g(x_k + h,\, y_k + h\varphi_3,\, z_k + h\psi_3).
\end{aligned}$$

This scheme is convenient because the step size h can be changed (reduced if the unknowns change rapidly or increased otherwise) starting from any k. In practice, the choice of the step size h can be controlled using the following simple technique. For each k, one calculates the parameters

$$\theta_1 = \left|\frac{\varphi_2 - \varphi_3}{\varphi_1 - \varphi_2}\right|, \qquad \theta_2 = \left|\frac{\psi_2 - \psi_3}{\psi_1 - \psi_2}\right|.$$

If θ_i ($i = 1,\, 2$) are of the order of a few hundredths of unity, the calculations are continued with the same step size. If they are over one tenth, the step size should be decreased. If they are less that one hundredth, the step size can be increased to speed up the calculations.

► Numerical integration of problems with blow-up solutions.

In problems having a blow-up solution,* the right-hand side of at least one of the equations (7.4.1.1), which determines the derivative y'_x (or/and z'_x), tends to infinity as $x \to x_*$. When either or both of the functions $f(x, y, z)$ and $g(x, y, z)$ become infinite at a finite value of the independent variable, x_*, unknown in advance, we see the main reason why standard numerical methods fail to provide an acceptable solution for such problems.

*Refer to Section 1.14.4 for details.

Autonomous systems of equations. Consider the Cauchy problem for the autonomous system of equations of general form, whose right-hand side is independent explicitly of x,

$$y'_x = f(y,z), \qquad z'_x = g(y,z) \qquad (x > x_0), \tag{7.4.1.3}$$

with the initial conditions (7.4.1.2).

Let us look at the equivalent autonomous system of equations

$$y'_t = \frac{f(y,z)}{\sqrt{f^2(y,z)+g^2(y,z)}}, \qquad z'_t = \frac{g(y,z)}{\sqrt{f^2(y,z)+g^2(y,z)}} \qquad (t > t_0) \tag{7.4.1.4}$$

with the initial conditions

$$y(t_0) = y_0, \qquad z(t_0) = z_0. \tag{7.4.1.5}$$

The initial value t_0 can be chosen arbitrarily (in particular, it is often convenient to set $t_0 = 0$).

Suppose we have found a solution $y = y(t)$, $z = z(t)$ to the Cauchy problem (7.4.1.4)–(7.4.1.5). Then the formulas

$$\begin{gathered} x = x(t), \qquad y = y(t), \qquad z = z(t), \\ x(t) = x_0 + \int_{t_0}^{t} \frac{d\tau}{\sqrt{f^2(y(\tau),z(\tau)) + g^2(y(\tau),z(\tau))}} \end{gathered} \tag{7.4.1.6}$$

determine a solution to the original problem (7.4.1.3) in parametric form.

Unlike the original system (7.4.1.2), the right-hand sides of system (7.4.1.3) do not have singularities, since the derivatives are always bounded: $|y'_t| \le 1$ and $|z'_t| \le 1$ (recall that, for blow-up solutions, at least one of the derivatives y'_x or z'_x tends to infinity as $x \to x_*$).

A numerical solution to problem (7.4.1.4)–(7.4.1.5) can be obtained using, for example, the Runge–Kutta method (see above). The desired value x_*, determining the point of singularity of the problem, is found by calculating the integral in (7.4.1.6): $x_* = \lim_{t\to\infty} x(t)$.

This method allows for various modifications and generalizations. For example, system (7.4.1.4) can be replaced with the autonomous system

$$y'_t = \frac{f(y,z)}{|f(y,z)| + |g(y,z)|}, \qquad z'_t = \frac{g(y,z)}{|f(y,z)| + |g(y,z)|} \qquad (t > t_0). \tag{7.4.1.7}$$

The modulus sign in the denominators is used for generality, to ensure that system (7.4.1.7) can be used for the numerical solution of problems with root singularities even when f and g have different signs.

