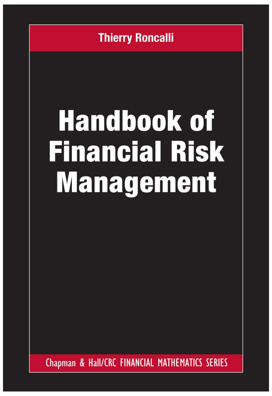

Handbook of Financial Risk Management

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Stress Testing and Scenario Analysis

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Chapter 14

Stress Testing and Scenario Analysis

In 1996, the Basel Committee proposed that banks regularly conduct stress testing programs in the case of market risk. The underlying idea was to identify events that could generate exceptional losses and understand the vulnerability of a bank. The use of stress tests has been increasing with the implementation of the Basel II Accord. Indeed, stress testing is the core of the Pillar 2 supervision, in particular for credit risk. At the same time, stress testing programs have been extended to the financial sector taken as a whole. In this case, they do not concern a given financial institution, but a set of banks or institutions. For example, the financial sector assessment program (FSAP) conducted by the International Monetary Fund and the World Bank measures the resilience of the financial sector of a given country or region. In Europe, EBA and ECB are in charge of the EU-wide stress testing. Since the 2008 Global Financial Crisis, they have conducted six stress testing surveys. In the US, the Fed performs every year a stress testing program that concerns the largest 30 banks. This annual assessment includes two related programs: The '*Comprehensive Capital Analysis and Review*' (CCAR) and the '*Dodd-Frank Act stress testing*' (DFAST). The objective of this last program is to evaluate the impact of stressful economic and financial market conditions on the bank capital. Recently, the Basel Committee on Banking Supervision has published a consultative document on stress testing principles. It highlights the growing importance of stress testing in the banking supervision model:

> "*Stress testing is now a critical element of risk management for banks and a core tool for banking supervisors and macroprudential authorities*" (BCBS, 2017a, page 5).

During long times, stress testing mainly concerned market risk, and later credit risk. These last years, it has been extended to other risks: funding risk, liquidity risk, and spillover risk (Tarullo, 2016). Moreover, stress testing is now extended to other financial sectors such as insurance and asset management. For instance, the Financial Stability Board (2017) encourages national financial regulators to conduct system-wide stress testing of asset managers, in particular for measuring the liquidity risk[1]. These views are also supported by IMF (Bouveret, 2017) and some national regulators (AMF, 2017; BaFin, 2017). The counterparty credit risk is another topic where stress testing could help. This explains that ESMA and CFTC have conducted specific stress testing of central counterparty clearing houses. We could then expect that the use of stress testing programs will increase across financial industries in the coming years, not only at the level of financial institutions, but also for regulatory purposes.

[1] "*Although such system-wide stress testing exercises are still in an exploratory stage, over time they may provide useful insights that could help inform both regulatory actions and funds' liquidity risk management practices*" (FSB, 2017, page 23).

14.1 Stress test framework

14.1.1 Definition

14.1.1.1 General objective

There are several definitions of stress testing, because stress tests can be used for different objectives. Lopez (2005) describes stress testing as "*a risk-management tool used to evaluate the potential impact on portfolio values of unlikely, although plausible, events or movements in a set of financial variables*". In this case, stress testing is a complementary tool for VaR analysis. Jorion (2007) considers that stress testing encompasses scenario analysis and the impact of stressed model parameters. Scenario analysis consists in measuring the potential loss due to a given economic or financial stress scenario. For example, the bank could evaluate the impact on its balance sheet if the world GDP decreases by 5% in the next two years. Stress testing of model parameters consists in evaluating the impact of stressed parameters on the P&L or the balance sheet of the bank. For example, the bank could evaluate the impact of more severe LGD parameters on its risk-weighted assets or the impact of higher correlations between banks on its CVA P&L. In the case of market risk, we can use stressed covariance matrices. In this context, stress testing can be viewed as an extension of the historical value-at-risk (Kupiec, 1998). More generally, stress testing aims to provide a forward-looking assessment of losses that would be suffered under adverse economic and financial conditions (BCBS, 2017a). In the case of a trading book, we recall that the loss of Portfolio w is equal to:

$$L_s(w) = P_t(w) - g(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s}; w)$$

where g is the pricing function and $(\mathcal{F}_{1,s}, \ldots, \mathcal{F}_{m,s})$ is the value of risk factors for the scenario s. When considering the historical value-at-risk, we calculate the quantile of the P&L obtained for n_S historical scenarios of risk factors ($s = 1, \ldots, n_S$). When considering the stress testing, we evaluate the portfolio loss for only one scenario:

$$L_{\text{stress}}(w) = P_t(w) - g(\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}}; w)$$

However, this scenario represented by the risk factors $(\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}})$ is supposed to be severe. Contrary to the value-at-risk, stress testing is then not built from a probability distribution.

14.1.1.2 Scenario design and risk factors

In the previous section, we feel that the stress scenario $\mathbb{S}$ is given by the set of risk factors $\mathcal{F}_{\text{stress}} = (\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}})$. It is only the case when we consider a historical scenario, e.g. the stock market crash in 1987 or the bond market crash in 1994. This type of approach is related to the concept of market price-based stress test, when the stress scenario is entirely defined by a set of market prices, for example the level of the VIX index, the return of the S&P 500 index, etc. However, most of the time, the scenario $\mathbb{S}$ is defined by a set $(\mathbb{S}_1, \ldots, \mathbb{S}_q)$ of q stress factors, which are not necessarily the market risk factors of the pricing function. This is particularly true when we consider hypothetical and macroeconomic stress tests. The difficulty is then to deduce the value of risk factors from the stress scenario:

$$\mathbb{S} = (\mathbb{S}_1, \ldots, \mathbb{S}_q) \Rightarrow \mathcal{F}_{\text{stress}} = (\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}})$$

Let us consider the FSAP stress scenarios used for the assessment of the stability of the French banking system (De Bandt and Oung, 2004). They tested 13 stress scenarios: 9 single- and multi-factor shocks ($F_1 - F_9$) and 4 macroeconomic shocks ($M_1 - M_4$). We report here the F_1, F_5 and F_9 shocks:

F_1 flattening of the yield curve due to an increase in interest rates: increase of 150 basis points (bp) in overnight rates, increase of 50 bp in 10-year rates, with interpolation for intermediate maturities;

F_5 share price decline of 30% in all stock markets;

F_9 flattening of the yield curve (increase of 150 basis points in overnight rates, increase of 50 bp in 10-year rates) together with a 30% drop in stock markets.

We denote by $\mathbb{S}_1$ and $\mathbb{S}_2$ the stress factors defined by the single-factor shocks F_1 and F_5. We have:

$$\begin{aligned} F_1 &: \mathbb{S}_1 \Rightarrow (\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}}) \\ F_5 &: \mathbb{S}_2 \Rightarrow (\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}}) \\ F_9 &: (\mathbb{S}_1, \mathbb{S}_2) \Rightarrow (\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}}) \end{aligned}$$

We notice that F_9 corresponds to the simultaneous shocks F_1 and F_5. It is obvious that the three shocks will impact differently the market risk factors $(\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}})$. However, the transformation of the stress $\mathbb{S}$ into $\mathcal{F}_{\text{stress}}$ is complex and depends on the modeling process of the financial institution. For instance, we can imagine that most of models will associate to the scenario $\mathbb{S}_1$ a negative impact on stock markets. For Bank A, it could be a 10% drop in stock markets while the model of Bank B may imply a share price decline of 20% in stock markets. It follows that stress testing is highly model-dependent. Let us now consider the M_2 macroeconomic shock:

M_2 increase to USD 40 in the price per barrel of Brent crude for two years (an increase of 48% compared with USD 27 per barrel in the baseline case), without any reaction from the central bank; the increase in the price of oil leads to an increase in the general rate of inflation and a decline in economic activity in France together with a drop in global demand.

Again, the stress factor $\mathbb{S}_3$ can produce different outcomes in terms of market risk factors depending on the model:

$$M_2 : \mathbb{S}_3 \Rightarrow (\mathcal{F}_{1,\text{stress}}, \ldots, \mathcal{F}_{m,\text{stress}})$$

Therefore, stress testing models are more sensitive to value-at-risk models. This is the main drawback of this approach. For instance, if we want to compare two banks, it is important to describe more precisely the stress scenarios than the shocks above. Moreover, having the stressed market risk factors of the two banks $\mathcal{F}^A_{\text{stress}} = \left(\mathcal{F}^A_{1,\text{stress}}, \ldots, \mathcal{F}^A_{m,\text{stress}}\right)$ and $\mathcal{F}^B_{\text{stress}} = \left(\mathcal{F}^B_{1,\text{stress}}, \ldots, \mathcal{F}^B_{m,\text{stress}}\right)$ is also relevant to understand how the initial shock spreads through the financial system, and the underlying assumptions of the models of banks A and B. The sensitivity to models and assumptions is even more pronounced in the case of liquidity stress tests. Indeed, the model must take into account spillover effects between financial institutions. In the case of funding liquidity, it requires modeling the network between banks, but also the monetary policy reaction function. In the case of market liquidity, the losses will depend on the behavior of all market participants, including asset managers and investors.

The previous introduction shows that we can classify stress scenarios into 4 main categories:

1. historical scenario: "*a stress test scenario that aims at replicating the changes in risk factor shocks that took place in an actual past episode*[2]" (BCBS, 2017a, page 60);

2. hypothetical scenario: "*a stress test scenario consisting of a hypothetical set of risk factor changes, which does not aim to replicate a historical episode of distress*" (BCBS, 2017a, page 60);

3. macroeconomic scenario: "*a stress test that implements a link between stressed macroeconomic factors [...] and the financial sustainability of either a single financial institution or the entire financial system*" (BCBS, 2017a, page 61);

4. liquidity scenario: "*a liquidity stress test is the process of assessing the impact of an adverse scenario on institution's cash flows as well as on the availability of funding sources, and on market prices of liquid assets*" (BCBS, 2017a, page 60).

Concerning hypothetical stress tests, we can also make the distinction between three types of scenarios: baseline, adverse and severely adverse. Since a baseline scenario corresponds to the best forecast of future economic conditions, it is not necessarily a stress scenario but serves as a benchmark. An adverse scenario is a scenario, where the economic conditions are assumed to be worse than for the baseline scenario. The distinction between an adverse and a severely adverse scenario is the probability of occurrence, which is very low for this latter. Therefore, we notice that defining a stress scenario is a two-step process. We first have to select the types of shocks, and then we have to calibrate the severity of the scenario. In Figures 14.1 and 14.2, we have reported the three scenarios of the 2017 Dodd-Frank Act stress test exercises[3] that were developed by the Board of Governors of the Federal Reserve System (2017). The baseline scenario for the United States is a moderate economic expansion, while the US economy experiences a moderate recession in the adverse scenario. The severely adverse scenario is characterized by a severe global recession that is accompanied by a period of heightened stress in corporate loan markets and commercial real estate markets. The baseline, adverse and severely adverse scenarios use the same set of stress factors, but the magnitude of the shocks are different.

14.1.1.3 Firm-specific versus supervisory stress testing

In the 1990s, stress tests were mainly conducted by banks in order to understand their hidden vulnerabilities:

> "*The art of stress testing should give the institution a deeper understanding of the specific portfolios that could be put in jeopardy given a certain situation. The question then would be: Would this be enough to bring down the firm? That way, each institution can know exactly what scenario they do not want to engage in*" (Dunbar and Irving, 1998).

More precisely, stress testing first emerged in trading activities. This explains that stress testing was presented by the 1996 amendment to the capital accord as an additional tool to the value-at-risk. It was an extreme risk measure, a tool for risk management, a requirement in order to validate internal models, but it was not used for calculating the regulatory

[2] According to BCBS (2017a), it may also result from "*a combination of changes in risk factor shocks observed during different past episodes*".

[3] The data are available at the following website: `www.federalreserve.gov/supervisionreg/dfast-archive.htm`.

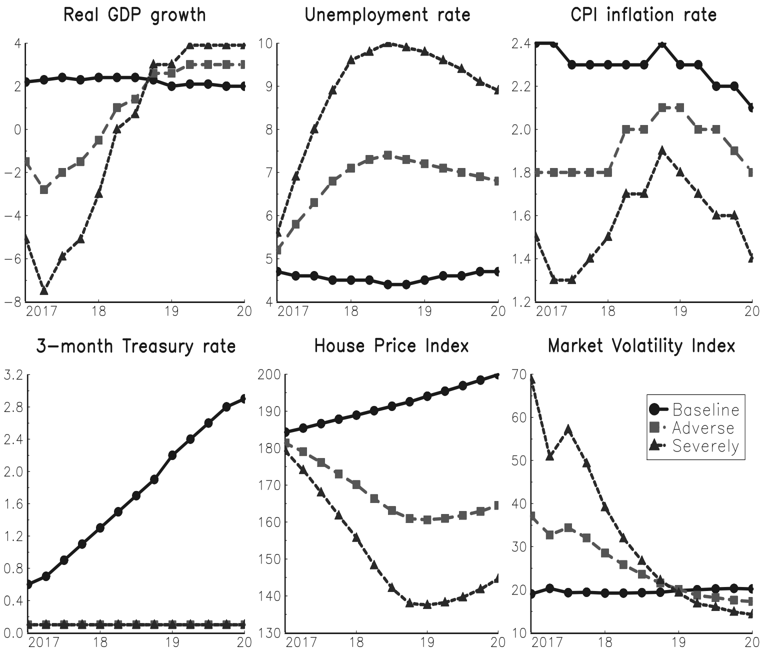

FIGURE 14.1: 2017 DFAST supervisory scenarios: Domestic variables

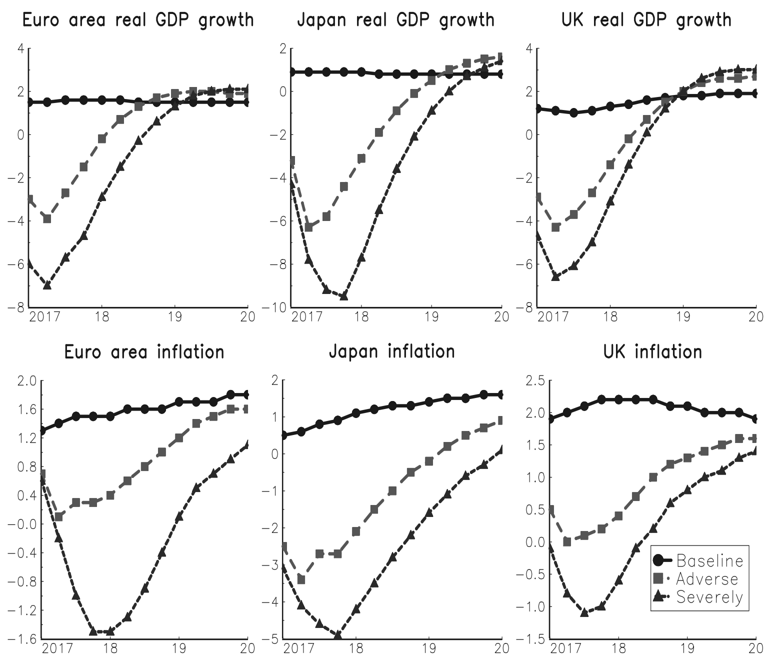

FIGURE 14.2: 2017 DFAST supervisory scenarios: International variables

capital. The Basel 2.5 framework has changed this situation since the capital depends on the stressed value-at-risk. In trading activities, stress scenarios are mainly historical. Besides the vulnerability analysis, stress testing have also been extensively used in the setting of trading limits. In the case of derivatives portfolios, trading limits are defined using sensitivities or VaR metrics. However, some situations can lead the bank to determine hard trading limits based on stress testing:

- some trading portfolios are sensitive to parameters that are unobservable or unstable; for example, a basket option depends on correlations, that can change faster in a crisis period;

- some underlying assets may become less liquid in a period of stress, for example volatility indices, dividends futures, small cap stocks, high yield bonds, etc.

In these cases, stress exposure limits are better than delta or vega exposure limits, because it is difficult to manage portfolios in non-normal situations. In the 2000s, the Basel II Accord has encouraged banks to apply stress testing techniques to credit risk, and some operational risk events such as rogue trading. However, firm-wide stress testing has made little progress before the development of supervisory stress tests (CGFS, 2001, 2005).

Supervisory stress tests starts in 1996 with the amendment to the Basel I Accord. However, they mainly concerned micro-prudential analysis. It was also the case with the Basel II Accord. The development of stress testing for macro-prudential purposes really begins to take off after the Global Financial Crisis. Before 2008, only the financial sector assessment program (FSAP), which was launched by the International Monetary Fund and the World Bank, can be considered as a system-wide stress testing exercise. Since the GFC, supervisory stress tests has become a standard for the different policymakers:

> "*Regulatory stress tests moved from being small-scale, isolated exercises within the broader risk assessment programme, to large-scale, comprehensive risk-assessment programmes in their own right leading directly to policy responses*" (Dent *et al.*, 2016, page 133).

Most of the time, policymakers and supervisors develop concurrent stress tests, meaning that the stress tests are applied to all banks of the system. Generally, these concurrent stress tests are used for setting capital buffers of banks. In this case, it is important to distinguish stress tests under the constant or dynamic balance sheet assumption (Busch *et al.*, 2017). Below, we review three supervisory stress testing frameworks:

- Financial sector assessment program (FSAP)[4]
  The FSAP exercise is conducted by the IMF and the World Bank. It is an in-depth assessment of a country's financial sector. According to the IMF, "*FSAPs analyze the resilience of the financial sector, the quality of the regulatory and supervisory framework, and the capacity to manage and resolve financial crises. Based on its findings, FSAPs produce recommendations of a micro- and macro-prudential nature, tailored to country-specific circumstances*". For instance, FSAPs have been tested for the following countries in 2017: Bulgaria, China, Finland, India, Indonesia, Japan, Lebanon, Luxembourg, Netherlands, New Zealand, Saudi Arabia, Spain, Sweden, Turkey and Zambia. Generally, the FSAP exercise includes one or two stress scenarios.

[4]The FSAP website is `www.imf.org/external/np/fsap/fsap.aspx`.

- Dodd-Frank Act stress test (DFAST)[5]
  According to the Fed, DFAST is a "*forward-looking quantitative evaluation of the impact of stressful economic and financial market conditions on bank holding companies' capital*". The results of DFAST are incorporated into the comprehensive capital analysis and review (CCAR), which evaluates the vulnerability of each bank on an annual basis. The DFAST exercise includes three types of scenario: baseline, adverse and severely adverse.

- EU-wide stress testing[6]
  EU-wide stress tests are conducted by the European Banking Authority (EBA), the European Systemic Risk Board (ESRB), the European Central Bank (ECB) and the European Commission (EC) in a regular basis, generally every two years[7]. According to the EBA, the aim of such tests is to "*assess the resilience of financial institutions to adverse market developments, as well as to contribute to the overall assessment of systemic risk in the EU financial system*".

Supervisory stress tests are not limited to these three examples, since most of developed central banks also use stress testing approaches, for instance the Bank of England (`www.bankofengland.co.uk/stress-testing`) or the Bank of Japan (`www.boj.or.jp/en/research/brp/fsr/index.htm`).

14.1.2 Methodologies

There are three main approaches for building stress scenarios. The historical approach can be viewed as an extension of the historical value-at-risk. The macroeconomic approach consists in developing hypothetical scenarios based on a macro-econometric model. Hypothetical scenarios can also be generated by the probabilistic approach. In this case, the probability distribution of risk factors is estimated and extreme scenarios are computed analytically or by Monte Carlo simulations.

14.1.2.1 Historical approach

This approach is the first method that have been used by banks in the early 1990s. It consists in identifying the worst period for a given risk factor. For instance, a stress scenario for equity markets may be the one that occurred during the Black Monday (1987) or the collapse of Lehman Brothers (2008). A typical adverse scenario for sovereign bonds is the US interest rate shock in 1994, also known as the '*great bond massacre*'. For currencies and commodities, historical stress scenarios can be calibrated using the Mexican peso crisis in 1994, the Asian crisis in 1997, or the commodity price crash in 2015. This approach is very simple and objective since it is based on past values of risk factors. However, it has two main drawbacks. First, the past worst scenario is not necessarily a good estimate of a future stress scenario. A typical example is the subprime crisis. Second, it is difficult to compare the severity of different historical stress scenarios.

The loss (or drawdown) function is defined by $\mathcal{L}(h) = \min_t R(t;h)$ where $R(t;h)$ is the asset return for the period $[t, t+h]$. In Table 14.1, we have reported the 5 maximum values of $\mathcal{L}(h)$ for the S&P 500 index and different values of h. For instance, the maximum of the daily drawdown is reached on 19 October 1987, where we observe a daily return of -20.47%. On 15 October 2008, a loss of -9% is observed. If we consider a monthly period, the maximum loss is about 30%. In Figure 14.3, we have reported the drawdown function

[5] The DFAST website is `www.federalreserve.gov/supervisionreg/dfa-stress-tests.htm`.
[6] The corresponding website is `www.eba.europa.eu/risk-analysis-and-data/eu-wide-stress-testing`.
[7] They took place in 2009, 2010, 2011, 2014, 2016 and 2018.

TABLE 14.1: Worst historical scenarios of the S&P 500 index

Sc.	1D		1W		1M	
1	1987-10-19	−20.47	1987-10-19	−27.33	2008-10-27	−30.02
2	2008-10-15	−9.03	2008-10-09	−18.34	1987-10-26	−28.89
3	2008-12-01	−8.93	2008-11-20	−17.43	2009-03-09	−22.11
4	2008-09-29	−8.79	2008-10-27	−13.85	2002-07-23	−19.65
5	1987-10-26	−8.28	2011-08-08	−13.01	2001-09-21	−16.89
Sc.	**2M**		**3M**		**6M**	
1	2008-11-20	−37.66	2008-11-20	−41.11	2009-03-09	−46.64
2	1987-10-26	−31.95	1987-11-30	−30.17	1974-09-13	−34.33
3	2002-07-23	−27.29	1974-09-13	−28.59	2002-10-09	−31.29
4	2009-03-06	−26.89	2002-07-23	−27.55	1962-06-27	−26.59
5	1962-06-22	−23.05	2009-03-09	−25.63	1970-05-26	−25.45

$\mathcal{L}(h)$. We notice that the drawdown increases with the time period at the beginning, but decreases when the time period is sufficiently long. The maximum loss is called the maximum drawdown:

$$\mathcal{MDD} = \min_{\Delta t} \mathcal{L}(\Delta t)$$

In the case of the S&P 500 index, the maximum drawdown is equal to −56.8% and has been observed between 9 October 2007 and 9 March 2009.

Remark 173 *In practice, the maximum drawdown is calculated using this formula:*

$$\mathcal{MDD} = -\max_t \left(\frac{\max_{[0,t]} P_t - P_t}{\max_{[0,t]} P_t} \right)$$

where P_t is the asset price or the risk factor.

The choice of the lag window h is important. Indeed, defining a stress scenario of −30% for US stocks is not the same if the time period is one day, one week or one month. Another important factor is the time period. For instance, a 50% drawdown for US stocks is observed many times in the last 50 years. However, it is not the same thing to consider the subprime crisis, the dot.com crisis or the 1973-1974 crisis of the stock market. Even if these three historical periods experience similar losses for stocks, the fixed income market reacts differently. It is then obvious that defining a stress scenario cannot be reduced to a single number for one risk factor. It is also important to define how the other risk factors will react and be impacted.