If a solution $y = y(t)$, $z = z(t)$ to the Cauchy problem (7.4.1.7), (7.4.1.5) has been found, the formulas

$$\begin{gathered} x = x(t), \qquad y = y(t), \qquad z = z(t), \\ x(t) = x_0 + \int_{t_0}^{t} \frac{d\tau}{|f(y(\tau),z(\tau))| + g|(y(\tau),z(\tau))|} \end{gathered} \tag{7.4.1.8}$$

define a solution to the original problem (7.4.1.3), (7.4.1.2) in parametric form.

The right-hand sides of system (7.4.1.7) do not have singularities, since the derivatives are always bounded: $|y'_t| \le 1$ and $|z'_t| \le 1$. The desired point of singularity is determined by calculating the integral in (7.4.1.8), $x_* = \lim_{t\to\infty} x(t)$.

Example 7.9. Consider the model Cauchy problem for the autonomous system of equations

$$\begin{gathered} y'_x = 1, \quad z'_x = z^2 \quad (x > 0); \\ y(0) = 0, \quad z(0) = 1. \end{gathered} \tag{7.4.1.9}$$

The problems has the exact solution

$$y = x, \quad z = \frac{1}{1-x}, \tag{7.4.1.10}$$

which only exists on a bounded interval, $0 \le x < x_* = 1$, and corresponds to a blow-up mode. As $x \to x_*$, we have $z \to \infty$ and $z'_x \to \infty$.

Instead of system (7.4.1.9), we will solve the special case of system 7.4.1.7 with $f(y,z) = 1$ and $g(y,z) = z^2$:

$$\begin{gathered} y'_t = \frac{1}{1+z^2}, \quad z'_t = \frac{z^2}{1+z^2} \quad (t > 0); \\ y(t=0) = 0, \quad z(t=0) = 1. \end{gathered} \tag{7.4.1.11}$$

The old independent variable x is expressed in terms of t as

$$x = \int_0^t \frac{d\tau}{1+z^2(\tau)}. \tag{7.4.1.12}$$

The solution of problem (7.4.1.11) followed by the computation of the integral (7.4.1.12) allows us to find a solution to the original problem (7.4.1.9) in parametric form

$$x = 1 + \tfrac{1}{2}t - \tfrac{1}{2}\sqrt{t^2+4}, \quad y = 1 + \tfrac{1}{2}t - \tfrac{1}{2}\sqrt{t^2+4}, \quad z = \tfrac{1}{2}t + \tfrac{1}{2}\sqrt{t^2+4}. \tag{7.4.1.13}$$

One can see that solution (7.4.1.13) exists for all $0 \le t < \infty$ and does not have singularities (unlike solution (7.4.1.10)). The functions $x = x(t)$, $y = y(t)$, and $z = z(t)$ all monotonically increase with t; moreover, the limit relations $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} y(t) = x_* = 1$ hold.

Nonautonomous systems of equations. In general, the Cauchy problem for nonautonomous systems of two equations (7.4.1.1) subject to the initial conditions (7.4.1.2) reduces the autonomous system of three equations

$$x'_\xi = 1, \quad y'_\xi = f(x,y,z), \quad z'_\xi = g(x,y,z) \tag{7.4.1.14}$$

with the initial conditions

$$x(\xi_0) = x_0, \quad y(\xi_0) = y_0, \quad z(\xi_0) = z_0, \tag{7.4.1.15}$$

where the initial value of the additional variable can be taken in the form $\xi_0 = 1$.

The numerical solution of the blow-up problem (7.4.1.14)–(7.4.1.15) is carried out using the method described in Section 7.4.2.

► Numerical integration of problems with root singularity.

Systems (7.4.1.4) and (7.4.1.7) can also be used for the numerical analysis of Cauchy problems of the form (7.4.1.3), (7.4.1.2) having solutions with a root singularity.