14.1.2.2 Macroeconomic approach

The macroeconomic approach consists in developing a macroeconometric model and considering an exogenous shock in order to generate adverse stress scenarios. The advantages of this approach are manifold. First, the macroeconomic model takes into account the current economic environment. Stress scenarios are then seen as more plausible than using the historical approach. For example, the drawdowns observed in the stock market in 1974, 2000 and 2008 are comparable in terms of magnitude, but not in terms of economic conditions. The origin of a financial market crisis is different each time. This is true for the stock market, but also for the other asset classes. Macroeconomic modeling may then help to develop the relationships between risk factors and the interconnectedness between asset classes for the next crisis. This is why the macroeconomic approach is certainly not better

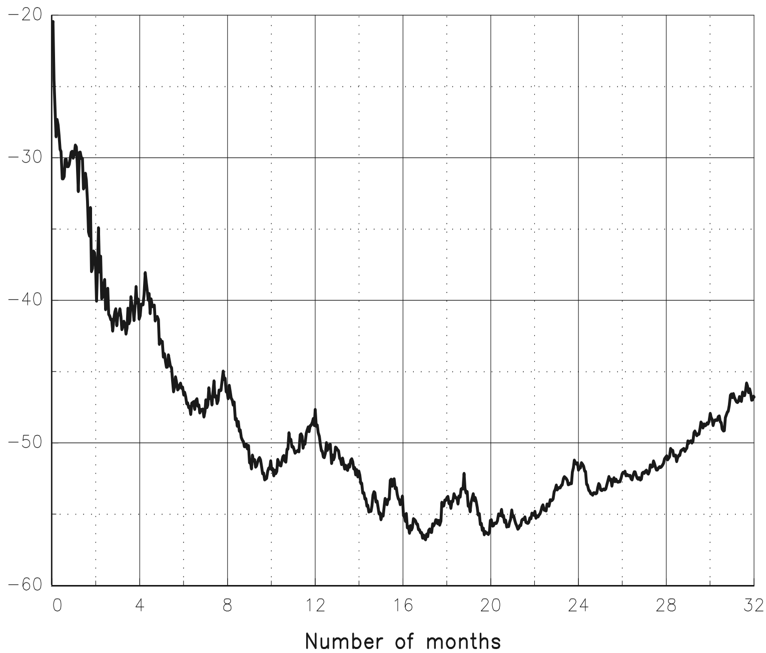

FIGURE 14.3: Loss function of the S&P 500 index

than the historical approach for defining single-factor stress testing, but it is more adapted for building multi-factor stress testing. Therefore, a second advantage is to describe the sequence of the crisis, and the dynamics between risk factors. Another advantage is that many scenarios can be generated by the model. For instance, we have previously seen that the DFAST program defines three scenarios: baseline, adverse and severely adverse. However, it is obvious that more scenarios are generated by the model. At the end, only two or three scenarios are selected, because some of them produce unrealistic outcomes, others may generate similar results, etc.

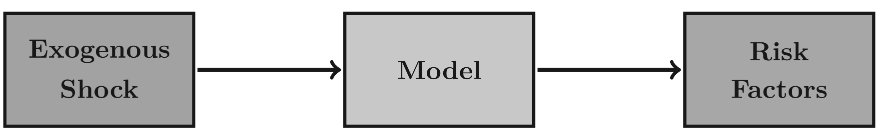

FIGURE 14.4: Macroeconomic approach of stress testing

However, we must be careful with the macroeconomic approach since it has also weaknesses. Stress testing always contains a side of uncertainty. In the case of the historical approach, this is obvious since there is no chance that the next crisis will look like a previous crisis. In the case of the macroeconomic approach, we generally expect to predict the future crisis, but we certainly expect too much (Borio *et al.*, 2014). In Figure 14.4, we have represented the traditional way to describe and think the macroeconomic approach of stress testing. The model uses input parameters (exogenous shocks) in order to produce output parameters (risk factors). In the real life, the impact of the risk factors on financial entities are not direct and deterministic. Indeed, we generally observe feedback effects from the stressed entities $(E_1, \ldots, E_n)$ on the economic situation (Figure 14.5). For instance, the default of one financial institution may lead monetary authorities to change their interest rate policy. These feedback effects are the most challenging point of the macroeconomic stress testing framework.

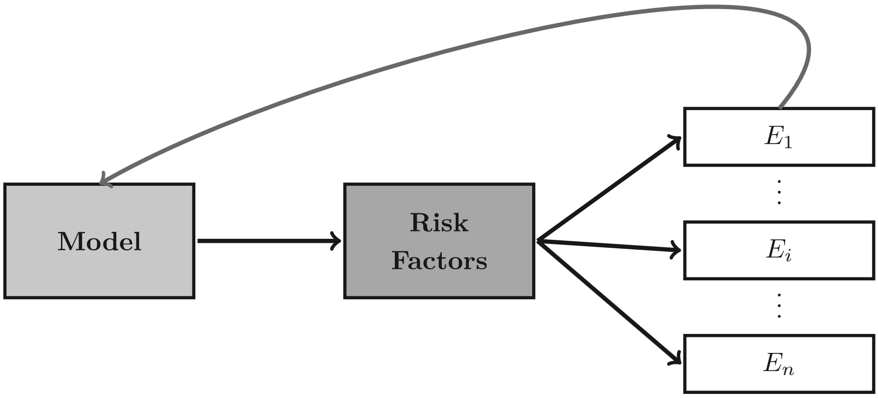

FIGURE 14.5: Feedback effects in stress testing models

A macroeconomic stress testing model is not only a macro-econometric model, that is based on a reduced form or a vector autoregressive process (Sims, 1980). Modeling activity (GDP, unemployment rate, etc.), interest rates and inflation (3M and 10Y interest rates, CPI, etc.) is the first step of the global process. It must also indicate the impact of the economic regime on credit risk parameters (default rates, CDS spreads, recovery rates, etc.) and fundamental variables (earnings, dividends, etc.). Finally, it must define the shocks on financial asset prices (stocks, bonds, commodities, real estate, etc.). For instance, the DFAST program defines 16 domestic and 12 international economic variables:

- Domestic variables: (1) Real GDP growth; (2) Nominal GDP growth; (3) Real disposable income growth; (4) Nominal disposable income growth; (5) Unemployment rate; (6) CPI inflation rate; (7) 3-month Treasury rate; (8) 5-year Treasury yield; (9) 10-year Treasury yield; (10) BBB corporate yield; (11) Mortgage rate; (12) Prime rate; (13) Dow Jones Total stock market index (Level); (14) House price index (Level); (15) Commercial real estate price index (Level); (16) Market volatility index (Level).

- International variables: (1) Euro area real GDP growth; (2) Euro area inflation; (3) Euro area bilateral dollar exchange rate (USD/euro); (4) Developing Asia real GDP growth; (5) Developing Asia inflation; (6) Developing Asia bilateral dollar exchange rate (F/USD, index); (7) Japan real GDP growth; (8) Japan inflation; (9) Japan bilateral dollar exchange rate (yen/USD); (10) UK real GDP growth; (11) UK inflation; (12) UK bilateral dollar exchange rate (USD/pound).

These variables concern activity, interest rates, inflation but also the prices of financial assets: the scenario for equities is given by the level of the Dow Jones index; the slope of the yield curve defines the scenario for fixed income instruments; the BBB corporate yield, the mortgage rate and the prime rate can be used to shape the scenario for credit products like corporate bonds or CDS; the scenario for real estate is given by house and commercial RE price indices; the level of the VIX indicates the scenario of the implied volatility for options and derivatives; the four exchange rates determine the stress scenario, which is valid for currency markets.

We notice that the DFAST program defines the major trends of the financial asset prices, but not a detailed scenario for each asset class. For equity markets, only the stress scenario for US large cap stocks is specified. Using this figure, one has to deduce the stress scenario for European equities, Japanese equities, EM equities, small cap equities, etc. So, there is room for interpretation. And there is a gap between the stress scenario given by the macroeconomic model and the outcome of the stress scenario. Contrary to the historical approach, the macroeconomic approach requires translating the big trends into the detailed path of risk factors. This step can only be done using parametric models: CAPM or APT models for stocks, Nelson-Siegel model for interest rates, Merton model for credit, etc.

14.1.2.3 Probabilistic approach

Until now, we have presented the outcome of a stress scenario as an extreme loss. However, the term '*extreme*' has little meaning and is not precise. It is obvious that a GDP growth of -10% is more extreme than a GDP growth of -5%. The extreme nature of a stress scenario can then be measured by its severity. However, we may wonder if a GDP growth of -50% is conceivable for instance. There is then a trade-off between the severity of a stress scenario and its probability or likelihood.

At first approximation, a stress scenario can be seen as an extreme quantile or value-at-risk. In this case, the aim of stress testing is not to estimate the maximum loss but an extreme loss. For instance, if we consider the univariate stress scenarios $F_1 - F_9$ of De Bandt and Oung (2004) presented on page 895, the authors indicate that the corresponding frequency is 1% over the last thirty years. In the case of multivariate stress scenarios, the probability of M_2 is equal to 1% while the probability of M_5 is equal to 5%. However, most of the time, the occurrence probability of a stress scenario is not discussed.

Let Y, X_1 and X_2 be three random variables that we would like to stress. These random variables may represent macroeconomic variables, market risk factors or parameters of risk models. We note $\mathbb{S}(Y)$, $\mathbb{S}(X_1)$ and $\mathbb{S}(X_2)$ the corresponding stressed values. Evaluating the likelihood of a stress scenario consists in calculating its probability of occurrence. The calculation depends on the relationship between the portfolio loss $L(w)$ and the random variable to stress. For instance, if the relationship between $L(w)$ and X_1 is decreasing, the probability of the stress $\mathbb{S}(X_1)$ is equal to:

$$\alpha_1 = \Pr\{X_1 \leq \mathbb{S}(X_1)\} = \mathbf{F}_1(\mathbb{S}(X_1))$$

If the relationship between $L(w)$ and X_2 is increasing, we have:

$$\alpha_2 = \Pr\{X_2 \geq \mathbb{S}(X_2)\} = 1 - \mathbf{F}_2(\mathbb{S}(X_2))$$

α_1 and α_2 measures the probability of univariate stress scenarios $\mathbb{S}(X_1)$ and $\mathbb{S}(X_2)$. Similarly, we may compute the joint probability of the stress scenario $(\mathbb{S}(X_1), \mathbb{S}(X_2))$:

$$\begin{aligned}
\alpha_{1,2} &= \Pr\{X_1 \leq \mathbb{S}(X_1), X_2 \geq \mathbb{S}(X_2)\} \\
&= \Pr\{X_1 \leq \mathbb{S}(X_1)\} - \Pr\{X_1 \leq \mathbb{S}(X_1), X_2 \leq \mathbb{S}(X_2)\} \\
&= \mathbf{F}_1(\mathbb{S}(X_1)) - \mathbf{C}_{1,2}(\mathbf{F}_1(\mathbb{S}(X_1)), \mathbf{F}_2(\mathbb{S}(X_2)))
\end{aligned}$$

While the univariate stress scenarios depend on the cumulative distribution functions $\mathbf{F}_1$ and $\mathbf{F}_2$, the bivariate stress scenario also depends on the copula function $\mathbf{C}_{1,2}$ between X_1 and X_2. If we assume that X_1 and X_2 are independent, we obtain:

$$\begin{aligned}
\alpha_{1,2} &= \alpha_1 - \alpha_1 \cdot (1 - \alpha_2) \\
&= \alpha_1 \cdot \alpha_2
\end{aligned}$$

If we assume that X_1 and X_2 are perfectly dependent — $\mathbf{C}_{1,2} = \mathbf{C}^{+}$, we have[8]:

$$\begin{aligned}\alpha_{1,2} &= \alpha_1 - \min(\alpha_1, 1-\alpha_2) \\ &= 0\end{aligned}$$

This result is perfectly normal because X_1 and X_2 impact $L(w)$ in an opposite way. If $\mathbf{C}_{1,2} = \mathbf{C}^{-}$, we have:

$$\begin{aligned}\alpha_{1,2} &= \alpha_1 - \max(0, \alpha_1 - \alpha_2) \\ &= \min(\alpha_1, \alpha_2)\end{aligned}$$

We deduce that the probability of the bivariate stress scenario is lower than the probability of the univariate stress scenarios:

$$0 \leq \alpha_{1,2} \leq \min(\alpha_1, \alpha_2)$$

We now consider that the stress scenario $\mathbb{S}(Y)$ is deduced from $\mathbb{S}(X_1)$ and $\mathbb{S}(X_2)$. The conditional probability of the stress scenario $\mathbb{S}(Y)$ is then given by:

$$\alpha = \Pr\{Y \leq \mathbb{S}(Y) \mid (X_1, X_2) = (\mathbb{S}(X_1), \mathbb{S}(X_2))\}$$

It follows that α depends on the conditional distribution of Y given X_1 and X_2. These three concepts of probability — univariate, joint and conditional — drive the quantitative approaches of stress testing that are presented below. They highlight the importance of quantifying the likelihood of the stress scenario, that is the probability of outcomes.

14.2 Quantitative approaches

The previous breakdown is used to classify the models into three main categories. The univariate case generally consists in modeling the probability distribution of a risk factor in an extreme situation. It is generally based on the extreme value theory. The multivariate case is a generalization of the first approach, and requires specifying the dependence between the risk factors. Copula functions are then the right tool for this task. The third approach uses more or less complex econometric models, in particular time series models.

14.2.1 Univariate stress scenarios

Let X be the random variable that produces the stress scenario $\mathbb{S}(X)$. If X follows the probability distribution $\mathbf{F}$, we have[9] $\Pr\{X \leq \mathbb{S}(X)\} = \mathbf{F}(\mathbb{S}(X))$. Given a stress scenario $\mathbb{S}(X)$, we may deduce its severity:

$$\alpha = \mathbf{F}(\mathbb{S}(X))$$

We may also compute the stressed value given the probability of occurrence α:

$$\mathbb{S}(X) = \mathbf{F}^{-1}(\alpha)$$

Even if this framework is exactly the approach used by the value-at-risk, there is a big difference between value-at-risk and stress testing. Indeed, the probability α used for stress testing is much lower than for value-at-risk.

[8] We recall that $\alpha_1 \approx 0$ and $\alpha_2 \approx 0$.

[9] We assume that the relationship between $L(w)$ and X is decreasing.

TABLE 14.2: Probability (in %) associated to the return period $\mathcal{T}$ in years

Return period	1	5	10	20	30	50
Daily	0.3846	0.0769	0.0385	0.0192	0.0128	0.0077
Weekly	1.9231	0.3846	0.1923	0.0962	0.0641	0.0385
Monthly	8.3333	1.6667	0.8333	0.4167	0.2778	0.1667
$1 - \alpha_{\text{GEV}}$	7.6923	1.5385	0.7692	0.3846	0.2564	0.1538

We recall that the return period $\mathcal{T}$ is related to the probability α by the relationship $\mathcal{T} = \alpha^{-1}$. We deduce that $\alpha = \mathcal{T}^{-1}$. In Table 14.2, we report the probability α for different return periods and different frequencies (daily, weekly and monthly). In the case where $\mathbf{F}$ is the cumulative distribution function of daily returns[10], the probability α is equal to 0.0769% when $\mathcal{T}$ is equal to 5 years, and 0.0128% when $\mathcal{T}$ is equal to 30 years. There are extreme probabilities in comparison to the confidence level $\alpha = 1\%$ for the value-at-risk. Therefore, we can use the extreme value theory to calculate these quantities. We reiterate that:

$$\mathcal{T} = \alpha^{-1} = n \cdot (1 - \alpha_{\text{GEV}})^{-1}$$

where n is the length of the block maxima[11].

TABLE 14.3: GEV parameter estimates (in %) of MSCI USA and MSCI EMU indices

Parameter	Long position		Short position	
	MSCI USA	MSCI EMU	MSCI USA	MSCI EMU
μ	1.242	1.572	1.317	1.599
σ	0.720	0.844	0.577	0.730
ξ	19.363	21.603	26.341	26.494

TABLE 14.4: Stress scenarios (in %) of MSCI USA and MSCI EMU indices

Year	Long position		Short position	
	MSCI USA	MSCI EMU	MSCI USA	MSCI EMU
5	−5.86	−7.27	5.69	7.16
10	−7.06	−8.83	7.01	8.84
25	−8.92	−11.29	9.17	11.60
50	−10.56	−13.49	11.18	14.17
75	−11.62	−14.94	12.54	15.91
100	−12.43	−16.05	13.59	17.26
Extreme statistic	−9.51	−10.94	11.04	10.87
$\mathcal{T}^{\star}$	32.49	22.24	47.87	20.03

Let us consider the MSCI USA and MSCI EMU indices from 1990 to 2017. We calculate the daily returns R_t. Then we take the block maxima ($X = R_t$) and the block minima ($X = -R_t$) for modeling short and long exposures. Finally, we estimate the parameters (μ, σ, ξ) by the method of maximum likelihood and calculate the corresponding stress scenario $\mathbb{S}(X) =$

[10]We assume that there are 260 trading days in one year.
[11]For instance, when $\mathcal{T}$ is equal to 5 years and n is equal to 20 days, we obtain $\alpha_{\text{GEV}} = 1.5385\%$.

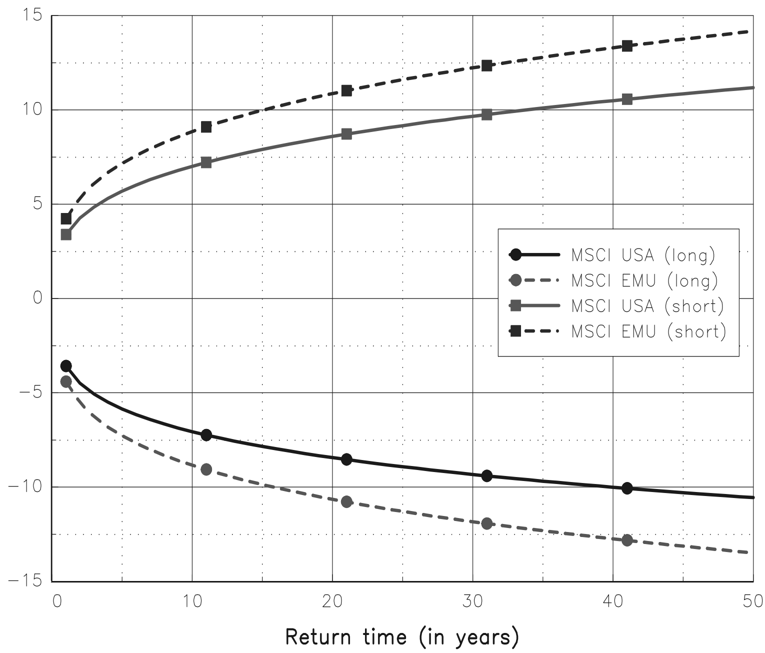

FIGURE 14.6: Stress scenarios (in %) of MSCI USA and MSCI EMU indices

$\hat{\mathbf{G}}^{-1}\left(1 - n\mathcal{T}^{-1}\right)$ where $\hat{\mathbf{G}}$ is the estimated GEV distribution. Results are given in Tables 14.3 and 14.4 and Figure 14.6 when the size of blocks is equal to 20 trading days. We notice that the magnitude of stress scenarios is higher for the MSCI EMU index than for the MSCI USA index. For each extreme statistic[12], we have reported the associated return period $\mathcal{T}^\star$. For the MSCI EMU index, $\mathcal{T}^\star$ is close to 20 years. For the MSCI USA index, we obtain a larger return period. This indicates that the stress scenarios for the MSCI USA index may be underestimated. Therefore, it may be appropriate to take the same stress scenario for the two indices, because the differences are not justified.

14.2.2 Joint stress scenarios

14.2.2.1 The bivariate case

Let $X_{n:n,1}$ and $X_{n:n,2}$ be the maximum order statistics of the random variables X_1 and X_2. We note $p = \Pr\left\{X_{n:n,1} > \mathbb{S}(X_1), X_{n:n,2} > \mathbb{S}(X_2)\right\}$ the joint probability of stress scenarios $(\mathbb{S}(X_1), \mathbb{S}(X_2))$. We have:

$$\begin{aligned} p &= 1 - \Pr\left\{X_{n:n,1} \leq \mathbb{S}(X_1)\right\} - \Pr\left\{X_{n:n,2} \leq \mathbb{S}(X_1)\right\} + \\ &\quad \Pr\left\{X_{n:n,1} \leq \mathbb{S}(X_1), X_{n:n,2} \leq \mathbb{S}(X_2)\right\} \\ &= 1 - \mathbf{F}_1(\mathbb{S}(X_1)) - \mathbf{F}_2(\mathbb{S}(X_2)) + \mathbf{C}(\mathbf{F}_1(\mathbb{S}(X_1)), \mathbf{F}_2(\mathbb{S}(X_2))) \\ &= \bar{\mathbf{C}}(\mathbf{F}_1(\mathbb{S}(X_1)), \mathbf{F}_2(\mathbb{S}(X_2))) \end{aligned}$$

where $\bar{\mathbf{C}}(u_1, u_2) = 1 - u_1 - u_2 + \mathbf{C}(u_1, u_2)$. We deduce that the failure area is represented by:

$$\left\{(\mathbb{S}(X_1), \mathbb{S}(X_2)) \in \mathbb{R}_+^2 \mid \bar{\mathbf{C}}(\mathbf{F}_1(\mathbb{S}(X_1)), \mathbf{F}_2(\mathbb{S}(X_2))) \leq \frac{n}{\mathcal{T}}\right\}$$

[12]They correspond to the minimum and maximum of daily returns.

Given a return period $\mathcal{T}$, we don't have a unique joint stress scenario $(\mathbb{S}(X_1), \mathbb{S}(X_2))$, but an infinite number of bivariate stress scenarios.

The previous result argues for computing the implied return period of a given scenario, and not the opposite:

$$\mathcal{T} = \frac{n}{\bar{\mathbf{C}}(\mathbf{F}_1(\mathbb{S}(X_1)), \mathbf{F}_2(\mathbb{S}(X_2)))}$$

In the univariate case, the implied return period of the stress $\mathbb{S}(X_i)$ is equal to:

$$\mathcal{T}_i = \frac{n}{1 - \mathbf{F}_i(\mathbb{S}(X_i))}$$

Since an extreme value copula satisfies the property $\mathbf{C}^{\perp} \prec \mathbf{C} \prec \mathbf{C}^{+}$, we deduce that:

$$\max(\mathcal{T}_1, \mathcal{T}_2) \le \mathcal{T} \le n\mathcal{T}_1\mathcal{T}_2$$

In Table 14.5, we report the upper and lower bounds of $\mathcal{T}$ for different values of n, $\mathcal{T}_1$ and $\mathcal{T}_2$ by assuming that a year contains 260 trading days. We observe that the range of $\mathcal{T}$ is wide. For instance, when n is equal to 20 days, and $\mathcal{T}_1$ and $\mathcal{T}_2$ are equal to 5 years, the return period of the joint stress scenario is equal to 5 years if the two scenarios are completely dependent and 325 years if they are independent.