Example 7.10. Consider the model Cauchy problem for the autonomous system equation

$$\begin{gathered} y'_x = 1, \quad z'_x = -\frac{1}{2z} \quad (x > 0); \\ y(0) = 0, \quad z(0) = 1. \end{gathered} \tag{7.4.1.16}$$

It follows from the second initial condition that $z = z(x)$ is positive and decreases with x. It is fairly easy to verify that problem (7.4.1.16) admits the exact solution with a root singularity

$$y = x, \qquad z = \sqrt{1-x}, \tag{7.4.1.17}$$

which only exists on a bounded interval, $0 \le x < x_* = 1$, since the radicand in (7.4.1.17) becomes negative for $x > x_*$. As $x \to x_*$, we have $|z'_x| \to \infty$.

For numerical solution, instead of system (7.4.1.16), we will use the special case of system 7.4.1.7 with $f(y,z) = 1$ and $g(y,z) = -(2z)^{-1}$:

$$\begin{gathered} y'_t = \frac{2z}{1+2z}, \qquad z'_t = -\frac{1}{1+2z} \qquad (t>0); \\ y(t=0) = 0, \qquad z(t=0) = 1. \end{gathered} \tag{7.4.1.18}$$

A solution to problem (7.4.1.18) is sought in the domain $z > 0$, where $|-z| = z$; it must stop at $z = 0$, when the denominator of the right-hand side of the second equation in (7.4.1.16) becomes zero.

The old independent variable x is expressed in terms of the new variable t as

$$x = 2\int_0^t \frac{z(\tau)\,d\tau}{1+2z(\tau)}. \tag{7.4.1.19}$$

The solution of problem (7.4.1.18) followed by the computation of the integral (7.4.1.19) allows us to find a solution to the original problem (7.4.1.16) in parametric form:

$$x = t + \tfrac{1}{2}\sqrt{9-4t} - \tfrac{3}{2}, \qquad y = t + \tfrac{1}{2}\sqrt{9-4t} - \tfrac{3}{2}, \qquad z = \tfrac{1}{2}\sqrt{9-4t} - \tfrac{1}{2}. \tag{7.4.1.20}$$

Solution (7.4.1.20) only exists in a bounded domain, $0 \le t < 2$, since $z(2) = 0$ (recall that the solution is sought in the domain $z > 0$), and does not have singularities in this domain (unlike solution (7.4.1.17)). The functions $x = x(t)$ and $y = y(t)$ both monotonically increase with t; moreover, the relations $\lim_{t\to 2} x(t) = \lim_{t\to 2} y(t) = x_* = 1$ hold. The function $z = z(t)$ monotonically decreases with t and vanishes at $t = 2$.

7.4.2 Systems Involving Three or More Equations

► **Form of the system.**

Consider the system of first-order equations of general form

$$y'_m = f_m(x, y_1, y_2, \dots, y_n), \qquad m = 1,\,2,\,\dots,\,n \tag{7.4.2.1}$$

subject to the initial conditions

$$y_m(x_0) = y_0^m \quad \text{with} \quad m = 1,\,2,\,\dots,\,n. \tag{7.4.2.2}$$

► **Method of Euler polygonal lines.**

The unknown quantities are calculated successively by the formulas

$$y_{k+1}^m = y_k^m + h f_m(x_k, y_k^1, y_k^2, \dots, y_k^{n-1}), \qquad m = 1,\,2,\,\dots,\,n,$$

where

$$x_k = x_0 + kh, \qquad y_k^m = y_m(x_k), \qquad k = 0,\,1,\,2,\,\dots\,.$$

▶ **Modified Euler method.**

First, one computes the intermediate quantities

$$x_{k+\frac{1}{2}} = x_k + \tfrac{1}{2}h, \qquad y^m_{k+\frac{1}{2}} = y^m_k + \tfrac{1}{2}hf_m(x_k, y^1_k, y^2_k, \dots, y^{n-1}_k).$$