TABLE 14.5: Upper and lower bounds of the return time $\mathcal{T}$ (in years)

n (in days)	$\mathcal{T}_1$	$\mathcal{T}_2$	Lower bound	Upper bound
1	5	5	5	6500
5	5	5	5	1300
20	5	5	5	325
260	5	5	5	25
260	10	5	10	50
260	1	1	1	1

We consider the previous example with MSCI USA and EMU indices. We have reported the failure area in Figure 14.7. For that, we have estimated the copula $\mathbf{C}$ by assuming a Gumbel copula function:

$$\mathbf{C}(u_1, u_2) = \exp\left(-\left((-\ln u_1)^{\theta} + (-\ln u_2)^{\theta}\right)^{1/\theta}\right)$$

We estimate θ by the method of maximum likelihood for each quadrant and obtain the following results:

MSCI USA MSCI EMU	Positive Positive	Positive Negative	Negative Negative	Negative Positive
$\hat{\theta}$	1.7087	1.4848	1.7430	1.4697

This means that $\hat{\theta}$ is equal to 1.7087 if the stress for MSCI USA and EMU indices are both positive. We have also reported the solution in the two extremes cases $\mathbf{C}^{\perp}$ and $\mathbf{C}^{+}$. We observe that the dependence plays a major role when considering joint scenarios. For instance, if we consider a scenario of −10% for the MSCI USA index and −10% for the MSCI EMU index, the return period is respectively equal to 39.9, 55.1 and 8 197 years for the product, Gumbel and Fréchet copulas.

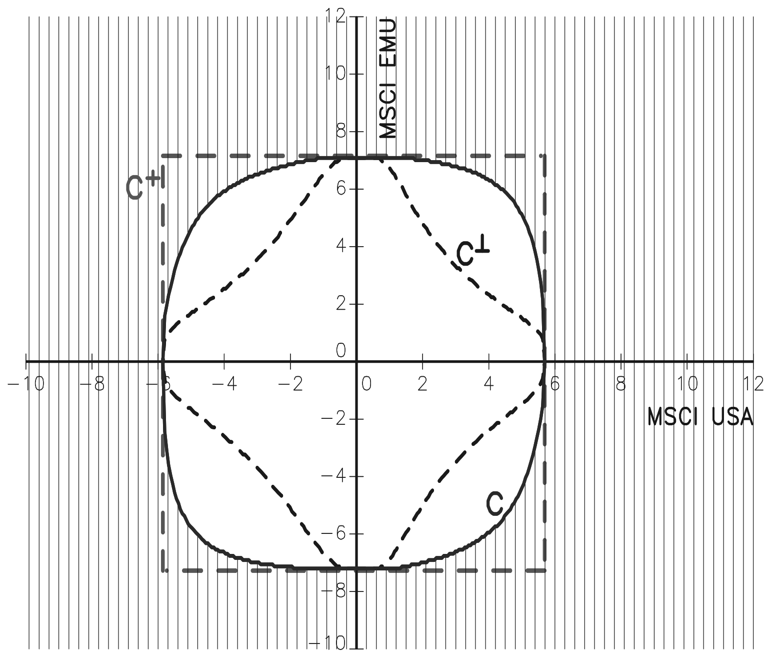

FIGURE 14.7: Failure area of MSCI USA and MSCI EMU indices (blockwise dependence)

Remark 174 *The previous exercise illustrates the limits of blockwise analysis. Let us consider the case of a negative stress for the MSCI USA index and a positive stress for the MSCI EMU index. When n is equal to 20 days, we calculate for each block the worst daily return for the first index and the best daily return for the second index. However, during 4 weeks, these two extreme returns do not certainly occur the same day. It follows that the dependence is overestimated for the two quadrants (Positive, Negative) and (Negative, Positive). This is why it is better to estimate the copula function using daily returns and not blockwise data. In this case, we obtain the results given in Figure 14.8.*

14.2.2.2 The multivariate case

In the multivariate case, the failure area is defined by:

$$\left\{ \left(\mathbb{S}(X_1), \ldots, \mathbb{S}(X_p) \right) \in \mathbb{R}_+^p \mid \bar{\mathbf{C}}\left(\mathbf{F}_1\left(\mathbb{S}(X_1)\right), \ldots, \mathbf{F}_m\left(\mathbb{S}(X_p)\right) \right) \leq \frac{n}{\mathcal{T}} \right\}$$

where:

$$\bar{\mathbf{C}}(u_1, \ldots, u_p) = \sum_{i=0}^{p} \left[(-1)^i \sum_{\mathbf{v} \in \mathcal{Z}(p-i,p)} \mathbf{C}(\mathbf{v}) \right]$$

and $\mathcal{Z}(m,p)$ denotes $\{\mathbf{v} \in [0,1]^p \mid \sum_{i=1}^{p} \mathbb{1}\{v_i = 1\} = m\}$. In the case $p = 2$, we retrieve the previous expression:

$$\bar{\mathbf{C}}(u_1, u_2) = 1 - u_1 - u_2 + \mathbf{C}(u_1, u_2)$$

When p is equal to 3, we obtain:

$$\begin{aligned} \bar{\mathbf{C}}(u_1, u_2, u_3) &= 1 - u_1 - u_2 - u_3 + \\ &\quad \mathbf{C}(u_1, u_2) + \mathbf{C}(u_1, u_3) + \mathbf{C}(u_2, u_3) - \\ &\quad \mathbf{C}(u_1, u_2, u_3) \end{aligned}$$

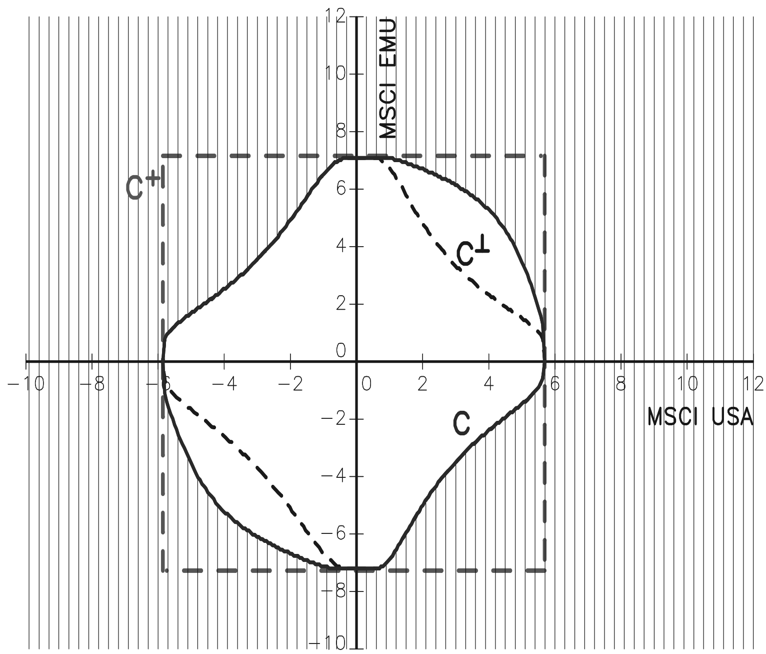

FIGURE 14.8: Failure area of MSCI USA and MSCI EMU indices (daily dependence)

Remark 175 *Bouyé et al. (2000) used this framework for evaluating stress scenarios associated to five commodities of the London Metal Exchange. Since commodity returns are not necessarily positively correlated, they showed that collecting univariate stress scenarios to form a multivariate stress scenario is completely biased. In particular, they presented an example where the return period of univariate stress scenarios is* 5 *years while the return period of the multivariate stress scenario is* 50 000 *years.*

14.2.3 Conditional stress scenarios

In supervisory stress testing, the goal is to impact the parameters of the risk model according to a given scenario. For example, these parameters may be the systematic risk factor in market risk factors, the probability of default and the loss given default in credit risk modeling, or the frequency of the Poisson distribution and the parameters of the severity distribution in operational risk modeling. Therefore, we have to estimate the relationship between these parameters and the variables of the scenario, and deduce their stressed values.

14.2.3.1 The conditional expectation solution

Let us assume a linear model between the independent variable Y and the explanatory variables $X = (X_1, \ldots, X_n)$:

$$Y_t = \beta_0 + \sum_{i=1}^{n} \beta_i X_{i,t} + \varepsilon_t$$

where $\varepsilon_t \sim \mathcal{N}\left(0, \sigma^2\right)$. By assuming that the standard properties of the linear regression model hold, we obtain:

$$\mathbb{E}\left[Y_t\right] = \beta_0 + \sum_{i=1}^{n} \beta_i \mathbb{E}\left[X_{i,t}\right]$$

We can also calculate the conditional expectation of Y_t:

$$\mathbb{E}\left[Y_t \mid X_t = (x_1, \ldots, x_n)\right] = \beta_0 + \sum_{i=1}^{n} \beta_i x_i$$

Given a joint stress scenario $\mathbb{S}(X) = (\mathbb{S}(X_1), \ldots, \mathbb{S}(X_n))$, we deduce the conditional stress scenario of Y and we have:

$$\begin{aligned}\mathbb{S}(Y) &= \mathbb{E}\left[Y_t \mid X_t = (\mathbb{S}(X_1), \ldots, \mathbb{S}(X_n))\right] \\ &= \beta_0 + \sum_{i=1}^{n} \beta_i \mathbb{S}(X_i)\end{aligned}$$

In some cases, assuming a linear relationship is not relevant, in particular for the probability of default or the loss given default. It is then common to use the following transformation (Dees *et al.*, 2017):

$$Z_t = \ln\left(\frac{Y_t}{1 - Y_t}\right)$$

We have:

$$\begin{aligned}Y_t &= \frac{\exp(Z_t)}{1 + \exp(Z_t)} \\ &= \frac{1}{1 + \exp(-Z_t)} \\ &= h(Z_t)\end{aligned}$$

where $h(z)$ is the logit transformation. We verify that $Y_t \in [0, 1]$. Since the statistical model becomes $Z_t = \beta_0 + \sum_{i=1}^{n} \beta_i X_{i,t} + u_t$, we deduce that:

$$\mathbb{E}\left[Y_t \mid X_t = (x_1, \ldots, x_n)\right] = \int_{-\infty}^{\infty} h\left(\beta_0 + \sum_{i=1}^{n} \beta_i X_{i,t} + \omega\right) \frac{1}{\sigma}\phi\left(\frac{\omega}{\sigma}\right) \mathrm{d}\omega \qquad (14.1)$$

This conditional expectation can be calculated thanks to numerical integration algorithms.

Remark 176 *The previous model can also be extended in order to take into account fixed effects (panel data) or lag dynamics. For instance, we can use an ARX(p) model:*

$$Y_t = \beta_0 + \sum_{i=1}^{p} \phi_i Y_{t-i} + \sum_{i=1}^{n} \beta_i X_{i,t} + u_t$$

Example 161 *We assume that the probability of default* PD_t *at time* t *is explained by the following linear regression model:*

$$\ln\left(\frac{\mathrm{PD}_t}{1 - \mathrm{PD}_t}\right) = -2.5 - 5g_t - 3\pi_t + 2u_t + \varepsilon_t$$

where $\varepsilon_t \sim \mathcal{N}(0, 0.25)$, g_t *is the growth rate of the GDP,* π_t *is the inflation rate, and* u_t *is the unemployment rate. The baseline scenario is defined by* $g_t = 2\%$, $\pi_t = 2\%$ *and* $u_t = 5\%$.

In Figure 14.9, we have reported the probability density function of PD_t for the baseline scenario and the following stress scenario: $g_t = -8\%$, $\pi_t = 5\%$ and $u_t = 10\%$. The conditional expectation[13] is respectively equal to 7.90% and 12.36%. The figure of 7.90% can

[13] We use a Gauss-Legendre quadrature method with an order of 512 for computing the conditional expectation given by Equation (14.1).

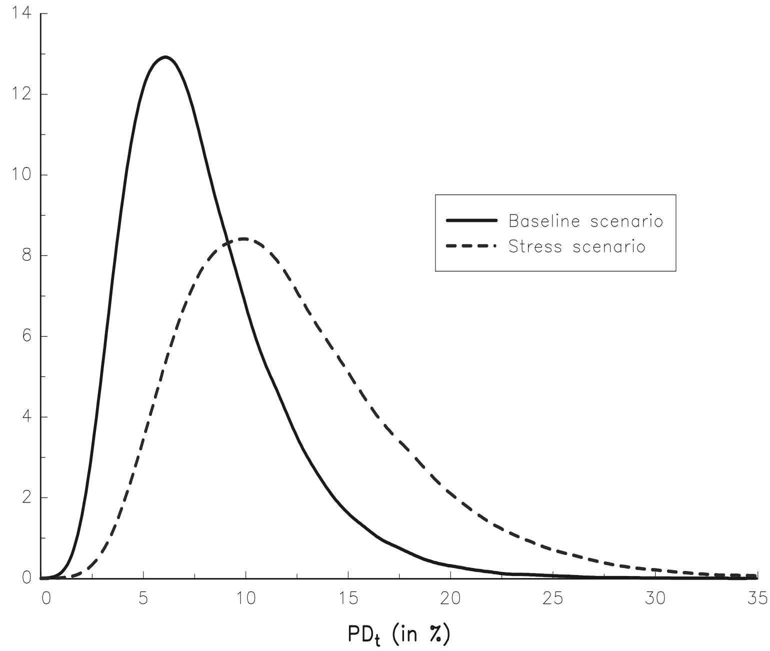

FIGURE 14.9: Probability density function of PD_t

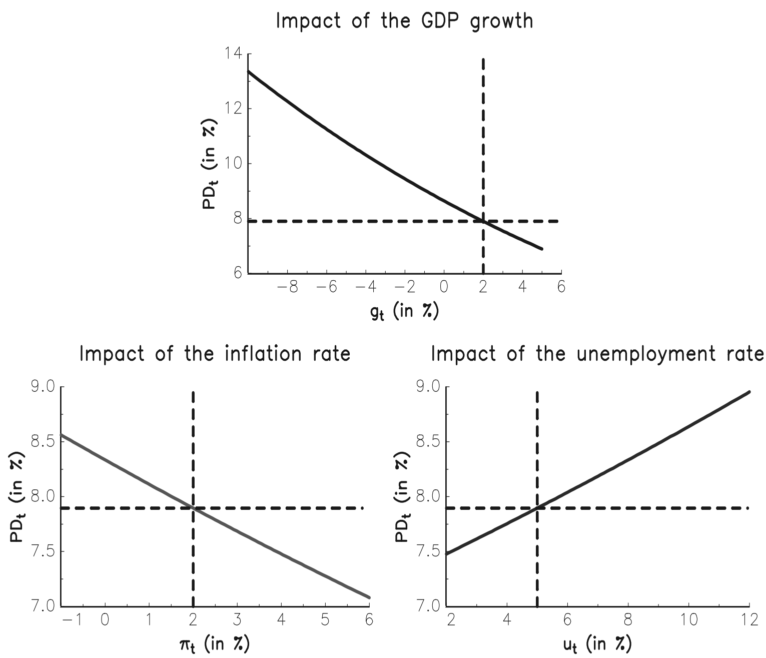

FIGURE 14.10: Relationship between the macroeconomic variables and PD_t

TABLE 14.6: Stress scenario of the probability of default

t	g_t	π_t	u_t	$\mathbb{E}[\text{PD}_t \mid \mathbb{S}(X)]$	$q_{90\%}(\mathbb{S}(X))$
0	2.00	2.00	5.00	7.90	12.78
1	−6.00	2.00	6.00	11.45	18.26
2	−7.00	1.00	7.00	12.47	19.79
3	−9.00	1.00	9.00	14.03	22.14
4	−7.00	1.00	10.00	13.12	20.78
5	−7.00	2.00	11.00	13.01	20.59
6	−6.00	2.00	10.00	12.26	19.49
7	−4.00	4.00	9.00	10.49	16.80
8	−2.00	3.00	8.00	9.70	15.58
9	−1.00	3.00	7.00	9.11	14.68
10	2.00	3.00	6.00	7.82	12.68
11	4.00	3.00	6.00	7.14	11.60
12	4.00	3.00	6.00	7.14	11.60

be interpreted as the long-run (or unconditional) probability of default that is used in the IRB formula. The relationship between the macroeconomic variables and the conditional expectation of PD_t is shown in Figure 14.10. For each panel, we consider the baseline scenario and we vary one parameter each time. In Table 14.6, we consider a stress scenario for the next 3 years, and we indicate the values taken by g_t, π_t and u_t for each quarter t. Then, we calculate the conditional expectation and the conditional quantile at the 90% confidence level of the probability of default PD_t. The stress scenario occurs at time $t = 1$ and propagates until $t = 12$. This is why we initially observe a jump in the probability of default, since it goes from 7.90% to 11.45%. The conditional expectation continues to increase and reaches a top at 14.03%. Then, it decreases and we obtain a new equilibrium after 3 years.

In the previous example, we have also reported the conditional quantile $q_{90\%}(\mathbb{S}(X))$. We observed that its values are larger than those given by the conditional expectation $\mathbb{E}[\text{PD}_t \mid \mathbb{S}(X)]$. These differences raise the question of defining a conditional stress scenario. Indeed, the previous framework defines the conditional stress scenario as the conditional expectation of the linear model $Y_t = \beta_0 + \sum_{i=1}^{n} \beta_i X_{i,t} + \varepsilon_t$. In this case, the vector of parameters $\beta = (\beta_0, \beta_1, \ldots, \beta_n)$ is estimated by ordinary least squares. We could also define the conditional stress scenario $\mathbb{S}(Y) = q_\alpha(\mathbb{S}(X))$ as the solution of the quantile regression:

$$\Pr\{Y_t \leq q_\alpha(\mathbb{S}) \mid X_t = \mathbb{S}\} = \alpha$$

In this case, we can use the tools presented on pages 613 (parametric approach) and 643 (non-parametric approach). The parametric approach assumes that the probability distribution between Y and X is Gaussian. The non-parametric approach is more adapted when this assumption is not satisfied, for example when the stochastic dependence is not linear.

14.2.3.2 The conditional quantile solution

In order to understand the impact of the dependence on the conditional stress scenario, we consider again the copula framework. If we consider the bivariate random vector (X, Y), using the linear regression is equivalent to assume that:

$$Y_t = \mathbb{E}[Y_t \mid X_t = x] + \varepsilon_t$$

This implies that (X, Y) is a bivariate Gaussian random vector. The average dependence structure between X and Y is then linear and can be represented by the parametric function $y = m(x)$ where $m(x)$ is the conditional expectation function $\mathbb{E}[Y_t \mid X_t = x]$. However, the conditional expectation is not appropriate when (X, Y) is not Gaussian.

The statistical framework We have defined the conditional quantile function $q_\alpha(x)$ as the solution of the equation $\Pr\{Y \le q_\alpha(x) \mid X = x\} = \alpha$. Let $\mathbf{F}(x, y)$ be the probability distribution of (X, Y). By using the integral transforms $U_1 = \mathbf{F}_x(X)$ and $U_2 = \mathbf{F}_y(Y)$ where $\mathbf{F}_x$ and $\mathbf{F}_y$ are the marginal distributions, we have:

$$\Pr\left\{Y \le \mathbf{F}_y^{-1}(u_2) \mid X = \mathbf{F}_x^{-1}(u_1)\right\} = \alpha$$

where $u_1 = \mathbf{F}_x(x)$ and $u_2 = \mathbf{F}_y(y) = \mathbf{F}_y(q_\alpha(x))$. It follows that the quantile regression of Y on X is equivalent to solve the following statistical problem:

$$\Pr\{U_2 \le u_2 \mid U_1 = u_1\} = \alpha$$

or:

$$\frac{\partial}{\partial u_1}\mathbf{C}(u_1, u_2) = \alpha$$

where $\mathbf{C}(u_1, u_2)$ is the copula function associated to probability distribution $\mathbf{F}(x, y)$. We have $u_2 = \mathbf{C}_{2|1}^{-1}(u_1, \alpha)$ where $\mathbf{C}_{2|1}(u_1, u_2) = \partial_1 \mathbf{C}(u_1, u_2)$. It follows that:

$$\mathbf{F}_y(y) = \mathbf{C}_{2|1}^{-1}(\mathbf{F}_x(x), \alpha)$$

Finally, we obtain $y = q_\alpha(x)$ where:

$$q_\alpha(x) = \mathbf{F}_y^{-1}\left(\mathbf{C}_{2|1}^{-1}(\mathbf{F}_x(x), \alpha)\right)$$

Remark 177 *In the case where X and Y are independent, we have $\mathbf{C}(u_1, u_2) = u_1 u_2$, $\partial_1 \mathbf{C}(u_1, u_2) = u_2$, $\mathbf{C}_{2|1}^{-1}(u_1, \alpha) = \alpha$ and:*

$$y = q_\alpha(x) = \mathbf{F}_y^{-1}(\alpha)$$

Therefore, the conditional quantile $q_\alpha(x)$ of Y with respect to $X = x$ is equal to the unconditional quantile $\mathbf{F}_y^{-1}(\alpha)$ of Y.

Some special cases Let us assume that the dependence structure is a Normal copula with parameter ρ. On page 737, we have shown that:

$$\mathbf{C}(u_1, u_2) = \int_0^{u_1} \Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u$$

We deduce that:

$$\partial_1 \mathbf{C}(u_1, u_2) = \Phi\left(\frac{\Phi^{-1}(u_2) - \rho\Phi^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$$

Solving the equation $\partial_1 \mathbf{C}(u_1, u_2) = \alpha$ gives:

$$u_2 = \Phi\left(\rho\Phi^{-1}(u_1) + \sqrt{1-\rho^2}\,\Phi^{-1}(\alpha)\right)$$

The conditional quantile function is then:

$$y = q_\alpha(x) = \mathbf{F}_y^{-1}\left(\Phi\left(\rho\Phi^{-1}(\mathbf{F}_x(x)) + \sqrt{1-\rho^2}\Phi^{-1}(\alpha)\right)\right)$$

In the case of the Student's t copula, we have demonstrated that[14]:

$$\mathbf{C}_{2|1}(u_1, u_2; \rho, \nu) = \mathbf{T}_{\nu+1}\left(\left(\frac{\nu+1}{\nu+\left[\mathbf{T}_\nu^{-1}(u_1)\right]^2}\right)^{1/2}\frac{\mathbf{T}_\nu^{-1}(u_2) - \rho\mathbf{T}_\nu^{-1}(u_1)}{\sqrt{1-\rho^2}}\right)$$

Solving the equation $\mathbf{C}_{2|1}(u_1, u_2; \rho, \nu) = \alpha$ gives:

$$u_2 = \mathbf{T}_\nu\left(\rho\mathbf{T}_\nu^{-1}(u_1) + \sqrt{1-\rho^2}\left(\frac{\nu+\left[\mathbf{T}_\nu^{-1}(u_1)\right]^2}{\nu+1}\right)^{1/2}\mathbf{T}_{\nu+1}^{-1}(\alpha)\right)$$

The conditional quantile function is then:

$$y = q_\alpha(x) = \mathbf{F}_y^{-1}\left(\mathbf{T}_\nu\left(\rho\mathbf{T}_\nu^{-1}(\mathbf{F}_x(x)) + \eta\sqrt{1-\rho^2}\right)\right)$$

where:

$$\eta = \left(\frac{\nu+\left[\mathbf{T}_\nu^{-1}(\mathbf{F}_x(x))\right]^2}{\nu+1}\right)^{1/2}\mathbf{T}_{\nu+1}^{-1}(\alpha)$$

Illustration Let us consider an example with two asset returns $(R_{1,t}, R_{2,t})$. We assume that they follow a bivariate Gaussian distribution with $\mu_1 = 3\%$, $\mu_2 = 5\%$, $\sigma_1 = 10\%$, $\sigma_2 = 20\%$ and $\rho = -20\%$. In Figure 14.11, we have reported the conditional quantile function $R_{2,t} = q_\alpha(R_{1,t})$ for different confidence levels α. We verify that the median regression corresponds to the linear regression. The quantile regression shifts the intercept below when $\alpha < 50\%$ and above when $\alpha > 50\%$. We now assume two variants of this example:

1. the dependence structure is the previous Normal copula, but the marginal distributions follow a Student's t_1 distribution[15];

2. the marginal distributions are the previous Gaussian distributions, but the dependence structure is a Student's t_1 copula.