Then, one finds the values y^m_{k+1} by the formulas

$$y^m_{k+1} = y^m_k + hf_m\bigl(x_{k+\frac{1}{2}}, y^1_{k+\frac{1}{2}}, y^2_{k+\frac{1}{2}}, \dots, y^{n-1}_{k+\frac{1}{2}}\bigr).$$

▶ **Fourth-order Runge–Kutta method.**

The unknown values y^m_k are successively found by the formulas

$$y^m_{k+1} = y^m_k + \tfrac{1}{6}h(\varphi^m_1 + 2\varphi^m_2 + 2\varphi^m_3 + \varphi^m_4), \qquad m = 1,\ 2,\ \dots,\ n,$$

where

$$\begin{aligned}
\varphi^m_1 &= f_m(x_k, y^1_k, y^2_k, \dots, y^{n-1}_k),\\
\varphi^m_2 &= f_m(x_k + \tfrac{1}{2}h,\ y^1_k + \tfrac{1}{2}h\varphi^1_1,\ y^2_k + \tfrac{1}{2}h\varphi^2_1,\ \dots,\ y^n_k + \tfrac{1}{2}h\varphi^n_1),\\
\varphi^m_3 &= f_m(x_k + \tfrac{1}{2}h,\ y^1_k + \tfrac{1}{2}h\varphi^1_2,\ y^2_k + \tfrac{1}{2}h\varphi^2_2,\ \dots,\ y^n_k + \tfrac{1}{2}h\varphi^n_2),\\
\varphi^m_4 &= f_m(x_k + h,\ y^1_k + h\varphi^1_3,\ y^2_k + h\varphi^2_3,\ \dots,\ y^n_k + h\varphi^n_3).
\end{aligned}$$

▶ **A system of special type resulting from a single nth-order ODE.**

Let us look at the system of first-order equations of the special form

$$\begin{aligned}
&y'_1 = y_2, \quad y'_2 = y_3, \quad \dots, \quad y'_{n-1} = y_n,\\
&y'_n = f(x, y_1, y_2, \dots, y_n),
\end{aligned} \tag{7.4.2.3}$$

which is obtained from the single nth-order ODE

$$y^{(n)}_x = f(x, y, y'_x, \dots, y^{(n-1)}_x),$$

with $y \equiv y_1$.

System (7.4.2.3) is a special case of system (7.4.2.1) with

$$\begin{aligned}
f_m(x, y_1, y_2, \dots, y_n) &\equiv y_{m+1}, \qquad m = 1,\ 2,\ \dots,\ n-1,\\
f_n(x, y_1, y_2, \dots, y_n) &\equiv f(x, y_1, y_2, \dots, y_n).
\end{aligned}$$

Hence, it is solvable using the numerical methods described previously in Section 7.4.2.

▶ **Numerical integration of problems with blow-up solutions.**

Autonomous systems of equations. Consider the Cauchy problem for the autonomous system of equations of general form, whose right-hand side is independent explicitly of x,

$$\frac{dy_m}{dx} = f_m(y_1, \dots, y_n), \qquad m = 1, \dots, n \qquad (x > x_0), \tag{7.4.2.4}$$

with the initial conditions (7.4.2.2).

In problems having blow-up solutions, the right-hand side of a least one of the equations (7.4.2.4) tends to infinity as $x \to x_*$, with x_* unknown in advance.

Instead of (7.4.2.4), we will be looking at the equivalent autonomous system of equations

$$\frac{dy_m}{dt} = \frac{f_m(y_1, \dots, y_n)}{\sqrt{\sum_{j=1}^n f_j^2(y_1, \dots, y_n)}}, \qquad m = 1, \dots, n, \qquad (t > t_0) \tag{7.4.2.5}$$

with the initial conditions

$$y_m(t_0) = y_0^m \quad \text{with} \quad m = 1,\, 2,\, \dots,\, n. \tag{7.4.2.6}$$

The initial value t_0 can be chosen arbitrarily (in particular, it is often convenient to set $t_0 = 0$).

Suppose a solution $y_m = y_m(t)$ $(m = 1, \dots, n)$ to the Cauchy problem (7.4.2.5), (7.4.2.6) has been found. Then the formulas

$$y_m = y_m(t), \qquad m = 1, \dots, n, \qquad x = x_0 + \int_{t_0}^{t} \frac{d\tau}{\sqrt{\sum_{j=1}^n f_j^2(y_1(\tau), \dots, y_n(\tau))}}$$

define a solution to the original problem (7.4.2.4), (7.4.2.2) in parametric form.

Unlike system (7.4.2.4), the right-hand sides of system (7.4.2.5) do not have singularities, since the derivatives are all bounded: $|(y_m)'_t| \le 1$ $(m = 1, \dots, n)$; recall that, for blow-up solutions, at least one of the derivatives $(y_m)'_t$ tends to infinity as $x \to x_*$.

A numerical solution to problem (7.4.2.5)–(7.4.2.6) can be obtained using, for example, the Runge–Kutta method (see above).

This presented method admits various modifications and generalizations. For example, instead of (7.4.2.5), one uses the following autonomous system for numerical solution:

$$\frac{dy_m}{dt} = \frac{f_m(y_1, \dots, y_n)}{\sum_{j=1}^n |f_j(y_1, \dots, y_n)|}, \qquad m = 1, \dots, n, \qquad (t > t_0) \tag{7.4.2.7}$$

If a solution $y_m = y_m(t)$ $(m = 1, \dots, n)$ to the Cauchy problem (7.4.2.7), (7.4.2.6) has been obtained, the formulas

$$y_m = y_m(t), \qquad m = 1, \dots, n, \qquad x = x_0 + \int_{t_0}^{t} \frac{d\tau}{\sum_{j=1}^n |f_j(y_1(\tau), \dots, y_n(\tau))|}$$

define a solution to the original problem (7.4.2.4), (7.4.2.2) in parametric form.

The right-hand sides of system (7.4.2.7) do not have singularities, since the derivatives are all bounded: $|(y_m)'_t| \le 1$ $(m = 1, \dots, n)$.

Nonautonomous systems of equations. In general, the Cauchy problem for the nonautonomous system of n equations (7.4.2.1) subject to the initial conditions (7.4.2.2) is first reduced to an autonomous system of $n + 1$ equation (see Section 7.1.3). Then, one constructs a numerical solution to one of the two equivalent auxiliary systems described above.

► Numerical integrations of problem having solutions with root singularity.

The autonomous systems (7.4.2.5) and (7.4.2.7) can also be used for the numerical analysis of Cauchy problems of the form (7.4.2.4), (7.4.2.2) having solutions with a root singularity (see Section 7.4.1 for systems of two equations).

The nonautonomous system of n equations of general form (7.4.2.1) subject to the initial conditions (7.4.2.2) is first reduced to an autonomous system consisting of $n + 1$ equations (see Section 7.1.3) and then replaced with a suitable equivalent autonomous system discussed above.

► Differential-algebraic equations.

Systems of differential-algebraic equations (DAEs for short) are systems in which one or more dependent variables occur without their derivatives. Numerical methods for the solution of DAEs can be found in the books by Hairer, Lubich, and Roche (1989), Schiesser (1994), Hairer and Wanner (1996), Brenan, Campbell, and Petzold (1996), Ascher and Petzold (1998), and Rabier and Rheinboldt (2002).

⊙ *Literature for Section 7.4*: N. S. Bakhvalov (1977), N. N. Kalitkin (1978), A. N. Tikhonov, A. B. Vasil'eva, and A. G. Sveshnikov (1985), J. C. Butcher (1987), E. Hairer, C. Lubich, and M. Roche (1989), W. E. Schiesser (1994), U. M. Ascher and L. R. Petzold (1998), G. A. Korn and T. M. Korn (2000), H. J. Lee and W. E. Schiesser (2004).