Results are given in Figures 14.12 and 14.13. We deduce that the linearity of the conditional quantile vanishes if the marginals are not Gaussian or the dependence structure is not Gaussian. In the first case, assuming a linear dependence between $R_{1,t}$ and $R_{2,t}$ implies to overestimate on average the conditional return $R_{2,t} \mid R_{1,t}$ when the first asset has high negative returns. In the second case, we obtain the contrary result.

[14] See page 738.

[15] We have

$$\frac{R_{i,t} - \mu_i}{\sigma_i} \sim t_1$$

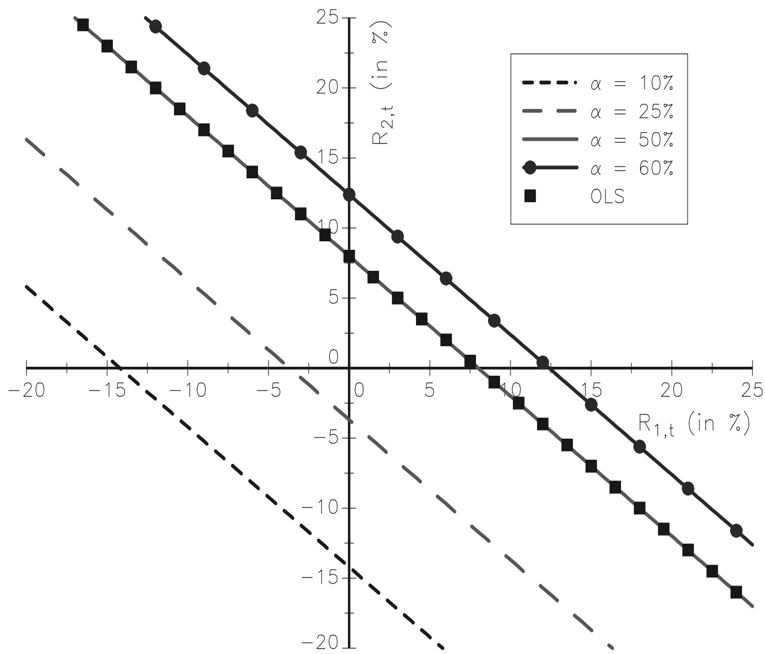

FIGURE 14.11: Conditional quantile (Gaussian distribution)

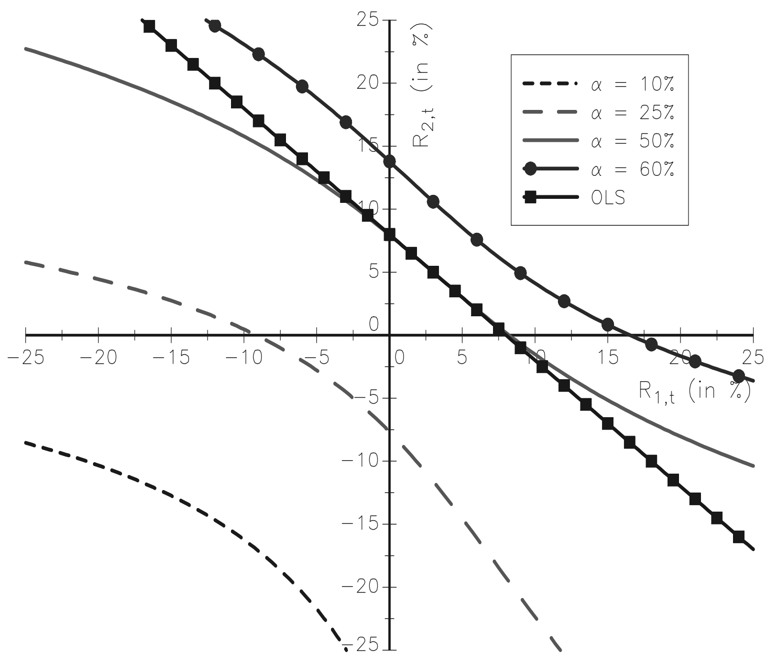

FIGURE 14.12: Conditional quantile (Normal copula and Student's t marginals)

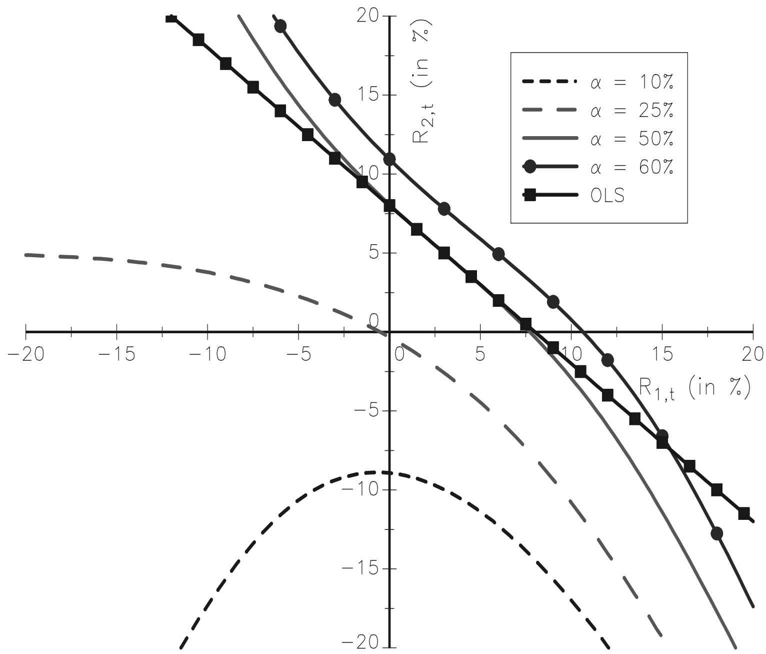

FIGURE 14.13: Conditional quantile (Student's t copula and Gaussian marginals)

14.2.4 Reverse stress testing

According to EBA (2018b), reverse stress test "*means an institution stress test that starts from the identification of the pre-defined outcome (e.g. points at which an institution business model becomes unviable, or at which the institution can be considered as failing or likely to fail) and then explores scenarios and circumstances that might cause this to occur*". The underlying idea is then to identify the set of risk factors that may cause the bankruptcy of the bank (or the financial institution). The difference between stress testing and reverse stress testing can be summarized as follows:

- In stress testing, extreme scenarios of risk factors are used to test the viability of the bank:
$$(\mathbb{S}(\mathcal{F}_1), \ldots, \mathbb{S}(\mathcal{F}_m)) \Rightarrow \mathbb{S}(L(w)) \Rightarrow \begin{cases} D = 0 & \text{if } \mathbb{S}(L(w)) < C \\ D = 1 & \text{otherwise} \end{cases}$$
Using the set of stressed risk factors, we then compute the corresponding loss $\mathbb{S}(L(w))$ of the portfolio. This stress can cause the default of the bank if the stressed loss is larger than its capital C.

- In reverse stress testing, extreme scenarios of risk factors are deduced from the bankruptcy scenario:
$$D = 1 \Rightarrow \mathbb{RS}(L(w)) \Rightarrow (\mathbb{RS}(\mathcal{F}_1), \ldots, \mathbb{RS}(\mathcal{F}_m))$$
We first assume that the bank defaults and compute the associated stressed loss. Then, we deduce the implied set of risk factors that has produced the bankruptcy.

Therefore, reverse stress testing can be viewed as an inverse problem, which can face very quickly a curse of dimensionality.

14.2.4.1 Mathematical computation of reverse stress testing

We assume that the portfolio loss is a function of the risk factors:

$$L(w) = \ell(\mathcal{F}_1, \ldots, \mathcal{F}_m; w)$$

Let $(\mathbb{S}(\mathcal{F}_1), \ldots, \mathbb{S}(\mathcal{F}_m))$ be the stress scenario. The associated loss is given by:

$$\mathbb{S}(L(w)) = \ell(\mathbb{S}(\mathcal{F}_1), \ldots, \mathbb{S}(\mathcal{F}_m); w)$$

Reverse stress testing assumes that the financial institution has calculated the reverse stressed loss $\mathbb{RS}(L(w))$ that may produce its bankruptcy. It follows that the reverse stress scenario $\mathbb{RS}$ is the set of risk factors that corresponds to this stressed loss:

$$\mathbb{RS} = \{(\mathbb{RS}(\mathcal{F}_1), \ldots, \mathbb{RS}(\mathcal{F}_m)) : \ell(\mathbb{S}(\mathcal{F}_1), \ldots, \mathbb{S}(\mathcal{F}_m); w) = \mathbb{RS}(L(w))\}$$

Since we have one equation with m unknows, there is not a unique solution except in some degenerate cases. The issue is then to choose the most plausible reverse stress scenario. For instance, we can consider the following optimization program[16]:

$$\begin{aligned} (\mathbb{RS}(\mathcal{F}_1), \ldots, \mathbb{RS}(\mathcal{F}_m)) &= \arg\max \ln f(\mathcal{F}_1, \ldots, \mathcal{F}_m) \\ &\text{s.t.} \quad \ell(\mathbb{S}(\mathcal{F}_1), \ldots, \mathbb{S}(\mathcal{F}_m); w) = \mathbb{RS}(L(w)) \end{aligned} \tag{14.2}$$

where $f(x_1, \ldots, x_m)$ is the probability density function of the risk factors $(\mathcal{F}_1, \ldots, \mathcal{F}_m)$.

The linear Gaussian case We assume that $\mathcal{F} \sim \mathcal{N}(\mu_{\mathcal{F}}, \Sigma_{\mathcal{F}})$ and $L(w) = \sum_{j=1}^{m} w_j \mathcal{F}_j = w^\top \mathcal{F}$. Problem (14.2) becomes:

$$\begin{aligned} \mathbb{RS}(\mathcal{F}) &= \arg\min \frac{1}{2} (\mathcal{F} - \mu_{\mathcal{F}})^\top \Sigma_{\mathcal{F}}^{-1} (\mathcal{F} - \mu_{\mathcal{F}}) \\ &\text{s.t.} \quad w^\top \mathcal{F} = \mathbb{RS}(L(w)) \end{aligned}$$

The Lagrange function is:

$$\mathcal{L}(\mathcal{F}; \lambda) = \frac{1}{2} (\mathcal{F} - \mu_{\mathcal{F}})^\top \Sigma_{\mathcal{F}}^{-1} (\mathcal{F} - \mu_{\mathcal{F}}) - \lambda \left(w^\top \mathcal{F} - \mathbb{RS}(L(w))\right)$$

We deduce the first-order condition:

$$\frac{\partial \mathcal{L}(\mathcal{F}; \lambda)}{\partial \mathcal{F}} = \Sigma_{\mathcal{F}}^{-1} (\mathcal{F} - \mu_{\mathcal{F}}) - \lambda w = \mathbf{0}$$

It follows that $\mathcal{F} = \mu_{\mathcal{F}} + \lambda \Sigma_{\mathcal{F}} w$. Since we have $w^\top \mathcal{F} = w^\top \mu_{\mathcal{F}} + \lambda w^\top \Sigma_{\mathcal{F}} w$, we obtain:

$$\lambda = \frac{\mathbb{RS}(L(w)) - w^\top \mu_{\mathcal{F}}}{w^\top \Sigma_{\mathcal{F}} w}$$

and:

$$\mathbb{RS}(\mathcal{F}) = \mu_{\mathcal{F}} + \frac{\Sigma_{\mathcal{F}} w}{w^\top \Sigma_{\mathcal{F}} w} \left(\mathbb{RS}(L(w)) - w^\top \mu_{\mathcal{F}}\right) \tag{14.3}$$

Another approach for solving the inverse problem is to consider the joint distribution of $\mathcal{F}$ and $L(w)$:

$$\left(\begin{array}{c} \mathcal{F} \\ L(w) \end{array}\right) \sim \mathcal{N}\left(\left(\begin{array}{c} \mu_{\mathcal{F}} \\ w^\top \mu_{\mathcal{F}} \end{array}\right), \left(\begin{array}{cc} \Sigma_{\mathcal{F}} & \Sigma_{\mathcal{F}} w \\ w^\top \Sigma_{\mathcal{F}} & w^\top \Sigma_{\mathcal{F}} w \end{array}\right)\right)$$

[16] We notice that maximizing the density is equivalent to maximizing its logarithm.

Using Appendix A.2.2.4 on page 1062, we deduce that the conditional distribution of $\mathcal{F}$ given $L(w) = \mathbb{RS}(L(w))$ is Gaussian:

$$\mathcal{F} \mid L(w) = \mathbb{RS}(L(w)) \sim \mathcal{N}\left(\mu_{\mathcal{F}|L(w)}, \Sigma_{\mathcal{F}|L(w)}\right)$$

where:

$$\mu_{\mathcal{F}|L(w)} = \mu_{\mathcal{F}} + \frac{\Sigma_{\mathcal{F}} w}{w^\top \Sigma_{\mathcal{F}} w}\left(\mathbb{RS}(L(w)) - w^\top \mu_{\mathcal{F}}\right)$$

and:

$$\Sigma_{\mathcal{F}|L(w)} = \Sigma_{\mathcal{F}} - \frac{\Sigma_{\mathcal{F}} w w^\top \Sigma_{\mathcal{F}}}{w^\top \Sigma_{\mathcal{F}} w}$$

We know that the maximum of the probability density function of the multivariate normal distribution is reached when the random vector is exactly equal to the mean. We deduce that:

$$\begin{aligned}\mathbb{RS}(X) &= \mu_{\mathcal{F}|L(w)} \\ &= \mu_{\mathcal{F}} + \frac{\Sigma_{\mathcal{F}} w}{w^\top \Sigma_{\mathcal{F}} w}\left(\mathbb{RS}(L(w)) - w^\top \mu_{\mathcal{F}}\right) \qquad (14.4)\end{aligned}$$

Example 162 *We assume that* $\mathcal{F} = (\mathcal{F}_1, \mathcal{F}_2)$*,* $\mu_{\mathcal{F}} = (5, 8)$*,* $\sigma_{\mathcal{F}} = (1.5, 3.0)$ *and* $\rho(\mathcal{F}_1, \mathcal{F}_2) = -50\%$*. The sensitivity vector* w *to the risk factors is equal to* $(10, 3)$*.*

The stress scenario is the collection of univariate stress scenarios at the 99% confidence level:

$$\begin{aligned}\mathbb{S}(\mathcal{F}_1) &= 5 + 1.5 \cdot \Phi^{-1}(99\%) = 8.49 \\ \mathbb{S}(\mathcal{F}_2) &= 8 + 3.0 \cdot \Phi^{-1}(99\%) = 14.98\end{aligned}$$

The stressed loss is then equal to:

$$\mathbb{S}(L(w)) = 10 \cdot 8.49 + 3 \cdot 14.98 = 129.53$$

We assume that the reverse stressed loss is equal to 129.53. Using Formula (14.4), we deduce that $\mathbb{RS}(\mathcal{F}_1) = 10.14$ and $\mathbb{RS}(\mathcal{F}_2) = 9.47$. The reverse stress scenario is very different than the stress scenario even if they give the same loss. In fact, we have $f(\mathbb{S}(\mathcal{F}_1), \mathbb{S}(\mathcal{F}_2)) = 0.8135 \cdot 10^{-6}$ and $f(\mathbb{RS}(\mathcal{F}_1), \mathbb{RS}(\mathcal{F}_2)) = 4.4935 \cdot 10^{-6}$, meaning that the occurrence probability of the reverse stress scenario is more than five times higher than the occurrence probability of the stress scenario.

The general case In the general case, we use a copula function $\mathbf{C}$ in order to describe the joint distribution of the risk factors. We have:

$$\ln f(\mathcal{F}_1, \ldots, \mathcal{F}_m) = \ln c(\mathbf{F}_1(\mathcal{F}_1), \ldots, \mathbf{F}_m(\mathcal{F}_m)) + \sum_{j=1}^{m} \ln f_j(\mathcal{F}_j)$$

where $c(u_1, \ldots, u_m)$ is the copula density, $\mathbf{F}_j$ is the cdf of $\mathcal{F}_j$ and f_j is the pdf of $\mathcal{F}_j$. Finally, we obtain a non-linear optimization problem subject to a non-linear constraint.

14.2.4.2 Practical solutions

There are very few articles on reverse stress testing, and a lack of statistical methods. However, we can cite Grundke (2011), Kopeliovich *et al.* (2015), Glasserman *et al.* (2015) and, Grundke and Pliszka (2018). In these research papers, the optimization problem is generally approximated. For instance, Kopeliovich *et al.* (2015) and Grundke and Pliszka (2018) consider the PCA method to reduce the problem dimension. Glasserman *et al.* (2015) propose to use the method of empirical likelihood in order to evaluate the probability of a reverse stress test.

From a practical point of view, banks generally use a fewer number of risk factors. This helps to reduce the problem dimension. They can also consider a Gaussian approximation. In fact, the main difficulty lies in the equality constraint. This is why they generally consider the following optimization problem:

$$\begin{aligned}\left(\mathbb{RS}\left(\mathcal{F}_1\right),\ldots,\mathbb{RS}\left(\mathcal{F}_m\right)\right) &= \arg\max \ln f\left(\mathcal{F}_1,\ldots,\mathcal{F}_m\right) \\ &\text{s.t.} \quad \ell\left(\mathbb{S}\left(\mathcal{F}_1\right),\ldots,\mathbb{S}\left(\mathcal{F}_m\right);w\right) \geq \mathbb{S}\left(L\left(w\right)\right)\end{aligned}$$

In this case, they can use the Monte Carlo simulation method to estimate the reverse stress scenario.

14.3 Exercises

14.3.1 Construction of a stress scenario with the GEV distribution

1. We note a_n and b_n the normalization constants and $\mathbf{G}$ the limit distribution of the Fisher-Tippet theorem.

   (a) Find the limit distribution $\mathbf{G}$ when $X \sim \mathcal{E}(\lambda)$, $a_n = \lambda^{-1}$ and $b_n = \lambda^{-1}\ln n$.

   (b) Same question when $X \sim \mathcal{U}_{[0,1]}$, $a_n = n^{-1}$ and $b_n = 1 - n^{-1}$.

   (c) Same question when X is a Pareto distribution $\mathcal{P}(\alpha,\theta)$:

   $$\mathbf{F}(x) = 1 - \left(\frac{\theta + x}{\theta}\right)^{-\alpha}$$

   and the normalization constants are $a_n = \theta\alpha^{-1}n^{1/\alpha}$ and $b_n = \theta n^{1/\alpha} - \theta$.

2. We denote by $\mathbf{G}$ the GEV probability distribution:

   $$\mathbf{G}(x) = \exp\left(-\left(1 + \xi\left(\frac{x-\mu}{\sigma}\right)\right)^{-1/\xi}\right)$$

   What is the interest of this probability distribution? Write the log-likelihood function associated to the sample $\{x_1,\ldots,x_T\}$.

3. Show that for $\xi \to 0$, the distribution $\mathbf{G}$ tends toward the Gumbel distribution:

   $$\mathbf{\Lambda}(x) = \exp\left(-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right)\right)$$

4. We consider the minimum value of daily returns of a portfolio for a period of n trading days. We then estimate the GEV parameters associated to the sample of the opposite of the minimum values. We assume that ξ is equal to 1.

   (a) Show that we can approximate the portfolio loss (in %) associated to the return period $\mathcal{T}$ with the following expression:

   $$R(\mathcal{T}) \simeq -\left(\hat{\mu} + \left(\frac{\mathcal{T}}{n} - 1\right)\hat{\sigma}\right)$$

   where $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates of the GEV parameters.

   (b) We set n equal to 21 trading days. We obtain the following results for two portfolios:

Portfolio	$\hat{\mu}$	$\hat{\sigma}$	ξ
#1	1%	3%	1
#2	10%	2%	1

   Calculate the stress scenario for each portfolio when the return period is equal to one year. Comment on these results.

14.3.2 Conditional expectation and linearity

We consider the bivariate Gaussian random vector (X, Y):

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & \rho_{x,y}\sigma_x\sigma_y \\ \rho_{x,y}\sigma_x\sigma_y & \sigma_y^2 \end{pmatrix}\right)$$

1. Using the conditional distribution theorem, show that:

   $$Y = \beta_0 + \beta X + \sigma U$$

   where $U \sim \mathcal{N}(0,1)$. Give the expressions of β_0, β and σ.

2. Deduce the conditional expectation function $m(x)$:

   $$m(x) = \mathbb{E}[Y \mid X = x]$$

3. Let $\left(\tilde{X}, \tilde{Y}\right)$ be the log-normal random vector such that $\tilde{X} = \exp(X)$ and $\tilde{Y} = \exp(Y)$. Find the conditional expectation function $\tilde{m}(x)$:

   $$\tilde{m}(x) = \mathbb{E}\left[\tilde{Y} \mid \tilde{X} = x\right]$$

4. Comment on these results.

14.3.3 Conditional quantile and linearity

Let X and Y be a $n \times 1$ random vector and a random variable. We assume that (X, Y) is Gaussian:

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \Sigma_{x,x} & \Sigma_{x,y} \\ \Sigma_{y,x} & \Sigma_{y,y} \end{pmatrix}\right)$$

We note $\mathbf{F}_x(x)$ and $\mathbf{F}_y(x)$ the marginal distributions, and $\mathbf{F}(x, y)$ the joint distribution.

1. Calculate the conditional distribution $\mathbf{F}(y \mid X = x)$ of the random variable $Y(x) = Y \mid X = x$. Deduce the conditional quantile defined by:
$$q_\alpha(x) = \inf\{q : \Pr(Y(x) \leq q) \geq \alpha\}$$

2. Show that:
$$q_\alpha(x) = \beta_0(\alpha) + \beta^\top x$$
where $\beta_0(\alpha)$ is a function that depends on the confidence level α.

3. Compare $q_\alpha(x)$ with the conditional expectation $m(x)$. Deduce the main difference between linear regression and quantile regression.

4. We consider an exponential default time $\tau \sim \mathcal{E}(\lambda)$ that depends on the risk factors X. Moreover, we assume that X is Gaussian $\mathcal{N}(\mu_x, \Sigma_{x,x})$ and the dependence between the default time τ and the risk factors X is a Normal copula. Find the conditional quantile function $q_\alpha^\tau(x)$ of the random variable $\tau(x) = \tau \mid X = x$.

5. We now consider the probability of default PD associated to the default time $\tau \sim \mathcal{E}(\lambda)$. Calculate the conditional quantile function $q_\alpha^{\mathrm{PD}}(x)$ of the random variable $\mathrm{PD}(x) = \mathrm{PD} \mid X = x$.

6. We consider the single factor case where $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and we assume that the parameter of the Normal copula between τ and X is equal to ρ. Show that:
$$q_\alpha^{\mathrm{PD}}(x) = \Phi\left(\Phi^{-1}(\alpha)\sqrt{1-\rho^2} + \rho\frac{(x-\mu_x)}{\sigma_x}\right)$$

7. Comment on these results and propose a quantile regression model to stress the probability of default.